Problem 1 Λ exterior alg of ran 8:

$$\Lambda = k\langle x_1, \dots, x_8\rangle / (x_i^2,\ x_i x_j + x_i x_j)$$

(k field) and take

$$w = x_1x_2 + x_3x_4 + x_5x_6 + x_7x_8$$

Show: (a) $\Lambda_i \xrightarrow{\cdot w} \Lambda_{i+2}$ is injective for $i \leq 3$ and surjective for $i \geq 3$

(b) For $\Lambda \xrightarrow{\cdot w} \Lambda$ we have

$$\dim_k \mathrm{Ker}(\cdot w) + \dim_k \mathrm{Coker}(w) = \binom{10}{5}$$

Pro

Problem 2

In a finite tensor cat $\mathcal{C}$ every object X has a left dual X^* a right dual ${}^*X$. Can show: $\forall X, Y, Z$ $\exists$ natural iso's

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*)$$
$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, {}^*X \otimes Z)$$
$$\mathrm{Hom}_{\mathcal{C}}(X^* \otimes Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, X \otimes Z)$$
$$\mathrm{Hom}_{\mathcal{C}}(X \otimes {}^*Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y)$$

Show that this implies all of the following:

(a) $\otimes$ is bi-exact (so $X \otimes -$ and $- \otimes X$ exact $\forall X \in \mathcal{C}$)

(b) The proj obj form a two-sided ideal in $\mathcal{C}$:

$$P \text{ proj} \Rightarrow P \otimes X \text{ proj and } X \otimes P \text{ proj } \forall X \in \mathcal{C}$$

(c) P proj $\Rightarrow$ P^* proj and then that

$$P \text{ proj} \Rightarrow P \text{ inj} \quad \text{(and vice versa)}$$

Problem 3

For a finite tensor cat $(\mathcal{C}, \otimes, \mathbb{1})$ look at

$$H^*(\mathcal{C}) = \bigoplus_{n=0}^{\infty} \mathrm{Ext}^n_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$$

Show that an element $\eta \in H^n(\mathcal{C})$ corresponds to an exact sequence

$$0 \to \mathbb{1} \to K_\eta \to P_{n-2} \to \cdots \to P_0 \to \mathbb{1} \to 0$$

with P_i proj. So we get a complex

$$\cdots \to 0 \to K_\eta \to P_{n-2} \to \cdots \to P_0 \to 0 \to \cdots$$

called C_η. For $\eta_1, \ldots, \eta_t \in H^*(\mathcal{C})$, show that the complex $\eta_1 \otimes \cdots \otimes \eta_t$ is "almost" a perfect complex.

If V is a complex rep^n of G

Calculate $\mathbb{P}(V)^G$ (fixed lines in V)

Note $\mathbb{P}(V) = S(V)/T$ $\quad (T = \{z \mid |z| = 1\})$

Now $L_p(V) = S(V)/C_p$ $\quad (C_p = \{z \mid z^p = 1\})$

& calculate $L_p(V)^G$

$\mathbb{R}P^n = S(\mathbb{R}^{n+1})/\pm 1$.

Calculate $K(0)_*(\mathbb{R}P^5)$ (at $p=2$!).

$K(1)_*(\mathbb{R}P^5)$ (Use $Q_0 = Sq^1 = \beta$, degree 1)

$K(2)_*(\mathbb{R}P^5)$ (Use $Q_1 = Sq^1 Sq^2 + Sq^2 Sq^1$, degree 3)

$H^*(X; K(n)^*) \Rightarrow K(n)^*(X)$

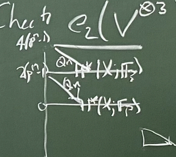
& Q_n is the first

H

(Kuhn-Lloyd 6.9) $e_2 \in \mathbb{Z}_{(2)}[\Sigma_3]$, $s = (12)$, $t = (123)$.

Let $e_2 = \frac{1}{3}(1+[s])(1-[t])$

Check $e_2(V^{\otimes 3}) = \begin{cases} 0 & \text{if } \dim V = 1 \\ 2\dim & \text{if } \dim V = 2 \\ \text{larger} & \text{if } \dim V \geq 3 \end{cases}$



CA betti numbers: $\quad \Omega_i(M) = \operatorname{im} \partial_i \cong \ker \partial_{i-1}$ $\quad F = \cdots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \cdots$ min free res of M

1) Show $\beta_i(M) = \dim_k \operatorname{Tor}_i^R(M,k) = \dim_k \operatorname{Ext}_R^i(M,k)$

2) Write the Koszul complex on 3 elements (modules + differentials)

3) Show that $\beta_i(M) = \operatorname{rk} \Omega_i(M) + \operatorname{rk} \Omega_{i+1}(M)$

4) Show that $\sum_{i=0}^{\operatorname{pdim} M} (-1)^i \beta_i(M) = \operatorname{rk} M$

5) Show that $\beta_2(R/I)$ can be arbitrarily large for $I = (f,g,h)$

OPEN Problem Is there an ideal of ht 5 with Betti numbers

1. Let P be a finite length complex of free R-modules w/ finite length homology.

(a). Prove the inequality $h(P \otimes_R P) \leq h(P)\,\mathrm{rank}(P)$

(b). Assume P is a resolution of R/I. Prove that the equality

$$h(P \otimes_R P) = h(P)\,\mathrm{rank}(P)$$

implies I/I^2 is a free R/I-module.

(c). Deduce that R/I is c.i. (Hint: Ask Ben)

2. Assume R is Cohen-Macaulay. Let $P \in \mathrm{Perf}^{fl}(R)$ be a complex of length n. Prove that $H_i(P) = 0$ for $i \geq n - \dim R$.

Hint: Tensor w/ the Čech complex $\check{C}(\underline{x})$ on a s.o.p.

3. Prove that if 2 is a unit, there is a direct sum decomp.

$$P \otimes_R P \simeq S^2(P) \oplus \Lambda^2 P,$$

where

$$S^2(P) - \{ v \otimes v' + v' \otimes v \in P \otimes_R P \},$$

$$\Lambda^2 P = \{ v \otimes v' - v' \otimes v \in P \otimes_R P \}$$

1) Figure out minimal Sullivan model for X and prove the Toral Rank Conj. for X.

a) $X = S^{2n+1}$ b) $X = S^{2n}$

(might want to ask me for ansatz)

2) $A = \Lambda(a, b, c)$ $da = 0 = db$, $dc = ab$

deg 1

Compute $H(A)$ and show that (A, d) is not formal

3) $X = S^3 \times S^3 \times S^3$, $T^2 \curvearrowright X$

$(s,t) \cdot (u, v, w) = (su, stv, tw)$

where $S^1 \curvearrowright S^3$ via Hopf action.

Compute model of X_T and show it is not formal

4) Suppose $H(X_T)$ is a free $\underset{\overset{\|}{R}}{H(BT)}$-module ("equivariantly formal"). Prove that action is mod. formal i.e. model $(R \otimes \Lambda V, D)$ for X_T is formal as R-module

(c). Deduce that R/I is c.i. (Hint: ASK BEN)

4. Assume char $R = 2$.

(a). Show that for every free R-module V,
there is a SES:

$$0 \to F^2(V) \to S^2(V) \to \Lambda^2 V \to 0$$

(still in char 2)
(b). In general, let $S^d(V) \xrightarrow{\varphi} \Lambda^d V$ be the canonical surj.
Show that $\ker \varphi$ admits a filtration w/ assoc. graded pieces of the form $\Lambda^{d-2i} V \otimes F^2 S^i(V)$.

• Suppose X, Y complexes of f.d. vector spaces, and there is a triangle

$$X \to Y \to Y[n] \to \cdots$$

some $n \neq 0$.

Show $H_*(X)$ grows exponentially iff so does $H_*(Y)$.

• Thm (André) If $R \to S$ is flat local, $R \searrow k \swarrow S$ ← res. field.

then $D_n(R,k) \to D_n(S,k)$ is injective for $n \gg 0$.

Using this, show that if $D_n(R) \to D_n(R,k)$ has exp. growing image, and $R \to S$ flat, then same for S.

Eg if $X = (S^n)^3 \setminus \{pt\}$. $\underline{n \text{ odd}}$. $|e_i| = n$

then $A = C^*(X;\mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(e_1, e_2, e_3) / (e_1 e_2 e_3)$
(exterior alg modulo top. elt).

Let $\Lambda = \Lambda_{\mathbb{Q}}(e_1, e_2, e_3)$.

Show that $A \overset{L}{\underset{\Lambda}{\otimes}} k \simeq k \ltimes V$ ← some ∞-dim gr. vect. sp.

Conclude that $HH_*(A)$ grows exp.

$\Rightarrow$ Same for $H^*(LX;\mathbb{Q})$

Let $R = k \ltimes V$ trivial algebra, $\dim V \geq 2$.

Using the bar construction, show that $HH_*(R)$ has exp growth.

bar construction: defn of HH

$HH_*(A)$ = homology of $\left[\bigoplus_{n \geq 0} (A_+[1])^{\otimes n} \otimes A,\ \partial(a_1 \otimes \cdots \otimes a_n \otimes a) = \sum \pm a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes a \pm a_2 \otimes \cdots \otimes a_n \otimes a a_1 \pm a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n a\right]$

$A_+ = A_{>0} = A/k$

Ben's hints: exp growth: $\exists\, c>1$ s.t ∞ many n $\quad a_n > c^n$.

$\iff \limsup \sqrt[n]{a_n} > 1$

$\iff \sum a_n t^n$ does not converge on unit disc.

Can use this for ③ $\rightsquigarrow$ long exact sequence $H_i(X) \to H_i(Y) \to H_{i-1}(Y) \to \cdots$

bound $\sum t^i \dim H_i(X)$ in terms of $\sum t^i \dim H_i(Y)$ and vice versa.

For ④. $\Lambda \to F \xrightarrow{\simeq} k$ min res (this is a dg algebra but you don't need this).

Show $\operatorname{cone}(F \to \Lambda)$ is the minimal res of $\Lambda/(e_1e_2e_3)$ over Λ

$F_0 = \Lambda,\ 1 \longmapsto e_1e_2e_3$ $\quad$ and it is a dg algebra with $F^2 = 0$.