Δ-critical quasi-hereditary algebras

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21 April 2018

Part 1: Introduction and Background

Introduction: Main References

- D. Happel and D. Vossieck, *Minimal algebras of infinite representation type with preprojective component*, Manuscr. Math., **42** (1983), 221–243.
- C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z., **208** (1991), 209–223.

Introduction: Preliminaries

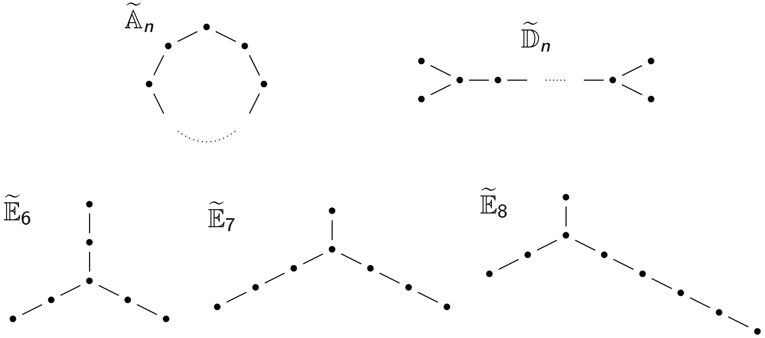
Throughout the talk,

- k is an algebraically closed field,
- A is a basic finite dimensional k-algebra, typically the path algebra kQ or kQ/I of a finite quiver without oriented cycles.

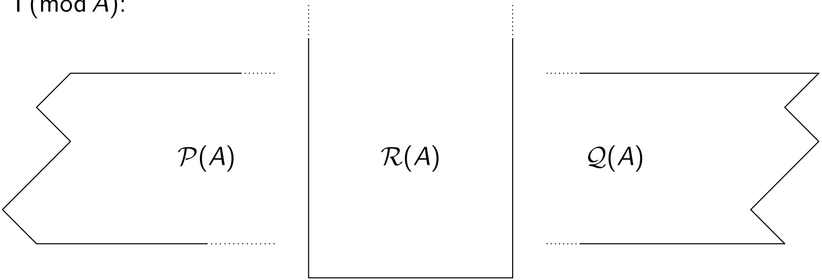
Introduction: Tame hereditary algebras

Theorem (Donovan–Freislich, Nazarova)

A connected hereditary k-algebra A is tame exactly if it is the path algebra kQ of a quiver Q whose underlying graph is a Euclidean diagram, i.e. a graph of type $\widetilde{\mathbb{A}}_n$ *(*$n \geq 1$*, no oriented cycle),* $\widetilde{\mathbb{D}}_n$ *(*$n \geq 4$*),* $\widetilde{\mathbb{E}}_6$*,* $\tilde{\mathbb{E}}_7$ *or* $\widetilde{\mathbb{E}}_8$*.*



$\Gamma(\mathrm{mod}\, A)$:



The Auslander–Reiten quiver $\Gamma(\mathrm{mod}\, A)$ of a tame hereditary algebra A consists of a preprojective component $\mathcal{P}(A)$, a preinjective component $\mathcal{Q}(A)$ and a regular part $\mathcal{R}(A)$.

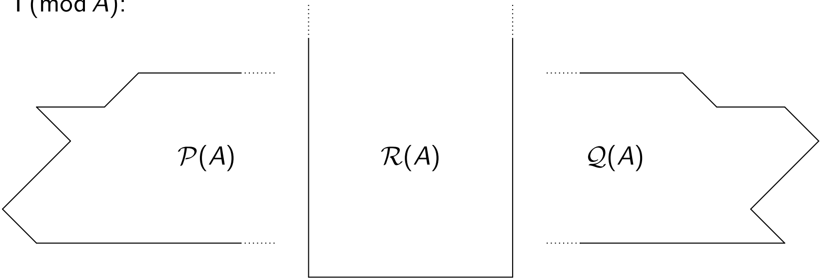
Introduction: Tame concealed algebras

Definition

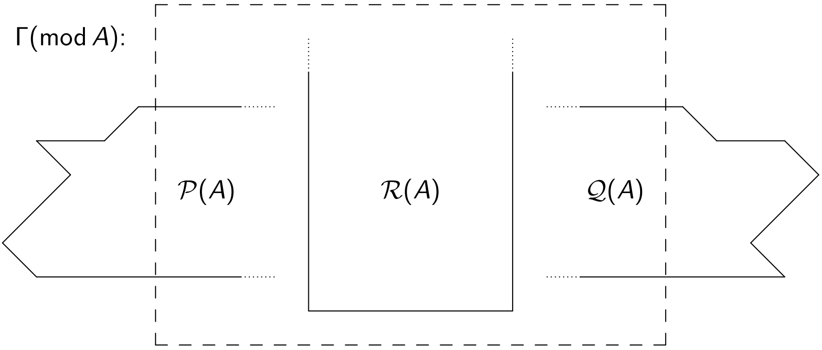
A k-algebra B is called *tame concealed* if there exists a tame connected hereditary algebra A and a preprojective (or preinjective) tilting module $T_A \in \operatorname{mod} A$ such that $B = \operatorname{End} T_A$.

Introduction: Tame concealed algebras

$\Gamma(\text{mod}\, A)$:



Question:
In which way are tame concealed algebras "concealed"?



Answer:
Apart from finite parts at the ends of the preprojective and the preinjective component, their Auslander-Reiten quiver looks like that of a tame hereditary algebra.

Introduction: Tame concealed algebras

Happel and Vossieck constructed all tame concealed k-algebras.

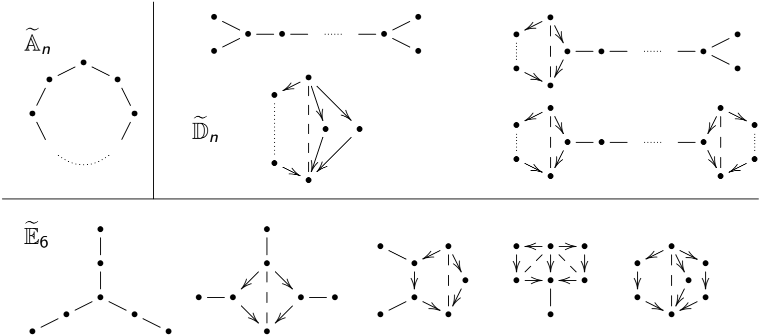
There are many of these, even for relatively small quivers:

n	1	2	3	4	5	6	7	8	
$\widetilde{\mathbb{A}}_n$	1	1	3	3	8	9	21	29	...
$\widetilde{\mathbb{D}}_n$				6	13	32	60	131	...
$\widetilde{\mathbb{E}}_n$						56	437	3801	

Introduction: Tame concealed algebras

Theorem (Happel–Vossieck)

A connected k-algebra A is tame concealed exactly if it is the path algebra kQ/I of a quiver Q (possibly with relations) given by one of 149 "frames" (1 for $\widetilde{\mathbb{A}}_n$, 4 for $\widetilde{\mathbb{D}}_n$, 5 for $\widetilde{\mathbb{E}}_6$, 22 for $\widetilde{\mathbb{E}}_7$, 117 for $\widetilde{\mathbb{E}}_8$).



Introduction: Quasi-hereditary algebras

Let Q be a finite quiver without oriented cycles (possibly with relations), let I be the two sided ideal in kQ generated by the relations of Q, and set $A = kQ/I$.

For every vertex $i \in Q_0$, let $S(i)$ be the simple module for the vertex i, $P(i)$ the projective cover of $S(i)$, $I(i)$ the injective envelope of $S(i)$.

Introduction: Quasi-hereditary algebras

Definition

An *enumeration* of A is a bijective map $\pi \colon Q_0 \to \{1, 2, \dots, |Q_0|\}$.

Given an enumeration π of A, let

- $\Delta_\pi(i)$ the maximal factor module $P(i)$,
- $\nabla_\pi(i)$ the maximal submodule of $I(i)$,

both with only composition factors $S(j)$ where $\pi(j) \leq \pi(i)$.

Set $\Delta_\pi = \{\Delta_\pi(i) \mid i \in Q_0\}$ and $\nabla_\pi = \{\nabla_\pi(i) \mid i \in Q_0\}$.

Definition

Two enumerations π and π' are *equivalent* if $\nabla_\pi = \nabla_{\pi'}$ (up to symmetries of the quiver).

Introduction: Quasi-hereditary algebras

Definition (Scott, Cline–Parshall–Scott)

Let $A = kQ/I$ as before and let π be an enumeration of A. The pair (A, π) is called a *quasi-hereditary algebra* if the following equivalent conditions hold:

- for each $i \in Q_0$, the module $P(i)$ has a Δ_π-filtration and $S(i)$ occurs only once as a composition factor of $\Delta_\pi(i)$,
- for each $i \in Q_0$, the module $I(i)$ has a ∇_π-filtration and $S(i)$ occurs only once as a composition factor of $\nabla_\pi(i)$.

In this case, the modules $\Delta_\pi(i)$ are called the *standard modules* and the $\nabla_\pi(i)$ are called the *costandard modules* of (A, π).

For convenience, also call π a *quasi-hereditary enumeration* of A if (A, π) is quasi-hereditary.

Introduction: Quasi-hereditary algebras

Remark

In the special case of Q being a finite quiver without oriented cycles, $S(i)$ occurs only once as a composition factor of $P(i)$ and of $Q(i)$, so the conditions of $S(i)$ occuring only once in $\Delta_\pi(i)$ and $\nabla_\pi(i)$ always hold.

Examples

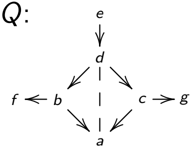
For a finite quiver Q without oriented cycles:

- There always exists an equivalence class of *trivial enumerations* π such that $\pi(j) > \pi(i)$ for all arrows $i \to j \in Q_1$. In this case, we have $\Delta_\pi(i) = S(i)$ and $\nabla_\pi(i) = I(i)$ for all $i \in Q_0$, so trivial enumerations are always quasi-hereditary.
- [Dlab–Ringel]: The algebra $A = kQ/I$ is hereditary if and only if all enumerations of A are quasi-hereditary.

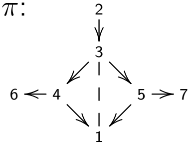
Introduction: Quasi-hereditary algebras

Now consider the algebra $A = kQ/I$ with enumeration π:

Q:



π:



$i \in Q_0$	$\pi(i)$	$S(i)$	$I(i)$	$\nabla_\pi(i)$
a	1	$\begin{smallmatrix}&&0&&\\0&0&0&0&0\\&&1&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&1&1&1&0\\&&1&&\end{smallmatrix}$	$\begin{smallmatrix}&&0&&\\0&0&0&0&0\\&&1&&\end{smallmatrix}$
b	4	$\begin{smallmatrix}&&0&&\\0&1&0&0&0\\&&0&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&1&1&0&0\\&&0&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&1&1&0&0\\&&0&&\end{smallmatrix}$
c	5	$\begin{smallmatrix}&&0&&\\0&0&0&1&0\\&&0&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&0&1&1&0\\&&0&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&0&1&1&0\\&&0&&\end{smallmatrix}$
d	3	$\begin{smallmatrix}&&0&&\\0&0&1&0&0\\&&0&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&0&1&0&0\\&&0&&\end{smallmatrix}$	$\begin{smallmatrix}&&1&&\\0&0&1&0&0\\&&0&&\end{smallmatrix}$

The module $I(a)$ does not have a ∇_π-filtration, so (A, π) is not quasi-hereditary.

Part 2: Δ-critical algebras

Δ-critical algebras: Definition and properties

Theorem (Ringel)

Let (A, π) be a quasi-hereditary algebra with standard modules Δ_π. Then the category $\mathcal{F}(\Delta_\pi)$ of A-modules that have a Δ_π-filtration is a functorially finite subcategory of A-mod which is closed under extensions and direct summands.

Corollary (Ringel, using Auslander–Smalø)

The category $\mathcal{F}(\Delta_\pi)$ has (relative) AR-sequences.

Definition (Ringel)

A *Δ-critical algebra* (A, π) is a tame concealed quasi-hereditary algebra for which all costandard modules $\nabla_\pi(i)$ are preinjective.

Δ-critical algebras: Definition and properties

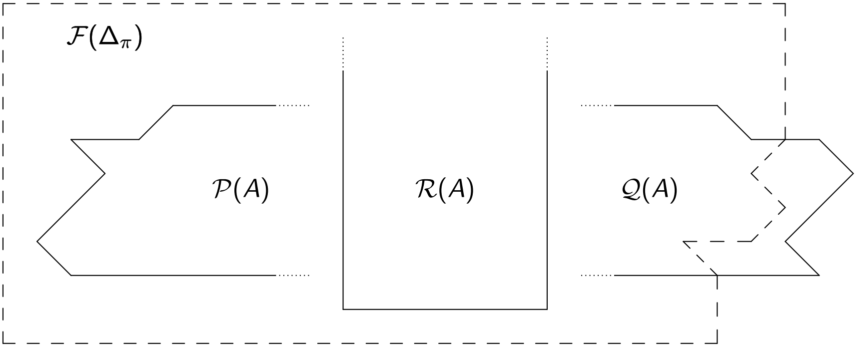
From now on, let (A, π) be a Δ-critical algebra.

The category $\mathcal{F}(\Delta_\pi)$ can then be described as follows:

$$\begin{aligned}\mathcal{F}(\Delta_\pi) &= \{M \in \operatorname{mod} A \,|\, \operatorname{Ext}^1(M, \nabla_\pi(i)) = 0 \text{ for all } i \in Q_0\} \\ &= \{M \in \operatorname{mod} A \,|\, \operatorname{Hom}(\nabla_\pi(i), \tau M) = 0 \text{ for all } i \in Q_0\}\end{aligned}$$

In particular, since all $\nabla_\pi(i)$ are preinjective, $\mathcal{F}(\Delta_\pi)$ contains all preprojective and all regular A-modules.

Δ-critical algebras: Definition and properties



In particular, since all $\nabla_\pi(i)$ are preinjective, $\mathcal{F}(\Delta_\pi)$ contains all preprojective and all regular A-modules.

Δ-critical algebras: Definition and properties

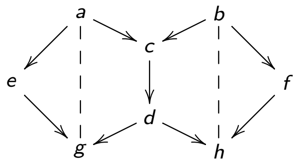
Theorem (Ringel)

Let (A, π) be a Δ-critical algebra. Then the following hold:

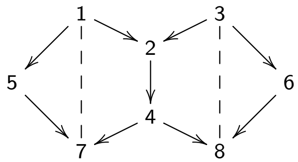
- *There exists a unique basic tilting module T_π such that $\mathcal{F}(\Delta_\pi) \cap \mathcal{F}(\Delta_\pi) = \mathrm{add}(T_\pi)$, called the* characteristic tilting module.
- *The characteristic module T_π admits a decomposition $T_\pi = \oplus_{i \in Q_0} T_\pi(i)$ such each $T_\pi(i)$ has a composition factor $S(i)$ and all other composition factors are of the form $S(j)$ with $\pi(j) < \pi(i)$.*
- *The category $\mathcal{F}(\Delta_\pi)$ has a preprojective component of type A and a preinjective component of type B, where B is the Ringel dual $B = \mathrm{End}(T_\pi)^{\mathrm{op}}$.*

Consider the algebra $A = kQ/I$ with enumeration π:

Q:

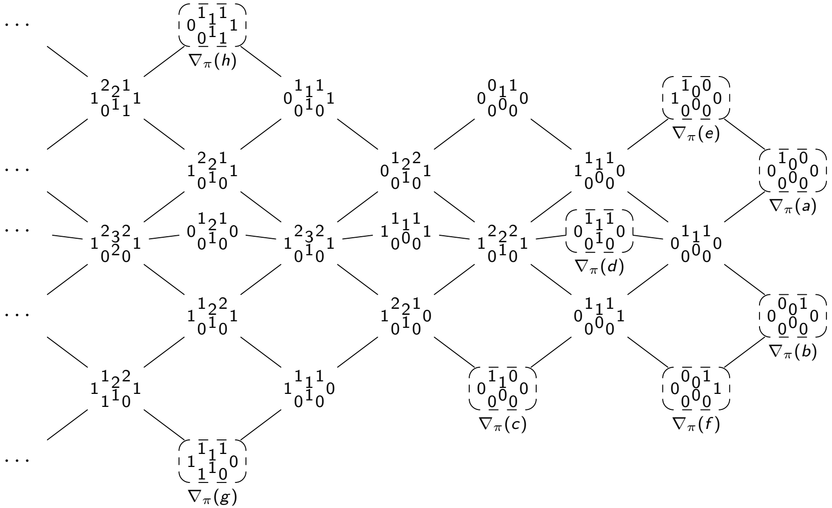


π:

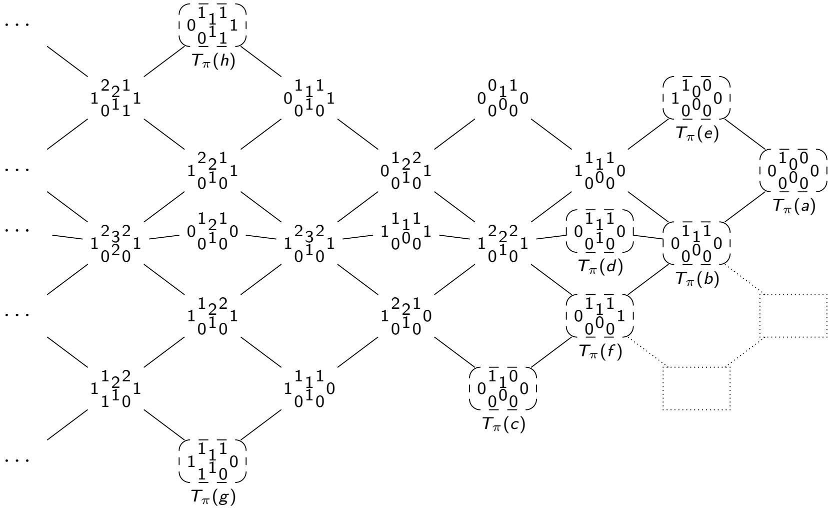


$i \in Q_0$	$\pi(i)$	$S(i)$	$P(i)$	$\Delta_\pi(i)$	$I(i)$	$\nabla_\pi(i)$	$T_\pi(i)$
a	1	$\begin{smallmatrix} & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & 1 & 1 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$
b	3	$\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 1 & 1 & 1 & \end{smallmatrix}$	$\begin{smallmatrix} & \mathbf{0} & & \mathbf{1} & \\ \mathbf{0} & & \mathbf{1} & & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 0 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & \mathbf{1} & & \mathbf{1} & \\ \mathbf{0} & & \mathbf{1} & & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{smallmatrix}$
c	2	$\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 1 & 1 & 1 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & \mathbf{1} & & \mathbf{0} & \\ \mathbf{0} & & \mathbf{1} & & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{smallmatrix}$	$\begin{smallmatrix} & \mathbf{1} & & \mathbf{0} & \\ \mathbf{0} & & \mathbf{1} & & \mathbf{0} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{smallmatrix}$
f	6	$\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & 0 & 1 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 0 & & 1 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & 0 & & 1 & \\ 0 & & 0 & & 1 \\ & 0 & 0 & 0 & \end{smallmatrix}$	$\begin{smallmatrix} & \mathbf{1} & & \mathbf{1} & \\ \mathbf{0} & & \mathbf{1} & & \mathbf{1} \\ & \mathbf{0} & \mathbf{0} & \mathbf{0} & \end{smallmatrix}$

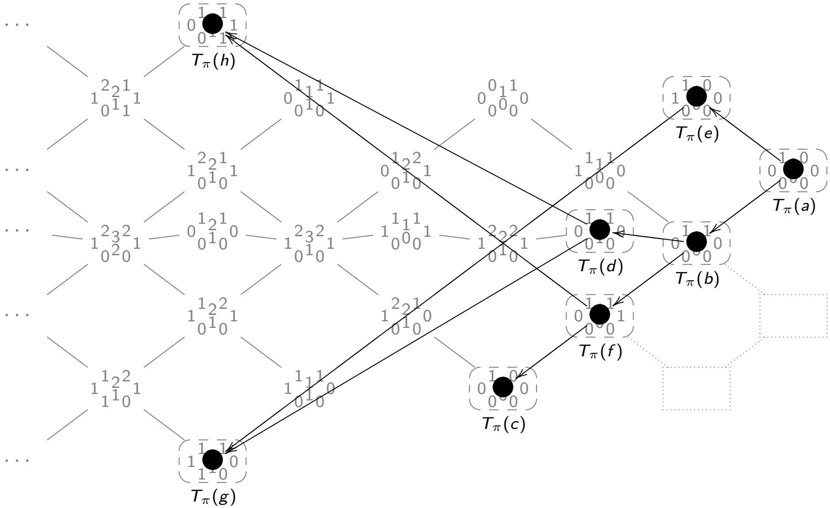
Δ-critical algebras: Definition and properties



Δ-critical algebras: Definition and properties

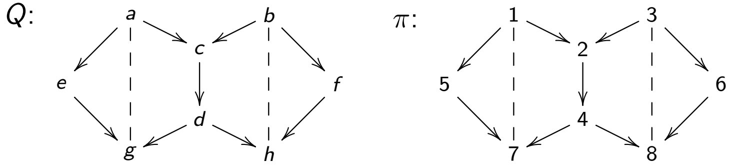


Δ-critical algebras: Definition and properties

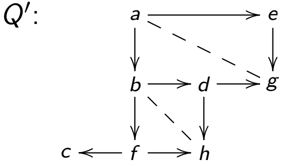


Δ-critical algebras: Definition and properties

Consider the algebra $A = kQ/I$ with enumeration π:



The Ringel dual $B = \mathrm{End}(T_\pi)^{\mathrm{op}}$ is of the form $B = kQ'/I'$ with Q' as follows:



Δ-critical algebras: Definition and properties

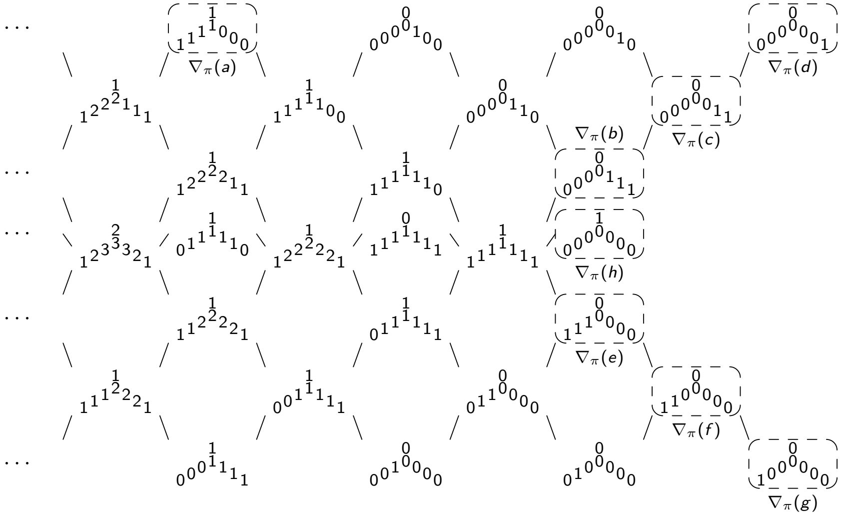
Proposition (Ringel)

If $A = kQ/I$ is a tame concealed algebra and U is a uniform (i.e. a module with a simple socle) preinjective A-module, then there exists a quasi-hereditary enumeration π of Q such that $U = \nabla_\pi(i)$ for some $i \in Q_0$.

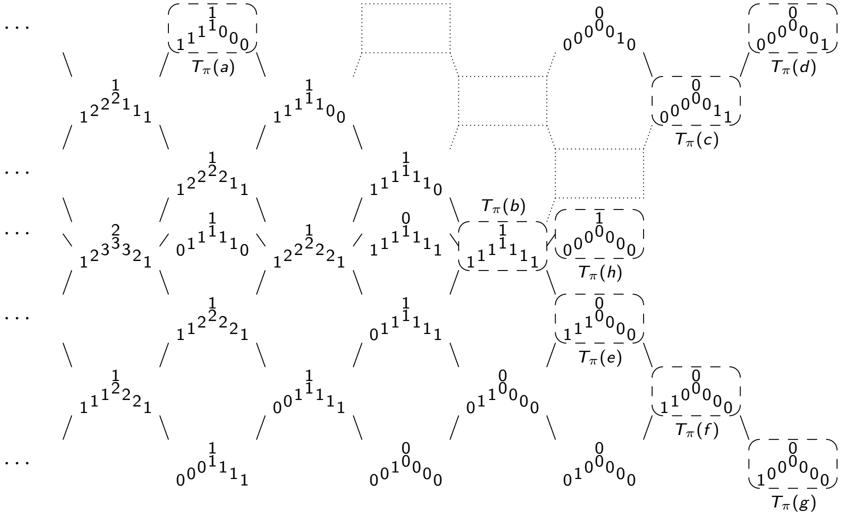
Proposition (Ringel)

If $A = kQ/I$ is a tame concealed algebra and π is an enumeration of Q such that $\nabla_\pi(i)$ is preinjective for all $i \in Q_0$, then (A, π) is quasi-hereditary (and thus also Δ-critical).

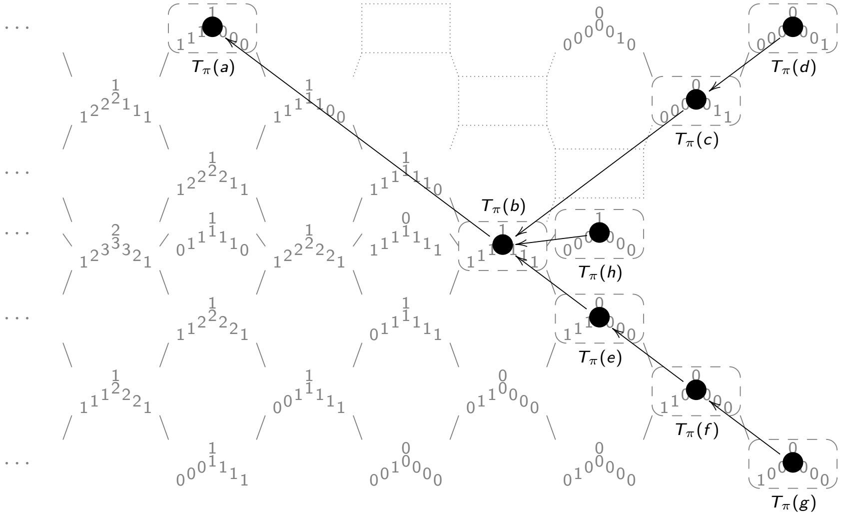
Δ-critical algebras: Definition and properties



Δ-critical algebras: Definition and properties



Δ-critical algebras: Definition and properties



Part 3: Classification

Classification: Type $\widetilde{\mathbb{A}}_n$

Aim: Classify all Δ-critical algebras

In case $\widetilde{\mathbb{A}}_n$, all irreducible maps in the injective component are surjective:

- This is obvious for the injective modules.
- If M is not injective, then there exists an AR-sequence $0 \to M \to X_1 \oplus X_2 \to N \to 0$ with indecomposable modules X_1 and X_2. By induction, $\dim X_1 > \dim N$ and $\dim X_2 > \dim N$ hold and imply $\dim M > \dim X_1$ and $\dim N > \dim X_2$.

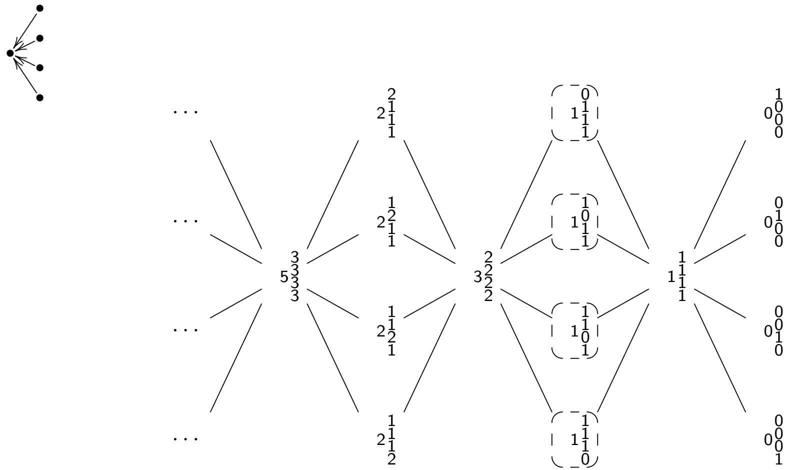
So the injective component does not contain any proper submodule of any $I(i)$ and the only possible enumerations are the trivial ones.

Classification: Type $\widetilde{\mathbb{A}}_n$

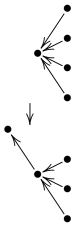
Aim: Classify all Δ-critical algebras:

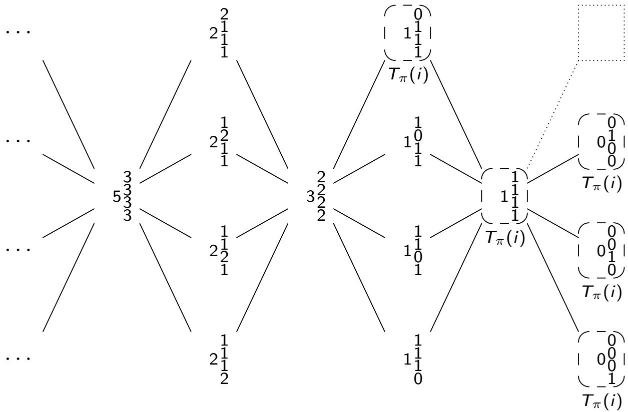
- Type $\widetilde{\mathbb{A}}_n$: ✓

Classification: Type $\widetilde{\mathbb{D}}_4$

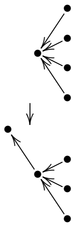


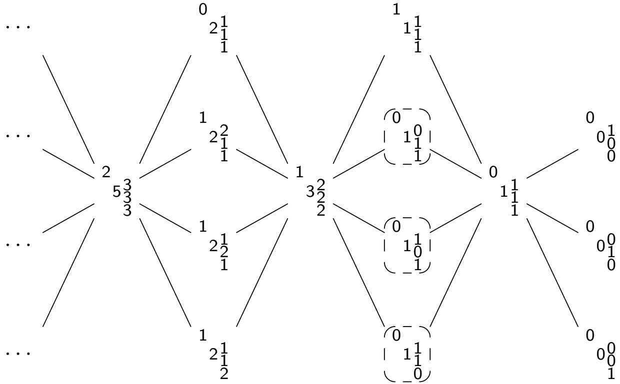
Classification: Type $\widetilde{\mathbb{D}}_4$



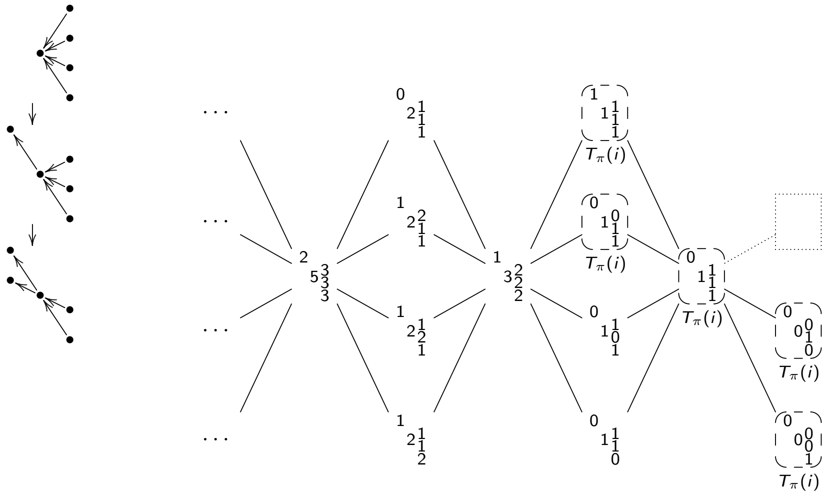


Classification: Type $\widetilde{\mathbb{D}}_4$

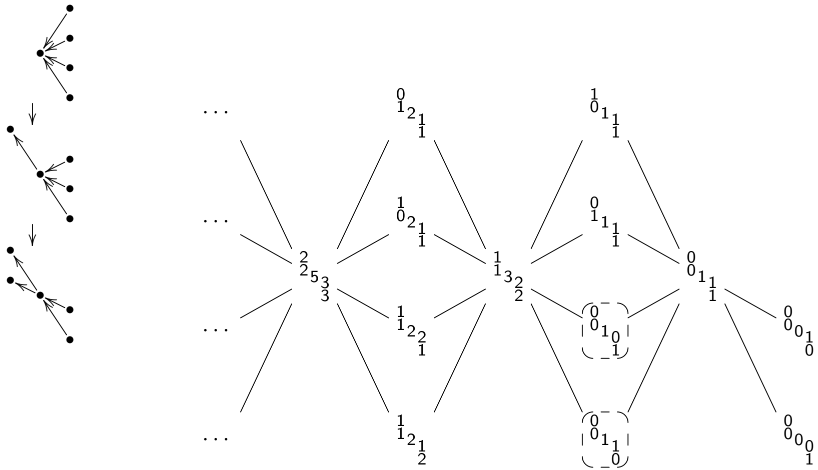




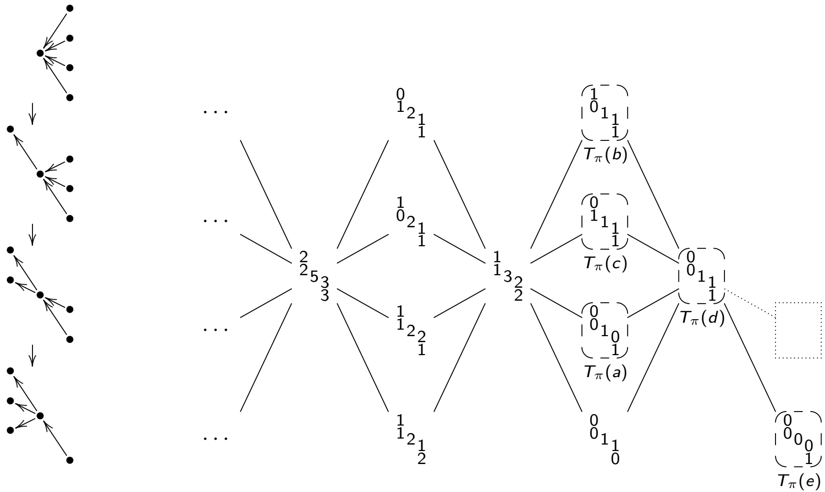
Classification: Type $\widetilde{\mathbb{D}}_4$



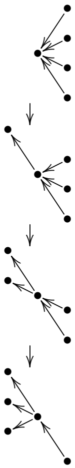
Classification: Type $\widetilde{\mathbb{D}}_4$

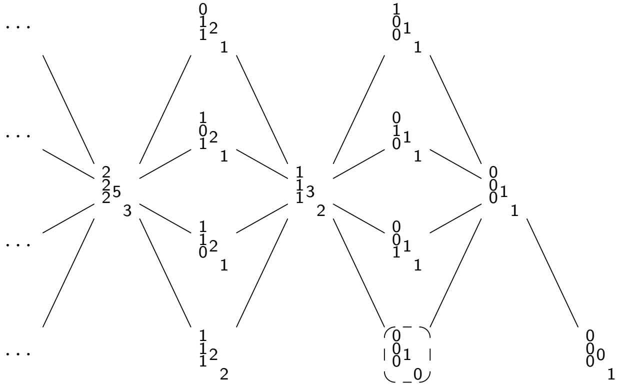


Classification: Type $\widetilde{\mathbb{D}}_4$

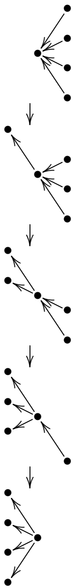


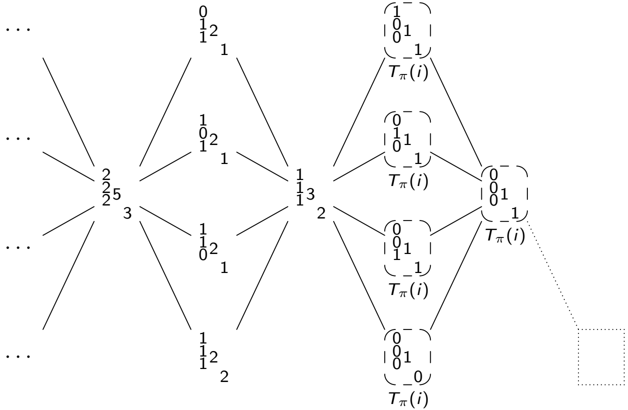
Classification: Type $\widetilde{\mathbb{D}}_4$



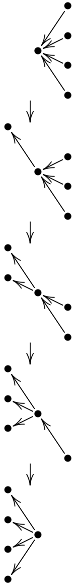


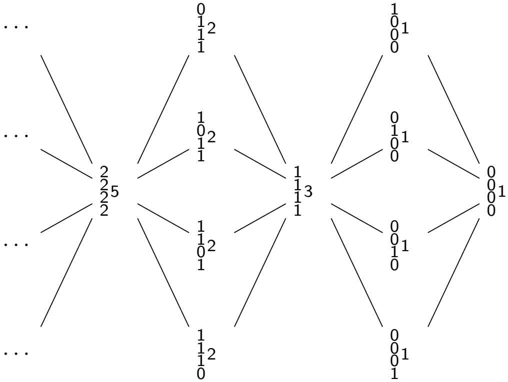
Classification: Type $\widetilde{\mathbb{D}}_4$



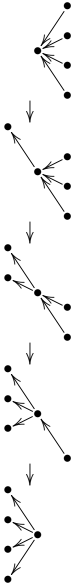


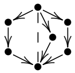
Classification: Type $\widetilde{\mathbb{D}}_4$

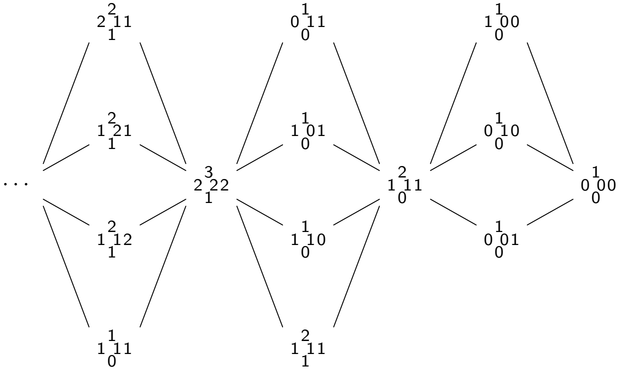




Classification: Type $\widetilde{\mathbb{D}}_4$







Classification: Type $\widetilde{\mathbb{D}}_4$

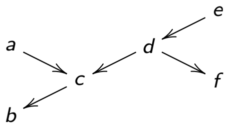
Aim: Classify all Δ-critical algebras:

- Type $\widetilde{\mathbb{A}}_n$: ✓
- Type $\widetilde{\mathbb{D}}_4$: ✓

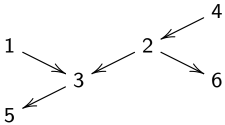
Classification: Type $\widetilde{\mathbb{D}}_5$

Consider the algebra $A = kQ/I$ with enumeration π:

Q:

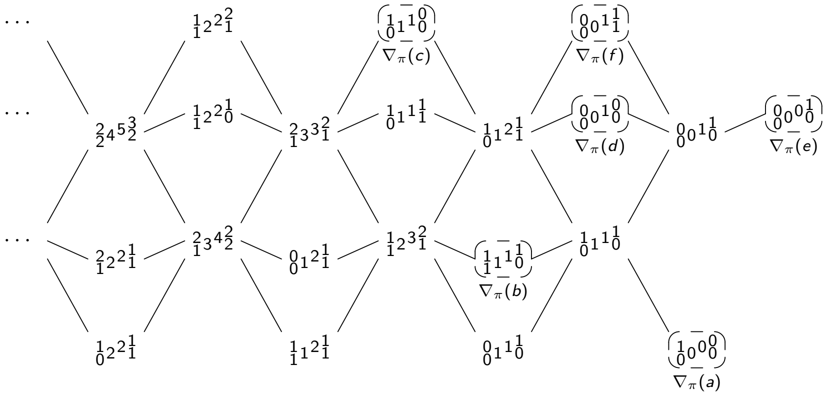


π:

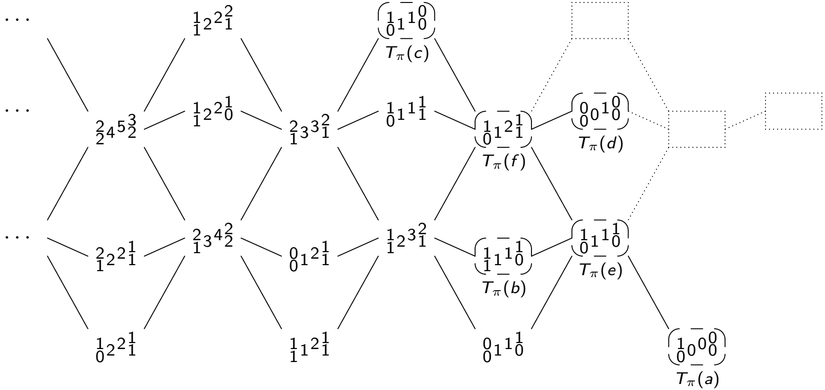


$i \in Q_0$	$\pi(i)$	$S(i)$	$P(i)$	$\Delta_\pi(i)$	$I(i)$	$\nabla_\pi(i)$	$T_\pi(i)$
c	3	$\substack{0\\0}1\,0\substack{0\\0}$	$\substack{0\\1}1\,0\substack{0\\0}$	$\substack{0\\0}1\,0\substack{0\\0}$	$\substack{1\\0}1\,1\substack{1\\0}$	$\mathbf{\substack{1\\0}1\,1\substack{0\\0}}$	$\mathbf{\substack{1\\0}1\,1\substack{0\\0}}$
d	2	$\substack{0\\0}0\,1\substack{0\\0}$	$\substack{0\\1}1\,1\substack{0\\1}$	$\substack{0\\0}0\,1\substack{0\\0}$	$\substack{0\\0}0\,1\substack{1\\0}$	$\mathbf{\substack{0\\0}0\,1\substack{0\\0}}$	$\mathbf{\substack{0\\0}0\,1\substack{0\\0}}$
e	4	$\substack{0\\0}0\,0\substack{1\\0}$	$\substack{0\\1}1\,1\substack{1\\1}$	$\mathbf{\substack{0\\0}1\,1\substack{1\\0}}$	$\substack{0\\0}0\,0\substack{1\\0}$	$\substack{0\\0}0\,0\substack{1\\0}$	$\mathbf{\substack{1\\0}1\,1\substack{1\\0}}$
f	6	$\substack{0\\0}0\,0\substack{0\\1}$	$\substack{0\\0}0\,0\substack{0\\1}$	$\substack{0\\0}0\,0\substack{0\\1}$	$\substack{0\\0}0\,1\substack{1\\1}$	$\substack{0\\0}0\,1\substack{1\\1}$	$\mathbf{\substack{1\\0}1\,2\substack{1\\1}}$

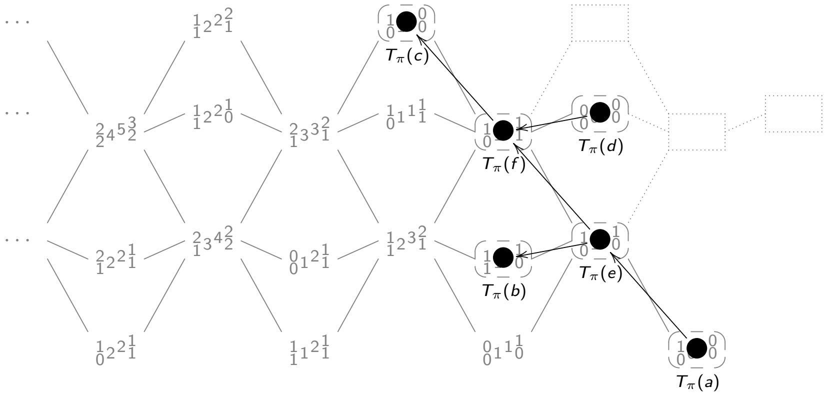
Classification: Type $\widetilde{\mathbb{D}}_5$



Classification: Type $\widetilde{\mathbb{D}}_5$

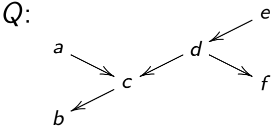


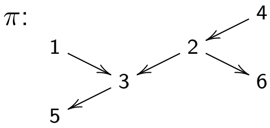
Classification: Type $\widetilde{\mathbb{D}}_5$



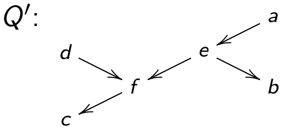
Classification: Type $\widetilde{\mathbb{D}}_5$

Consider the algebra $A = kQ/I$ with enumeration π:



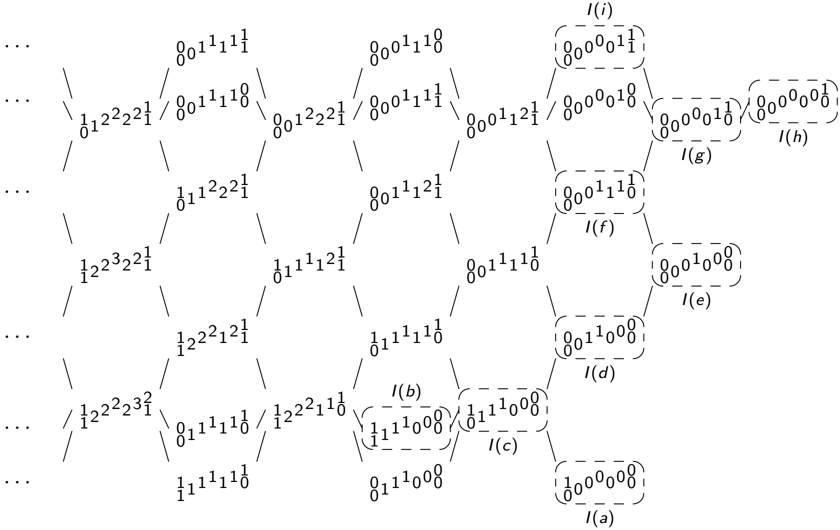


The Ringel dual $B = \mathrm{End}(T_\pi)^{\mathrm{op}}$ is of the form $B = kQ'/I'$ with Q' as follows:

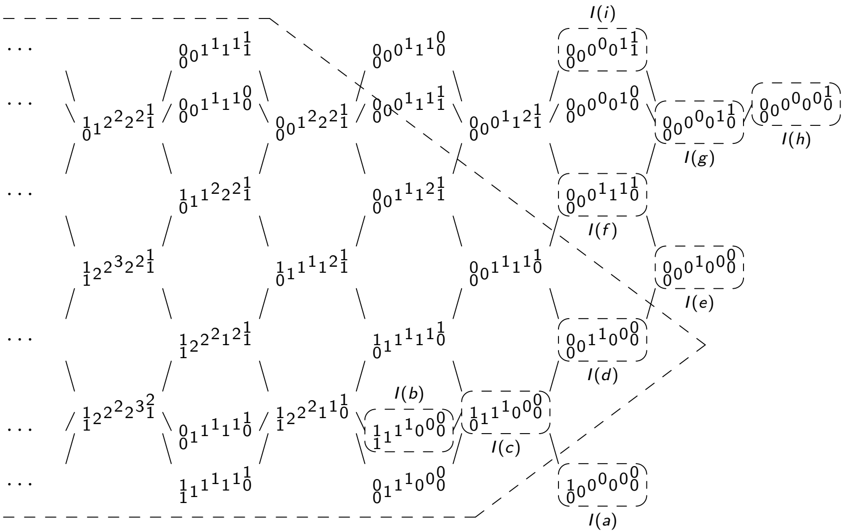


The quiver Q' is the same quiver again and B is isomorphic to A.

Classification: Type $\widetilde{\mathbb{D}}_n$

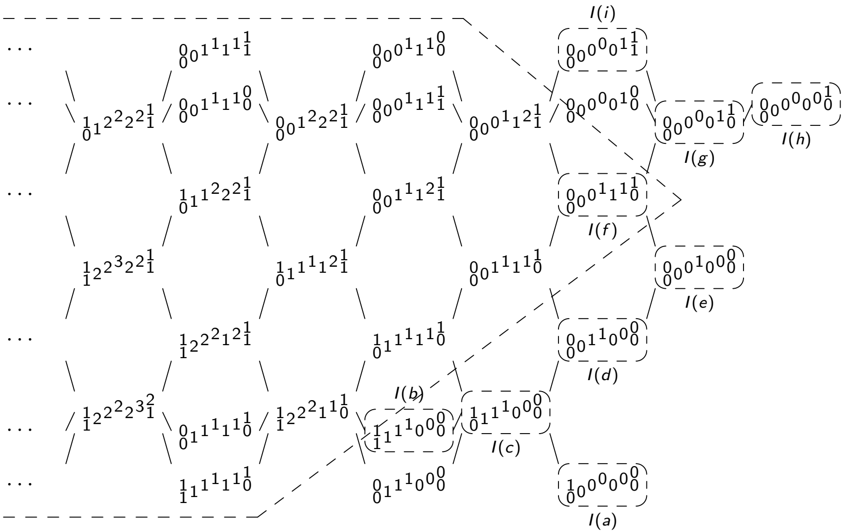


Classification: Type $\widetilde{\mathbb{D}}_n$



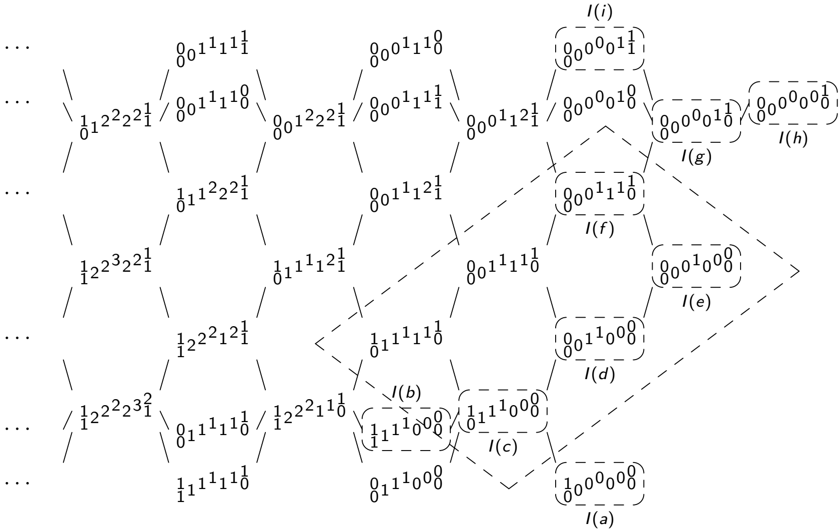
Every module in the marked area has $S(d)$ as a composition factor.

Classification: Type $\widetilde{\mathbb{D}}_n$



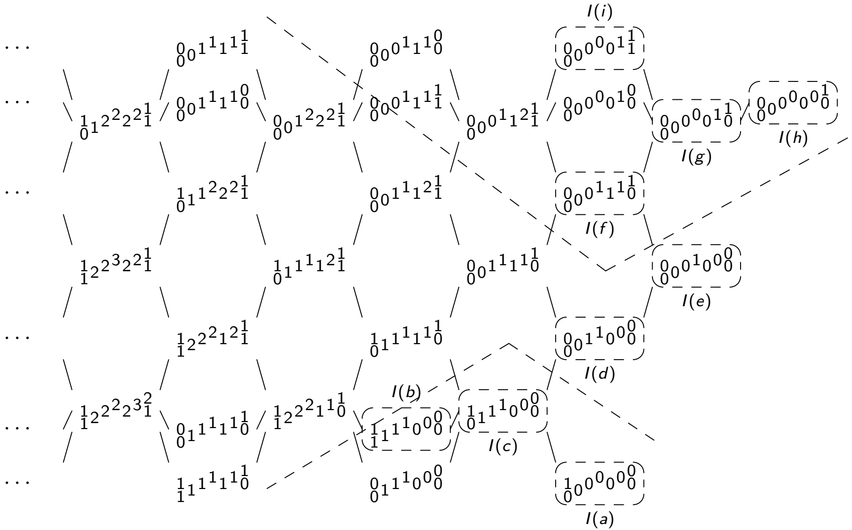
Every module in the marked area has $S(f)$ as a composition factor.

Classification: Type $\widetilde{\mathbb{D}}_n$



Every irreducible map between modules in the marked area is surjective.

Classification: Type $\widetilde{\mathbb{D}}_n$

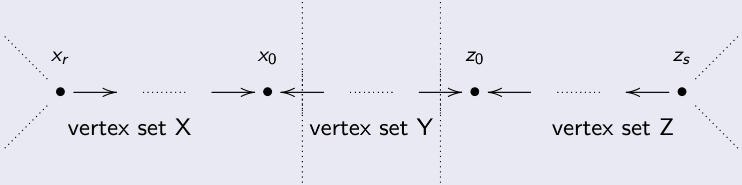


All $\nabla_\pi(j)$ for vertices f, g, h, i and a, b, c lie in the respective wings, and $T_\pi(j) = \nabla_\pi(j) = I(j)$ holds for vertices d and e.

Classification: Type $\widetilde{\mathbb{D}}_n$

Proposition

Let $(kQ/I, \pi)$ be a Δ-critical algebra, where Q is a quiver of type $\widetilde{\mathbb{D}}_n$ of the form



with $X \cup Y \cup Z = Q_0$, where $x_r = x_0$, $x_0 = z_0$ and $z_0 = z_s$ are possible.

Then $\nabla_\pi(x)$ lies in the wing given by $\{I(i) \mid i \in X\}$ for all $x \in X$, $\nabla_\pi(z)$ lies in the wing given by $\{I(i) \mid i \in Z\}$ for all $z \in Z$ and $\nabla_\pi(y) = I(y) = T_\pi(y)$ holds for all $y \in Y$.

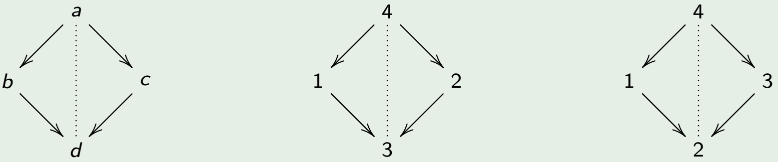
Further on, $I(x_0)$ and $I(z_0)$ are direct summands of T_π.

Classification: Type $\widetilde{\mathbb{D}}_n$

Definition

Let Q be a finite quiver without oriented cycles and let $\pi : Q_0 \to \{1, 2, \ldots, |Q_0|\}$ be an enumeration of Q. Two vertices $i, j \in Q_0$ are said to be *relevantly misordered by* π if $\pi(i) < \pi(j)$ hold and the relative order of $\pi(i)$ and $\pi(j)$ matters for $\nabla_\pi(i)$.

Example



Vertices d and a are: relevantly misordered — not relevantly misordered

Classification: Type $\widetilde{\mathbb{D}}_n$

Proposition

Let (kQ/I) be a tame concealed algebra and let X be a subquiver of Q that is connected to $Q \backslash X$ only through a single vertex $x_0 \in X_0$.

(i) Let π be an enumeration of Q such that vertices $x \in X_0$, $y \in Q_0 \backslash X_0$ are never relevantly misordered. Then there exist enumerations π', π'' of Q such that

$$\nabla_{\pi'}(i) = \begin{cases} \nabla_\pi(i) & \text{if } i \in Q_0 \backslash X_0 \\ I(i) & \text{if } i \in X_0 \end{cases} \text{ and } \nabla_{\pi''}(i) = \begin{cases} I(i) & \text{if } i \in Q_0 \backslash X_0 \\ \nabla_\pi(i) & \text{if } i \in X_0 \end{cases}.$$

(ii) Let π', π'' be enumerations of Q such that all vertices $i, j \in Q_0$ satisfy $i, j \in Q_0 \backslash X_0$ if they are relevantly misordered by π' and $i, j \in X_0$ if they are relevantly misordered by π''. Then there exists an enumeration π of Q such that

$$\nabla_\pi(i) = \begin{cases} \nabla_{\pi'}(i) & \text{if } i \in Q_0 \backslash X_0 \\ \nabla_{\pi''}(i) & \text{if } i \in X_0 \end{cases}.$$

Classification: Type $\widetilde{\mathbb{D}}_n$

Construction

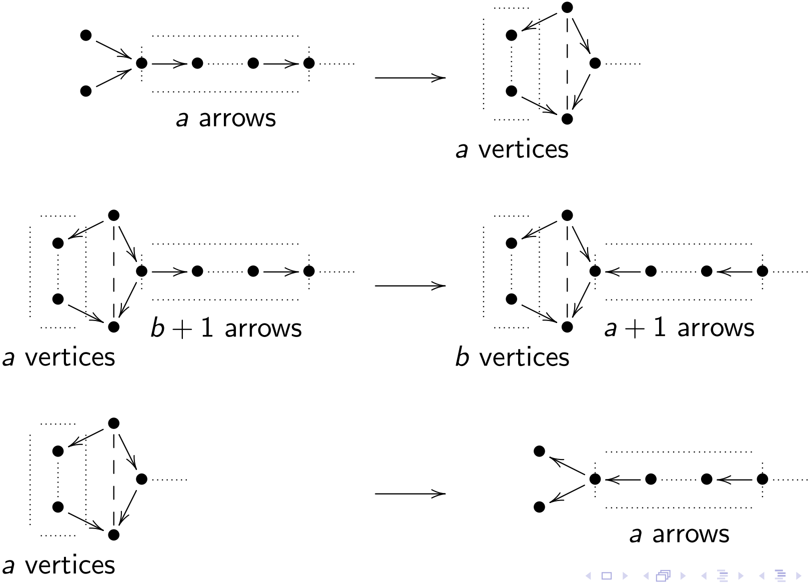
(i) Let ρ be a trivial enumeration of Q.

Let
$$\rho'(i) = \begin{cases} \pi(i) & \text{if } i \in Q_0 \setminus X_0 \\ (\pi(x_0), \rho(i)) & \text{if } i \in X_0 \end{cases}$$
and let $\pi' = \iota' \circ \rho'$, where ι' maps $\rho'(Q_0)$ to $\{1, 2, \ldots, |Q_0|\}$ in lexicographical order.

Let
$$\rho''(i) = \begin{cases} \rho(i) & \text{if } i \in Q_0 \setminus X_0 \\ (\rho(x_0), \pi(i)) & \text{if } i \in X_0 \end{cases}$$
and let $\pi'' = \iota'' \circ \rho''$, where ι'' maps $\rho''(Q_0)$ to $\{1, 2, \ldots, |Q_0|\}$ in lexicographical order.

(ii) Let
$$\rho(i) = \begin{cases} \pi'(i) & \text{if } i \in Q_0 \setminus X_0 \\ (\pi'(x_0), \pi''(i)) & \text{if } i \in X_0 \end{cases}$$
and let $\pi = \iota \circ \rho$, where ι maps $\rho(Q_0)$ to $\{1, 2, \ldots, |Q_0|\}$ in lexicographical order.

Classification: Type $\widetilde{\mathbb{D}}_n$



Classification: Type $\widetilde{\mathbb{D}}_n$

Aim: Classify all Δ-critical algebras:

- Type $\widetilde{\mathbb{A}}_n$: ✓
- Type $\widetilde{\mathbb{D}}_4$: ✓
- Type $\widetilde{\mathbb{D}}_n, n \geq 5$: ✓

Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

There are only finitely tame concealed algebras of types $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$, so a computer can be used to check all 4,302 such algebras.

Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

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This yields:

- Case $\widetilde{\mathbb{E}}_6$: 424 equivalence classes for 56 quivers

Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

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- Case $\widetilde{\mathbb{E}}_6$: 424 equivalence classes for 56 quivers
- Case $\widetilde{\mathbb{E}}_7$: 8,824 equivalence classes for 437 quivers

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This yields:

- Case $\widetilde{\mathbb{E}}_6$: 424 equivalence classes for 56 quivers
- Case $\widetilde{\mathbb{E}}_7$: 8,824 equivalence classes for 437 quivers
- Case $\widetilde{\mathbb{E}}_8$: 179,302 equivalence classes for 3,809 quivers

Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

Aim: Classify all Δ-critical algebras:

- Type $\widetilde{\mathbb{A}}_n$: ✓
- Type $\widetilde{\mathbb{D}}_4$: ✓
- Type $\widetilde{\mathbb{D}}_n$, $n \geq 5$: ✓
- Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$: ✓?

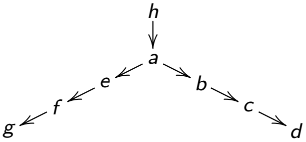
Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

New aim: Reduce the number of cases to be considered for $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$.

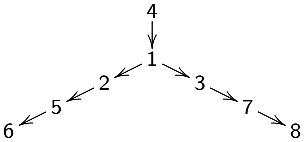
Just like in type $\widetilde{\mathbb{D}}_n$, enumerations can be composed and decomposed – and maybe we can restrict to wings again.

So consider the algebra $A = kQ/I$ with enumeration π:

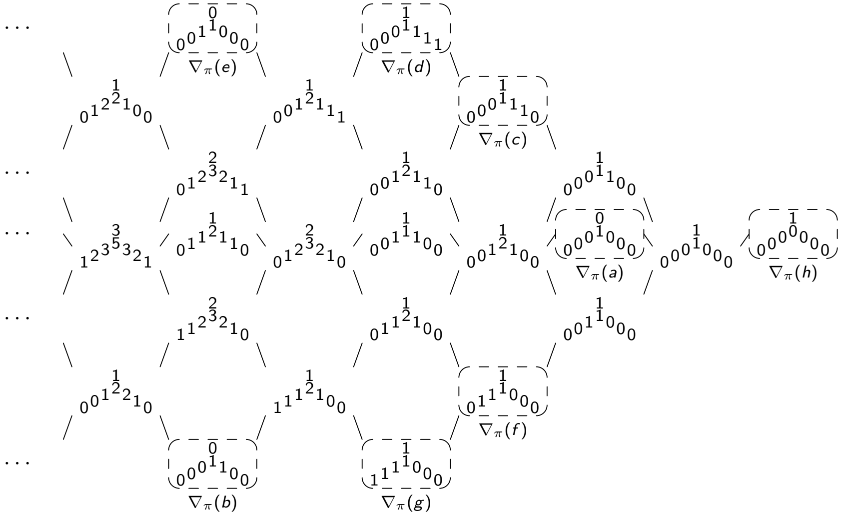
Q:



π:



Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$



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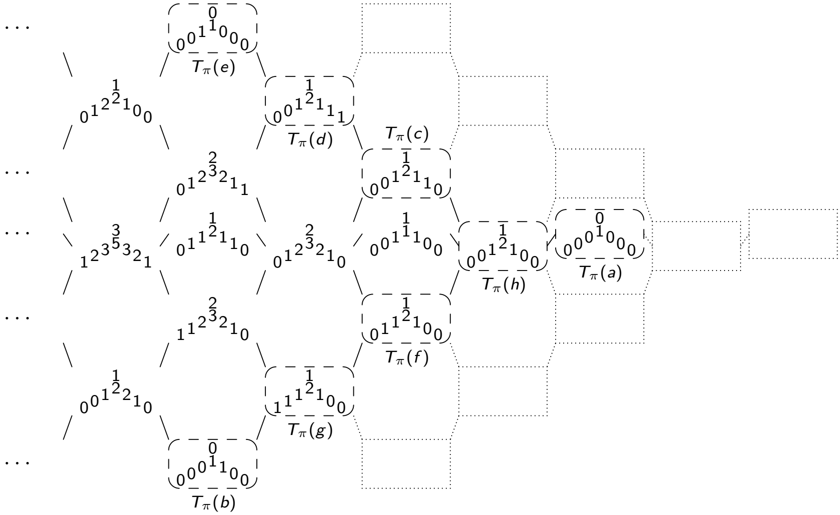


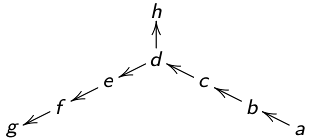
Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

There are modules $\nabla_\pi(i)$ outside the wings and there is once again a non-trivial enumeration that gives the same quiver again.

The wings are still useful, though:

Consider the algebra $A = kQ/I$:

Q:



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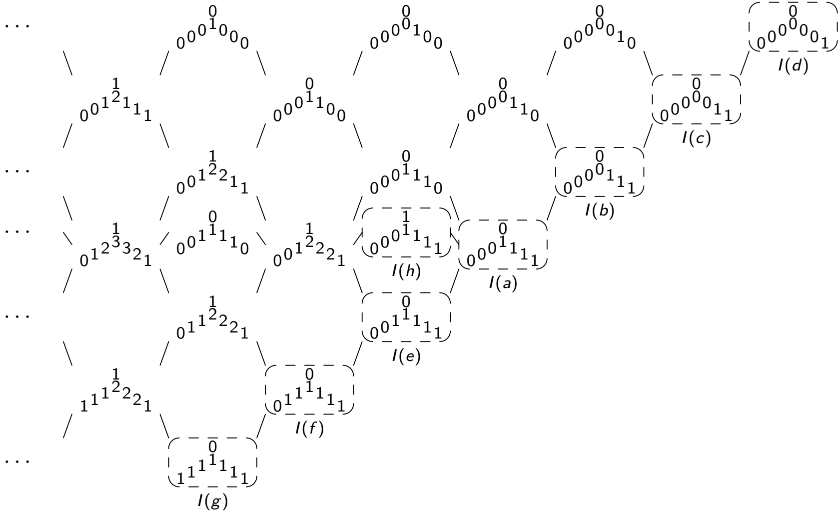

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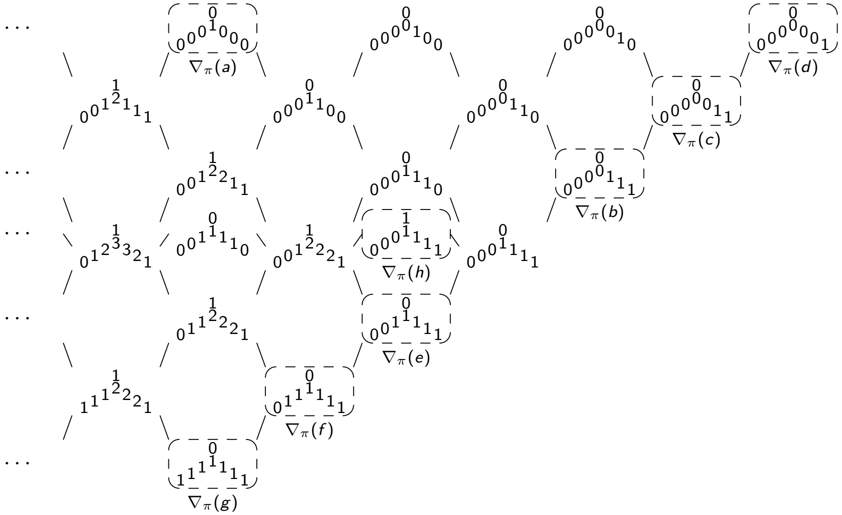

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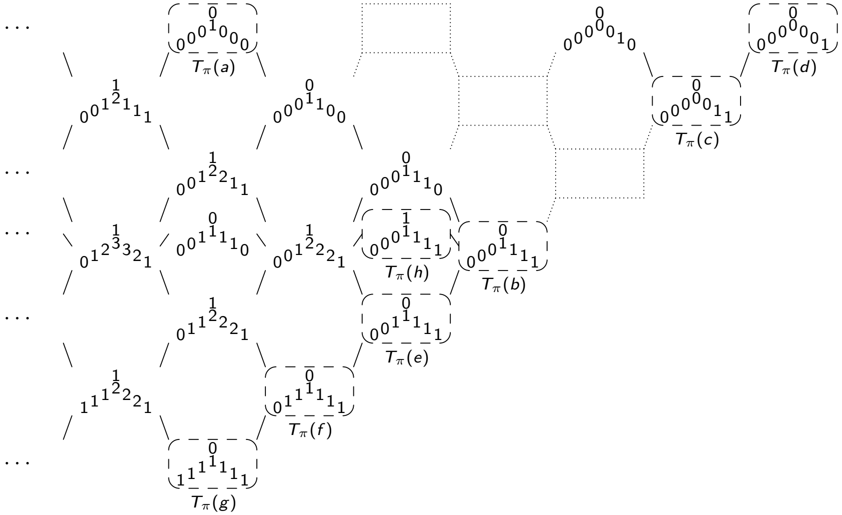

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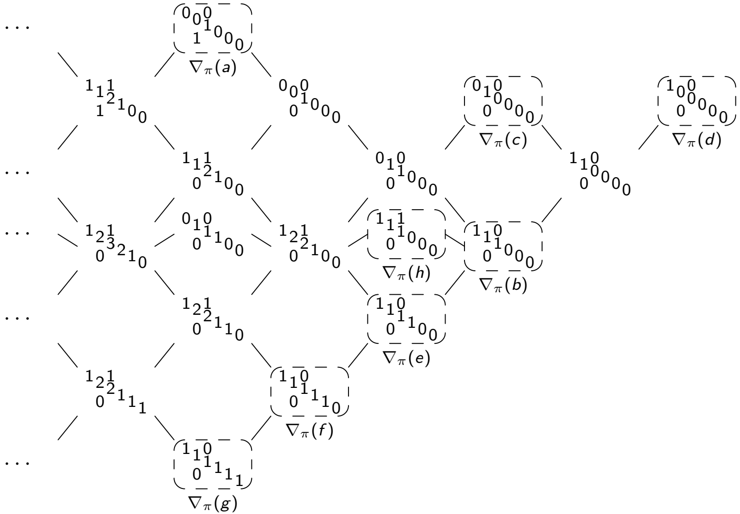

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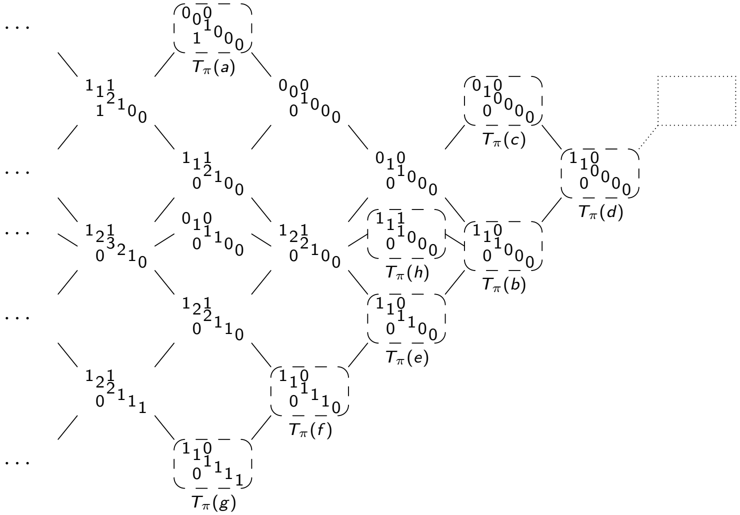

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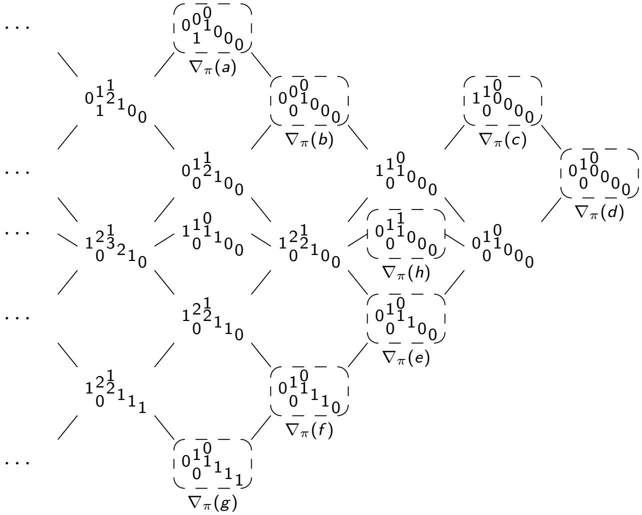

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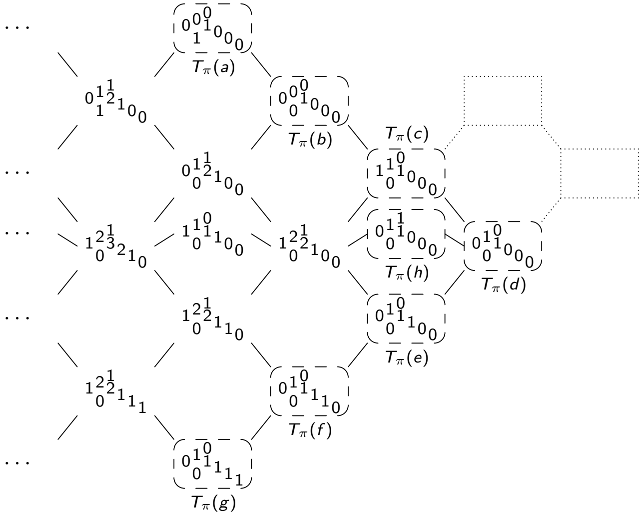
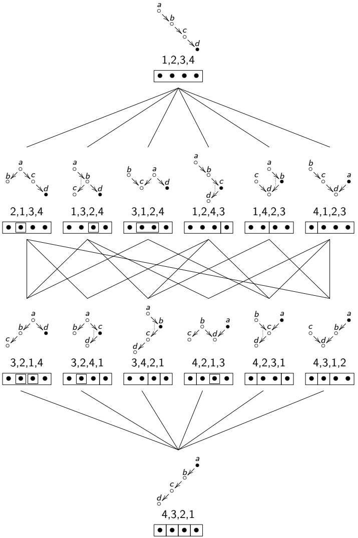


Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

Wings of branches may contain many submodules of the $I(i)$.

In particular, a linearly (towards the rest of the quiver) ordered branch of Q can be enumerated to yield any chosen orientation of that branch.

Further on, the different equivalence classes are labelled by non-crossing partitions and can be generally described.



Classification: Type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$

After splitting off misorderings in wings where possible, the numbers of cases to be considered are as follows:

- Case $\widetilde{\mathbb{E}}_6$: 68 cases (424 equivalence classes) for 56 quivers,
- Case $\widetilde{\mathbb{E}}_7$: 1,487 cases (8,824 equivalence classes) 437 quivers,
- Case $\widetilde{\mathbb{E}}_8$: 37,306 cases (179,302 equivalence classes) for 3,809 quivers