Part 1: Introduction and Background

Throughout the talk,

- $k$ is an algebraically closed field,
- $A$ is a basic finite dimensional $k$-algebra, typically the path algebra $kQ$ or $kQ/I$ of a finite quiver without oriented cycles.
Introduction: Tame hereditary algebras

Theorem (Donovan–Freislich, Nazarova)

A connected hereditary $k$-algebra $A$ is tame exactly if it is the path algebra $kQ$ of a quiver $Q$ whose underlying graph is a Euclidean diagram, i.e. a graph of type $\tilde{A}_n$ ($n \geq 1$, no oriented cycle), $\tilde{D}_n$ ($n \geq 4$), $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. 

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Delta-critical quasi-hereditary algebras
The Auslander–Reiten quiver $\Gamma(\text{mod } A)$ of a tame hereditary algebra $A$ consists of a preprojective component $\mathcal{P}(A)$, a preinjective component $\mathcal{Q}(A)$ and a regular part $\mathcal{R}(A)$.
Introduction: Tame concealed algebras

Definition

A $k$-algebra $B$ is called *tame concealed* if there exists a tame connected hereditary algebra $A$ and a preprojective (or preinjective) tilting module $T_A \in \text{mod } A$ such that $B = \text{End } T_A$. 
Question:
In which way are tame concealed algebras “concealed”?
Answer:
Apart from finite parts at the ends of the preprojective and the preinjective component, their Auslander-Reiten quiver looks like that of a tame hereditary algebra.
Happel and Vossieck constructed all tame concealed $k$-algebras.

There are many of these, even for relatively small quivers:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{A}_n$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>9</td>
<td>21</td>
<td>29</td>
</tr>
<tr>
<td>$\tilde{D}_n$</td>
<td></td>
<td></td>
<td>6</td>
<td>13</td>
<td>32</td>
<td>60</td>
<td>131</td>
<td></td>
</tr>
<tr>
<td>$\tilde{E}_n$</td>
<td></td>
<td></td>
<td></td>
<td>56</td>
<td>437</td>
<td>3801</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Theorem (Happel–Vossieck)

A connected $k$-algebra $A$ is tame concealed exactly if it is the path algebra $kQ/I$ of a quiver $Q$ (possibly with relations) given by one of 149 “frames” (1 for $\tilde{A}_n$, 4 for $\tilde{D}_n$, 5 for $\tilde{E}_6$, 22 for $\tilde{E}_7$, 117 for $\tilde{E}_8$).
Let $Q$ be a finite quiver without oriented cycles (possibly with relations), let $I$ be the two sided ideal in $kQ$ generated by the relations of $Q$, and set $A = kQ/I$.

For every vertex $i \in Q_0$, let $S(i)$ be the simple module for the vertex $i$, $P(i)$ the projective cover of $S(i)$, $I(i)$ the injective envelope of $S(i)$. 
An enumeration of $A$ is a bijective map $\pi : Q_0 \to \{1, 2, \ldots, |Q_0|\}$.

Given an enumeration $\pi$ of $A$, let
- $\Delta_\pi(i)$ the maximal factor module $P(i)$,
- $\nabla_\pi(i)$ the maximal submodule of $I(i)$,
both with only composition factors $S(j)$ where $\pi(j) \leq \pi(i)$.

Set $\Delta_\pi = \{\Delta_\pi(i) | i \in Q_0\}$ and $\nabla_\pi = \{\nabla_\pi(i) | i \in Q_0\}$.

Two enumerations $\pi$ and $\pi'$ are equivalent if $\nabla_\pi = \nabla_{\pi'}$ (up to symmetries of the quiver).
Introduction: Quasi-hereditary algebras

Definition (Scott, Cline–Parshall–Scott)

Let $A = kQ/I$ as before and let $\pi$ be an enumeration of $A$. The pair $(A, \pi)$ is called a quasi-hereditary algebra if the following equivalent conditions hold:

- for each $i \in Q_0$, the module $P(i)$ has a $\Delta_{\pi}$-filtration and $S(i)$ occurs only once as a composition factor of $\Delta_{\pi}(i)$,
- for each $i \in Q_0$, the module $I(i)$ has a $\nabla_{\pi}$-filtration and $S(i)$ occurs only once as a composition factor of $\nabla_{\pi}(i)$.

In this case, the modules $\Delta_{\pi}(i)$ are called the standard modules and the $\nabla_{\pi}(i)$ are called the costandard modules of $(A, \pi)$.

For convenience, also call $\pi$ a quasi-hereditary enumeration of $A$ if $(A, \pi)$ is quasi-hereditary.
Remark

In the special case of $Q$ being a finite quiver without oriented cycles, $S(i)$ occurs only once as a composition factor of $P(i)$ and of $Q(i)$, so the conditions of $S(i)$ occurring only once in $\Delta_\pi(i)$ and $\nabla_\pi(i)$ always hold.

Examples

For a finite quiver $Q$ without oriented cycles:

- There always exists an equivalence class of trivial enumerations $\pi$ such that $\pi(j) > \pi(i)$ for all arrows $i \rightarrow j \in Q_1$. In this case, we have $\Delta_\pi(i) = S(i)$ and $\nabla_\pi(i) = I(i)$ for all $i \in Q_0$, so trivial enumerations are always quasi-hereditary.

- [Dlab–Ringel]: The algebra $A = kQ/I$ is hereditary if and only if all enumerations of $A$ are quasi-hereditary.
Now consider the algebra $A = kQ/I$ with enumeration $\pi$:

<table>
<thead>
<tr>
<th>$i \in Q_0$</th>
<th>$\pi(i)$</th>
<th>$S(i)$</th>
<th>$I(i)$</th>
<th>$\nabla_{\pi}(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>0 0 0 0 1 1 0</td>
<td>0 1 1 1 0 0</td>
<td>0 0 0 1 0 0</td>
</tr>
<tr>
<td>$b$</td>
<td>4</td>
<td>0 0 1 0 0 1 0</td>
<td>1 0 1 0 0 0</td>
<td>1 0 1 0 0 0</td>
</tr>
<tr>
<td>$c$</td>
<td>5</td>
<td>0 0 0 1 0 0 1</td>
<td>1 0 0 1 0 0</td>
<td>1 0 0 1 0 0</td>
</tr>
<tr>
<td>$d$</td>
<td>3</td>
<td>0 0 1 0 0 1 0</td>
<td>1 0 0 1 0 0</td>
<td>1 0 0 1 0 0</td>
</tr>
</tbody>
</table>

The module $I(a)$ does not have a $\nabla_{\pi}$-filtration, so $(A, \pi)$ is not quasi-hereditary.
Part 2: $\Delta$-critical algebras
Theorem (Ringel)

Let \((A, \pi)\) be a quasi-hereditary algebra with standard modules \(\Delta_{\pi}\). Then the category \(\mathcal{F}(\Delta_{\pi})\) of \(A\)-modules that have a \(\Delta_{\pi}\)-filtration is a functorially finite subcategory of \(A\)-mod which is closed under extensions and direct summands.

Corollary (Ringel, using Auslander–Smalø)

The category \(\mathcal{F}(\Delta_{\pi})\) has (relative) AR-sequences.

Definition (Ringel)

A \(\Delta\)-critical algebra \((A, \pi)\) is a tame concealed quasi-hereditary algebra for which all costandard modules \(\nabla_{\pi}(i)\) are preinjective.
From now on, let \((A, \pi)\) be a \(\Delta\)-critical algebra.

The category \(\mathcal{F}(\Delta_{\pi})\) can then be described as follows:

\[
\mathcal{F}(\Delta_{\pi}) = \{ M \in \text{mod} \ A \mid \text{Ext}^1(M, \nabla_{\pi}(i)) = 0 \text{ for all } i \in Q_0 \} \\
= \{ M \in \text{mod} \ A \mid \text{Hom}(\nabla_{\pi}(i), \tau M) = 0 \text{ for all } i \in Q_0 \}
\]

In particular, since all \(\nabla_{\pi}(i)\) are preinjective, \(\mathcal{F}(\Delta_{\pi})\) contains all preprojective and all regular \(A\)-modules.
In particular, since all $\nabla_\pi(i)$ are preinjective, $\mathcal{F}(\Delta_\pi)$ contains all preprojective and all regular $A$-modules.
Theorem (Ringel)

Let \((A, \pi)\) be a \(\Delta\)-critical algebra. Then the following hold:

- There exists a unique basic tilting module \(T_\pi\) such that 
  \[ \mathcal{F}(\Delta_\pi) \cap \mathcal{F}(\Delta_\pi) = \text{add}(T_\pi), \]
  called the characteristic tilting module.

- The characteristic module \(T_\pi\) admits a decomposition 
  \[ T_\pi = \bigoplus_{i \in Q_0} T_\pi(i) \]
  such each \(T_\pi(i)\) has a composition factor \(S(i)\) and all other composition factors are of the form \(S(j)\) with \(\pi(j) < \pi(i)\).

- The category \(\mathcal{F}(\Delta_\pi)\) has a preprojective component of type \(A\) and a preinjective component of type \(B\), where \(B\) is the Ringel dual \(B = \text{End}(T_\pi)^{\text{op}}\).
Consider the algebra $A = kQ/I$ with enumeration $\pi$:

\[
Q:\quad \begin{array}{cccc}
& a & \rightarrow & b \\
& e & \rightarrow & c & \rightarrow & f \\
& g & \rightarrow & d & \rightarrow & h
\end{array}
\pi:\quad \begin{array}{cccc}
& 1 & \rightarrow & 2 \\
& 5 & \rightarrow & 3 & \rightarrow & 6 \\
& 4 & \rightarrow & 7 & \rightarrow & 8
\end{array}
\]

<table>
<thead>
<tr>
<th>$i \in Q_0$</th>
<th>$\pi(i)$</th>
<th>$S(i)$</th>
<th>$P(i)$</th>
<th>$\Delta_\pi(i)$</th>
<th>$I(i)$</th>
<th>$\nabla_\pi(i)$</th>
<th>$T_\pi(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>$0^{100}$</td>
<td>$0^{110}$</td>
<td>$0^{100}$</td>
<td>$0^{100}$</td>
<td>$0^{100}$</td>
<td>$0^{100}$</td>
</tr>
<tr>
<td>$b$</td>
<td>3</td>
<td>$0^{011}$</td>
<td>$0^{111}$</td>
<td>$0^{011}$</td>
<td>$0^{011}$</td>
<td>$0^{011}$</td>
<td>$0^{111}$</td>
</tr>
<tr>
<td>$c$</td>
<td>2</td>
<td>$0^{011}$</td>
<td>$0^{110}$</td>
<td>$0^{110}$</td>
<td>$0^{110}$</td>
<td>$0^{110}$</td>
<td>$0^{110}$</td>
</tr>
<tr>
<td>$f$</td>
<td>6</td>
<td>$0^{001}$</td>
<td>$0^{001}$</td>
<td>$0^{001}$</td>
<td>$0^{001}$</td>
<td>$0^{001}$</td>
<td>$0^{111}$</td>
</tr>
</tbody>
</table>
\[ \nabla \pi(a) \]
\[ \nabla \pi(b) \]
\[ \nabla \pi(c) \]
\[ \nabla \pi(d) \]
\[ \nabla \pi(e) \]
\[ \nabla \pi(f) \]
\[ \nabla \pi(g) \]
Δ-critical algebras: Definition and properties

Illustration of Δ-critical algebras with a network of nodes and edges representing the structure and properties of these algebras.
Δ-critical algebras: Definition and properties
Consider the algebra $A = kQ/I$ with enumeration $\pi$:

$$Q: \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{c} \\
\text{d} \\
\end{array}
\begin{array}{c}
\downarrow \\
\downarrow \\
\downarrow \\
\text{e} \\
\text{g} \\
\text{f} \\
\end{array}
\begin{array}{c}
\text{b} \\
\text{f} \\
\text{h} \\
\end{array}
\begin{array}{c}
\text{c} \\
\text{d} \\
\text{g} \\
\end{array}
\begin{array}{c}
\text{e} \\
\text{f} \\
\text{h} \\
\end{array}
\end{array}$$

$$\pi: \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
3 \\
\end{array}
\begin{array}{c}
\leftarrow \\
\leftarrow \\
\leftarrow \\
\text{5} \\
\text{4} \\
\text{6} \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{1} \\
\text{2} \\
\text{3} \\
\end{array}
\begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\text{5} \\
\text{4} \\
\text{6} \\
\end{array}
\end{array}$$

The Ringel dual $B = \text{End}(T_\pi)^{\text{op}}$ is of the form $B = kQ'/I'$ with $Q'$ as follows:

$$Q': \begin{array}{c}
\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c} \\
\end{array}
\begin{array}{c}
\text{e} \\
\text{d} \\
\text{f} \\
\end{array}
\begin{array}{c}
\text{g} \\
\text{h} \\
\end{array}
\end{array}$$
Proposition (Ringel)

If \( A = kQ/I \) is a tame concealed algebra and \( U \) is a uniform (i.e. a module with a simple socle) preinjective \( A \)-module, then there exists a quasi-hereditary enumeration \( \pi \) of \( Q \) such that \( U = \nabla_{\pi}(i) \) for some \( i \in Q_0 \).

Proposition (Ringel)

If \( A = kQ/I \) is a tame concealed algebra and \( \pi \) is an enumeration of \( Q \) such that \( \nabla_{\pi}(i) \) is preinjective for all \( i \in Q_0 \), then \( (A, \pi) \) is quasi-hereditary (and thus also \( \Delta \)-critical).
Δ-critical algebras: Definition and properties

\[ \Delta \pi(a) \]
\[ \Delta \pi(b) \]
\[ \Delta \pi(c) \]
\[ \Delta \pi(d) \]
\[ \Delta \pi(e) \]
\[ \Delta \pi(f) \]
\[ \Delta \pi(g) \]

Andre Beineke

Δ-critical quasi-hereditary algebras

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Δ-critical algebras: Definition and properties

\[ T_\pi(a) \]

\[ T_\pi(b) \]

\[ T_\pi(c) \]

\[ T_\pi(d) \]

\[ T_\pi(e) \]

\[ T_\pi(f) \]

\[ T_\pi(g) \]

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Part 3: Classification
Aim: Classify all $\Delta$-critical algebras

In case $\widetilde{A}_n$, all irreducible maps in the injective component are surjective:

- This is obvious for the injective modules.
- If $M$ is not injective, then there exists an AR-sequence $0 \to M \to X_1 \oplus X_2 \to N \to 0$ with indecomposable modules $X_1$ and $X_2$. By induction, $\dim X_1 > \dim N$ and $\dim X_2 > \dim N$ hold and imply $\dim M > \dim X_1$ and $\dim N > \dim X_2$.

So the injective component does not contain any proper submodule of any $I(i)$ and the only possible enumerations are the trivial ones.
Aim: Classify all $\Delta$-critical algebras:

- Type $\tilde{A}_n$: ✓
Classification: Type $\tilde{D}_4$
Classification: Type \( \widetilde{D}_4 \)
Classification: Type $\tilde{D}_4$
Classification: Type $\widetilde{D}_4$
Classification: Type $\tilde{D}_4$
Classification: Type $\widetilde{D}_4$
Classification: Type $\tilde{D}_4$
Classification: Type $\tilde{D}_4$
Classification: Type $\tilde{D}_4$

Aim: Classify all $\Delta$-critical algebras:

- Type $\tilde{A}_n$: ✓
- Type $\tilde{D}_4$: ✓
Consider the algebra $A = kQ/I$ with enumeration $\pi$:

\[
\begin{array}{ccccccc}
Q: & a & \rightarrow & c & \rightarrow & d & \rightarrow & e \\
\downarrow & b & \rightarrow & c & \rightarrow & f & \rightarrow & 4 \\
& 1 & \rightarrow & 3 & \rightarrow & 2 & \rightarrow & 6 \\
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
i \in Q_0 & \pi(i) & S(i) & P(i) & \Delta_\pi(i) & I(i) & \nabla_\pi(i) & T_\pi(i) \\
\hline
c & 3 & 0100 & 0100 & 0100 & 1110 & 0110 & 0110 \\
d & 2 & 0010 & 0110 & 0010 & 0010 & 0010 & 0010 \\
e & 4 & 0001 & 0111 & 0110 & 0001 & 0001 & 0110 \\
f & 6 & 0000 & 0000 & 0000 & 0011 & 0011 & 0121 \\
\hline
\end{array}
\]
Classification: Type $\tilde{D}_5$

$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
$$

$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
$$

$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
$$

$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
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\begin{array}{c}
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\end{array}
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$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
$$

$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
$$

$$
\begin{array}{c}
\ldots \\
\ldots
\end{array}
$$

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$\Delta$-critical quasi-hereditary algebras

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Classification: Type $\widehat{D}_5$
Consider the algebra $A = kQ/I$ with enumeration $\pi$:

\[ Q: \]
\[
\begin{array}{cccc}
  a & \rightarrow & d & \rightarrow \\
  c & \leftarrow & e & \leftarrow \\
  b & \rightarrow & f & \rightarrow \\
\end{array}
\]

\[ \pi: \]
\[
\begin{array}{cccc}
  1 & \rightarrow & 2 & \rightarrow \\
  3 & \leftarrow & 4 & \leftarrow \\
  5 & \rightarrow & 6 & \rightarrow \\
\end{array}
\]

The Ringel dual $B = \text{End}(T_\pi)^{\text{op}}$ is of the form $B = kQ'/I'$ with $Q'$ as follows:

\[ Q': \]
\[
\begin{array}{cccc}
  d & \rightarrow & f & \rightarrow \\
  e & \leftarrow & a & \leftarrow \\
  c & \rightarrow & b & \rightarrow \\
\end{array}
\]

The quiver $Q'$ is the same quiver again and $B$ is isomorphic to $A$. 
Classification: Type $\widetilde{D}_n$
Every module in the marked area has $S(d)$ as a composition factor.
Every module in the marked area has $S(f)$ as a composition factor.
Every irreducible map between modules in the marked area is surjective.
All $\nabla_\pi(j)$ for vertices $f, g, h, i$ and $a, b, c$ lie in the respective wings, and $T_\pi(j) = \nabla_\pi(j) = I(j)$ holds for vertices $d$ and $e$. 
Let \((kQ/I, \pi)\) be a \(\Delta\)-critical algebra, where \(Q\) is a quiver of type \(\tilde{D}_n\) of the form

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\vdots & \ddots & \vdots \\
x_r & \rightarrow & x_0 \\
\bullet & \leftarrow & \bullet \\
\vdots & \ddots & \vdots \\
z_0 & \leftarrow & z_s
\end{array}
\]

with \(X \cup Y \cup Z = Q_0\), where \(x_r = x_0, x_0 = z_0\) and \(z_0 = z_s\) are possible.

Then \(\nabla_\pi(x)\) lies in the wing given by \(\{I(i) \mid i \in X\}\) for all \(x \in X\), \(\nabla_\pi(z)\) lies in the wing given by \(\{I(i) \mid i \in Z\}\) for all \(z \in Z\) and \(\nabla_\pi(y) = I(y) = T_\pi(y)\) holds for all \(y \in Y\).

Further on, \(I(x_0)\) and \(I(z_0)\) are direct summands of \(T_\pi\).
Definition

Let \( Q \) be a finite quiver without oriented cycles and let \( \pi : Q_0 \to \{1, 2, \ldots, |Q_0|\} \) be an enumeration of \( Q \). Two vertices \( i, j \in Q_0 \) are said to be \textit{relevantly misordered by} \( \pi \) if \( \pi(i) < \pi(j) \) hold and the relative order of \( \pi(i) \) and \( \pi(j) \) matters for \( \nabla_{\pi}(i) \).

Example

Vertices \( d \) and \( a \) are: relevantly misordered  not relevantly misordered
Proposition

Let \((kQ/I)\) be a tame concealed algebra and let \(X\) be a subquiver of \(Q\) that is connected to \(Q \setminus X\) only through a single vertex \(x_0 \in X_0\).

(i) Let \(\pi\) be an enumeration of \(Q\) such that vertices \(x \in X_0, y \in Q_0 \setminus X_0\) are never relevantly misordered. Then there exist enumerations \(\pi', \pi''\) of \(Q\) such that

\[
\nabla_{\pi'}(i) = \begin{cases} \nabla_{\pi}(i) & \text{if } i \in Q_0 \setminus X_0 \\ l(i) & \text{if } i \in X_0 \end{cases}
\]

and

\[
\nabla_{\pi''}(i) = \begin{cases} l(i) & \text{if } i \in Q_0 \setminus X_0 \\ \nabla_{\pi}(i) & \text{if } i \in X_0 \end{cases}.
\]

(ii) Let \(\pi', \pi''\) be enumerations of \(Q\) such that all vertices \(i, j \in Q_0\) satisfy \(i, j \in Q_0 \setminus X_0\) if they are relevantly misordered by \(\pi'\) and \(i, j \in X_0\) if they are relevantly misordered by \(\pi''\). Then there exists an enumeration \(\pi\) of \(Q\) such that

\[
\nabla_{\pi}(i) = \begin{cases} \nabla_{\pi'}(i) & \text{if } i \in Q_0 \setminus X_0 \\ \nabla_{\pi''}(i) & \text{if } i \in X_0 \end{cases}.
\]
Classification: Type \( \widetilde{D}_n \)

**Construction**

(i) Let \( \rho \) be a trivial enumeration of \( Q \).

Let

\[
\rho'(i) = \begin{cases} 
\pi(i) & \text{if } i \in Q_0 \setminus X_0 \\
(\pi(x_0), \rho(i)) & \text{if } i \in X_0 
\end{cases}
\]

and let \( \pi' = \iota' \circ \rho' \), where \( \iota' \) maps \( \rho'(Q_0) \) to \( \{1, 2, \ldots, |Q_0|\} \) in lexicographical order.

Let

\[
\rho''(i) = \begin{cases} 
\rho(i) & \text{if } i \in Q_0 \setminus X_0 \\
(\rho(x_0), \pi(i)) & \text{if } i \in X_0 
\end{cases}
\]

and let \( \pi'' = \iota'' \circ \rho'' \), where \( \iota'' \) maps \( \rho''(Q_0) \) to \( \{1, 2, \ldots, |Q_0|\} \) in lexicographical order.

(ii) Let

\[
\rho(i) = \begin{cases} 
\pi'(i) & \text{if } i \in Q_0 \setminus X_0 \\
(\pi'(x_0), \pi''(i)) & \text{if } i \in X_0 
\end{cases}
\]

and let \( \pi = \iota \circ \rho \), where \( \iota \) maps \( \rho(Q_0) \) to \( \{1, 2, \ldots, |Q_0|\} \) in lexicographical order.
Classification: Type $\tilde{D}_n$

\begin{align*}
&\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \quad \xrightarrow{a \text{ arrows}} \quad \begin{array}{c}
\bullet \\
\quad \quad \quad \quad \\
\end{array} \\
&\begin{array}{c}
\bullet \\
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \quad \xrightarrow{a \text{ vertices}} \quad \begin{array}{c}
\bullet \\
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \\
&\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \quad \xrightarrow{a \text{ vertices}} \quad \begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \\
&\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \quad \xrightarrow{b + 1 \text{ arrows}} \quad \begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \\
&\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \quad \xrightarrow{b \text{ vertices}} \quad \begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \\
&\begin{array}{c}
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \quad \bullet \\
\end{array} \quad \xrightarrow{a \text{ vertices}} \quad \begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \\
&\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array} \quad \xrightarrow{a \text{ arrows}} \quad \begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
\quad \quad \quad \quad \\
\end{array}
\end{align*}
Aim: Classify all $\Delta$-critical algebras:

- Type $\tilde{A}_n$: ✓
- Type $\tilde{D}_4$: ✓
- Type $\tilde{D}_n$, $n \geq 5$: ✓
There are only finitely tame concealed algebras of types $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$, so a computer can be used to check all 4,302 such algebras.
Classification: Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

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This yields:

- Case $\tilde{E}_6$: 424 equivalence classes for 56 quivers
  - Case $\tilde{E}_7$: 8,824 equivalence classes for 437 quivers
  - Case $\tilde{E}_8$: 179,302 equivalence classes for 3,809 quivers
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Aim: Classify all $\Delta$-critical algebras:

- Type $\tilde{A}_n$: ✓
- Type $\tilde{D}_4$: ✓
- Type $\tilde{D}_n$, $n \geq 5$: ✓
- Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$: ✓?
New aim: Reduce the number of cases to be considered for $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

Just like in type $\tilde{D}_n$, enumerations can be composed and decomposed – and maybe we can restrict to wings again.

So consider the algebra $A = kQ/I$ with enumeration $\pi$:
Classification: Type $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$
Classification: Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

There are modules $\nabla_\pi(i)$ outside the wings and there is once again a non-trivial enumeration that gives the same quiver again.

The wings are still useful, though:

Consider the algebra $A = kQ/I$:

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\text{Q:} \\
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g \\
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\begin{array}{c}
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\]

Andre Beineke
$\Delta$-critical quasi-hereditary algebras
21 April 2018 66 / 75
Classification: Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$
Classification: Type $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$

\[
\begin{align*}
\Delta \pi(a) & \quad 0001000 \\
\Delta \pi(b) & \quad 000100 \\
\Delta \pi(c) & \quad 000010 \\
\Delta \pi(d) & \quad 0000001 \\
\Delta \pi(e) & \quad 00000011 \\
\Delta \pi(f) & \quad 00000011 \\
\Delta \pi(g) & \quad 00000011 \\
\end{align*}
\]
Classification: Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

\[ \cdots \]
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Classification: Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$
Classification: Type $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$
Classification: Type \( \tilde{E}_6, \tilde{E}_7, \tilde{E}_8 \)
Classification: Type $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$
Wings of branches may contain many sub-modules of the $I(i)$.

In particular, a linearly (towards the rest of the quiver) ordered branch of $Q$ can be enumerated to yield any chosen orientation of that branch.

Further on, the different equivalence classes are labelled by non-crossing partitions and can be generally described.
After splitting off misorderings in wings where possible, the numbers of cases to be considered are as follows:

- Case $\tilde{E}_6$: 68 cases (424 equivalence classes) for 56 quivers,
- Case $\tilde{E}_7$: 1,487 cases (8,824 equivalence classes) 437 quivers,
- Case $\tilde{E}_8$: 37,306 cases (179,302 equivalence classes) for 3,809 quivers