A Primer On Affine Group Schemes

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Let k be a commutative ring. The goal of this lecture is to introduce:

- Affine group schemes over k.
- Their representations.
- Functors on and features of category of G-modules.

Eventually, k will be assumed to be Noetherian.

Caveats

- Background needed varies by person.
- Generality poses challenge to accuracy and simplicity.
- Concepts and terms will be many, proofs will be few/none.

Basic Notions

An **affine group scheme** G (over k) is a representable functor from commutative k-algebras to groups.

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$k[G]$ = representing algebra for G (or **coordinate algebra**).

i.e. for all commutative k-algebras A,

$$G(A) \cong \mathrm{Hom}_{k-alg}(k[G], A)$$

First Examples

The **additive group scheme** $\mathbb{G}_a$ assigns to A the group $(A, +)$.

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The **multiplicitive group scheme** $\mathbb{G}_m$ assigns to A the group $(A^\times, \cdot)$.

$k[\mathbb{G}_m] \cong k[t, t^{-1}]$.

$k[G]$ is a commutative Hopf algebra over k, with co-structure

comultiplication: $\Delta : k[G] \to k[G] \otimes_k k[G]$

counit (augmentation): $\varepsilon : k[G] \to k$

coinverse (antipode): $\eta : k[G] \to k[G]$

Can obtain affine group schemes from many directions:

- Commutative Hopf algebras.
- Abelian groups.
- Finite duals of co-commutative Hopf algebras.
- Finite groups.
- Group functors that are representable.
- Affine algebraic groups (k a field). These correspond to affine group schemes such that $k[G]$ is finitely generated and nilpotent free.

Constant Group Schemes

If Γ is an abstract finite group, then G_Γ is the **constant group scheme corresponding to** Γ, where

$$G_\Gamma(A) \cong \Gamma \times \cdots \times \Gamma \qquad (\quad \pi_0(A)\text{-times} \quad)$$

$k[G_\Gamma]$ is the dual of the group algebra $k\Gamma$.

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Example The trivial group $\{1\}$ is represented by k.

Remark: The truly constant group functor that assigns Γ to every A is not representable if $1 < |\Gamma|$.

General Linear Groups

Let $n \geq 1$. The affine group scheme GL_n is defined by

$$GL_n(A) = \{n \times n \text{ invertible matrices with entries in } A\}$$

$k[GL_n] \cong k[X_{11}, X_{12}, \ldots, X_{nn}, Y]/(\det(X_{ij})Y - 1)$.

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- SL_n is defined analogously.
- These are primary examples of **reductive group schemes**.

More Examples

- If G and H are affine group schemes, then $G \times H$ is also, with $k[G \times H] \cong k[G] \otimes_k k[H]$.

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- If $\mathfrak{g}$ is a p-restricted Lie algebra over a field of characteristic p, then its restricted enveloping algebra is a finite dimensional cocommutative Hopf algebra, thus its dual represents an affine group scheme over k.

Homomorphisms

A homomorphism between affine group schemes

$$\phi : H \to G$$

is a natural transformation between group functors.

It is equivalent to a homomorphism of Hopf algebras

$$\phi^* : k[G] \to k[H].$$

(ϕ^* called the comorphism of ϕ)

(examples: $\mathbb{G}_a$ to GL_2, GL_N, and $\mathbb{G}_m$, split tori to $\mathbb{G}_m$).

Subgroup Schemes

The image of H is a **closed subgroup scheme** of G if

$$\phi^* : k[G] \to k[H]$$

is surjective.

i.e. $k[H] \cong k[G]/I$, where I is a Hopf ideal of $k[G]$, meaning

$$\Delta(I) \subseteq I \otimes k[G] + k[G] \otimes I$$

Example

Let $n \geq 1$.

The n-**th roots of unity** μ_n is the group scheme where

$\mu_n(A) = \{a \in A \mid a^n = 1\}$

We have $k[\mu_n] \cong k[t](t^n - 1) \cong k\mathbb{Z}/n\mathbb{Z}$.

Since $k[\mathbb{G}_m] \cong k\mathbb{Z}$, there is a surjective Hopf algebra homomorphism

$$k[\mathbb{G}_m] \to k[\mu_n]$$

so that $\mu_n \leq \mathbb{G}_m$.

Base Change and Ground Rings

If k' is a commutative k-algebra, we get affine group scheme $G_{k'}$ over k' where

$$k'[G_{k'}] = k[G] \otimes_k k'$$

If k_1 a subring of k, then G **is defined over** k_1 if $k[G] \cong k_1[G_1]$ for some affine group scheme G_1 over k_1.

Example: The groups GL_n, SL_n, $\mathbb{G}_a$, $\mathbb{G}_m$, μ_n are all defined over $\mathbb{Z}$.

Frobenius Morphisms

If k has characteristic p and G is defined over $\mathbb{F}_p$, then

$$k[G] \cong \mathbb{F}_p[G] \otimes_{\mathbb{F}_p} k.$$

There is a Frobenius morphism

$$F : G \to G$$

having comorphism

$$F^*(x \otimes_{\mathbb{F}_p} a) = x^p \otimes_{\mathbb{F}_p} a.$$

We denote by $G_{(r)}$ the kernel of F^r

(the r**-th Frobenius kernel of** G)

Examples

- $\mathbb{G}_a$
- $\mathbb{G}_m$
- GL_n

Remark: $\mathbb{G}_{a(1)}$ is defined over $\mathbb{Z}$ as an affine scheme, but not as a affine group scheme.

An affine group scheme G over k is

flat - if $k[G]$ is flat as an k-module.

algebraic - if $k[G]$ is finitely generated as a k-algebra.

finite - if $k[G]$ is finitely generated as an k-module.

Finally, an **affine algebraic group** will be an affine algebraic group scheme over k a field such that $k[G]$ is reduced.

Example

μ_n is a finite flat group scheme over k since

$k[\mu_n]$ is free k-module on $\{1, t, t^2, \ldots, t^{n-1}\}$

All G_Γ, and the Frobenius kernels encountered earlier, are also examples.

Algebra of Measures

If G is finite group scheme and $k[G]$ is projective k-module:

$k[G]^*$ is a cocommutative Hopf algebra, projective and finitely generated as a k-module.

$k[G]^*$ is sometimes denoted $M(G)$, called **algebra of measures**

$k[G]^*$ is sometimes denoted kG, called **the group algebra**

Indeed, for the constant group scheme G_Γ,

$k[G_\Gamma]^* \cong k\Gamma$.

Cartier's Theorem

If k is a field of characteristic 0, then $k[G]$ has no nilpotent elements.

Consequence: if $k = \bar{k}$ of char. 0 and G is finite group scheme, then $G \cong G_\Gamma$ for some Γ.

Representations

Let M be a k-module.

For each commutative k-algebra A we get an A-module

$$M \otimes_k A.$$

M becomes a G-module if for each A we have A-linear

$$G(A) \times (M \otimes_k A) \to (M \otimes_k A)$$

i.e. given by a natural transformation $G \times M \to M$.

(here M refers to set-functor defined above)

A G-module M is equivalent to the data of an $k[G]$-**comodule**:

A k-module M with an k-linear map

$$\Delta_M : M \to M \otimes_k k[G]$$

that is compatible with Δ and ε on $k[G]$.

Equivalences

$\{G\text{-modules}\} \xrightarrow{\sim} \{k[G]\text{-comodules}\}$

If G is a finite flat group scheme over k, then there is an equivalence:

$\{G\text{-modules}\} \xrightarrow{\sim} \{k[G]^*\text{-modules}\}$

We assume from now on that

k - is a Noetherian commutative ring.

G - is a flat affine group scheme over k.

Important Modules

- Simple and indecomposable modules.
- k - the trivial module
- $k[G]$ - G-module via left and right regular representations (isomorphic to each other)
- Any homomorphism $G \to GL_n$ makes k^n into G-module.
- G-module structures on k correspond to character group

$$X(G) := \mathrm{Hom}(G, \mathbb{G}_m)$$

Fixed Points

Let M be a G-module, so a $k[G]$-comodule.

The submodule of fixed points $M^G \subseteq M$ is all $m \in M$ such that either hold:

- $\Delta_M(m) = m \otimes 1$.
- $g(m \otimes_k 1) = m \otimes 1$ for all $g \in G(A)$ and all A.

The Category of G-modules

Derived Functors

The category of G-modules (not necessarily finitely generated over k)

- Has enough injectives.
- If G is finite, also has enough projectives.

In all cases we can define right derived functors to left exact functors.

Ext Functors

Let M be a G-module. Then $\mathrm{Hom}_G(M, \_)$ is left exact, having right derived functors

$\mathrm{Ext}^i_G(M, \_)$

These

- Can be computed via injective resolutions.
- Can also be computed via projective resolutions (when available).
- Can be interpreted as Yoneda Extension groups.

Induction Functor

Given $H \leq G$, there is a restriction functor

$$\mathrm{res}_G^H : G\text{-mod} \to H\text{-mod}$$

(describe it in terms of restricting the action, and in terms of comodules.)

it has a right-adjoint

$$\mathrm{ind}_H^G : H\text{-mod} \to G\text{-mod}$$

(You will hear more about these in Antoine's talk.)