Representations of Reductive Groups

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Spillover From Yesterday's Talk

Algebra of Measures

If G is finite group scheme and $k[G]$ is projective k-module:

$k[G]^*$ is a cocommutative Hopf algebra, projective and finitely generated as a k-module.

$k[G]^*$ is sometimes denoted $M(G)$, called **algebra of measures**

$k[G]^*$ is sometimes denoted kG, called **the group algebra**.

Indeed, for the constant group scheme G_Γ,

$k[G_\Gamma]^* \cong k\Gamma$.

Cartier's Theorem

If k is a field of characteristic 0, then $k[G]$ has no nilpotent elements.

Consequence: if $k = \bar{k}$ of char. 0 and G is finite group scheme, then $G \cong G_\Gamma$ for some Γ.

Representations

Assume G flat. A G-module M is equivalent to the data of an $k[G]$-**comodule**:

A k-module M with a k-linear map

$$\Delta_M : M \to M \otimes_k k[G]$$

that is compatible with Δ and ε on $k[G]$.

Equivalences

$$\{G\text{-modules}\} \xrightarrow{\sim} \{k[G]\text{-comodules}\}$$

If G is a finite flat group scheme over k, then there is an equivalence:

$$\{G\text{-modules}\} \xrightarrow{\sim} \{k[G]^*\text{-modules}\}$$

Defining Reductive

Let k be an algebraically closed field.

Reductive algebraic groups over k include some familiar groups:

- GL_n, SL_n, PGL_n
- SO_{2n+1}
- Sp_{2n}
- SO_{2n}
- Simple algebraic groups of exceptional type
- $(\mathbb{G}_m)^{\times r}$

If $\text{char}(k) = 0$, then G reductive $\iff$ **linearly reductive**.

G linearly reductive: all G-modules are semisimple.

In arbitrary characteristic, the following are equivalent

1. G is reductive
2. G is geometrically reductive
3. G is power reductive

For now, we denote as follows various subgroups of GL_n:

- $T_n =$ diagonal
- $B_n =$ lower-triangular
- $U_n =$ strictly lower-triangular (1's on diagonal)

Definition via group structure

An affine algebraic group U over k is called **unipotent** if for every rational U-module $M \neq \{0\}$, we have

$$M^U \neq \{0\}.$$

Examples: $\mathbb{G}_a$ and U_n are unipotent groups.

Fact: U is unipotent if and only if there is a closed embedding

$$U \to U_n$$

for some n.

There is a maximal connected normal unipotent subgroup $R_u(G) \leq G$, called the **unipotent radical of** G.

Example

$R_u(B_n) = U_n$.

G is called **reductive** if $R_u(G) = \{1\}$.

Every affine algebraic group G has a **reductive quotient** $G/R_u(G)$.

In addition to earlier remarks, reductive groups are nice because:

- Their group structure is well understood.
- Are definable as group schemes over $\mathbb{Z}$.
- Their representation theory is:
  - well understood in char. 0
  - somewhat understood in char. p
- Roots, weights, and character formulas.

Group Structure

Can fix subgroups in G:

A **maximal torus** $T \leq G, \quad T \cong (\mathbb{G}_m)^{\times n}$

A **Borel subgroup** B with $T \leq B \leq G$

The **Weyl group** $W = N_G(T)/T$

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Example: for $G = GL_n$, $T = T_n$, $B = B_n$, $U = U_n$

$N_G(T) =$ monomial matrices. $N_G(T)/T \cong S_n$.

T-modules

Every T-module is a direct sum of 1-dimensional modules.

The character group (or weight lattice) is

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$$

If $0 \neq \lambda \in X(T)$, we also write λ for k with T-action via λ.

Some Strategies

The completely understood nature of T-modules is exploited by:

- Restricting G-modules down to T (character theory).

- Building G-modules from simple T-modules.

Group Structure II

The action of T on $Lie(G)$ determines a set $\Phi \subseteq X(T)$ of **roots**.

$\Phi = \Phi^+ \cup \Phi^-$, where $\Phi^- = -\Phi^+$ (positive and negative roots)

For each $\alpha \in \Phi$ there is a root subgroup $U_\alpha \leq G$ such that

- $U_\alpha \cong \mathbb{G}_a$.
- T normalizes U_α.
- The negative root subgroups lie in B.
- G generated by T all U_α.

G-modules as T-modules

If M is a G-module, then $M \cong \bigoplus_{\lambda \in X(T)} M_\lambda$

where $M_\lambda = \{m \in M \mid t.m = \lambda(t)m \quad \forall t \in T\}$

If $u \in U_\alpha, m \in M_\lambda$, then

$$u.m = m + \left(\text{stuff in } \sum_{n>0} M_{\lambda+n\alpha}\right)$$

Partial Ordering On $X(T)$

In view of the above, it is relevant to define partial order $\leq$ on $X(T)$ where

$$\lambda \leq \mu$$

if $\mu - \lambda$ is non-negative sum of positive roots.

Dominant Weights

For each α, the root isogeny $\varphi_\alpha : SL_2 \to \langle U_{-\alpha}, U_\alpha \rangle$ defines a **coroot** $\alpha^\vee \in \mathrm{Hom}(\mathbb{G}_m, T)$ according to

$$s \mapsto \varphi_\alpha\left(\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\right).$$

For $\lambda \in X(T)$, define

$$\langle \lambda, \alpha^\vee \rangle = \lambda \circ \alpha^\vee \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}.$$

Define $X_+(T) = \{\lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \quad \forall \alpha \in \Phi^+\}$.

GL_n Example

This group and module structure can be seen explicitly for GL_n, acting in its natural representation.

Each $\lambda \in X(T)$ defines a B-module by pulling back via

$$B \to B/U \cong T$$

Define the **costandard/induced module** $\nabla(\lambda) = \mathrm{ind}_B^G \lambda$.

Properties of Induced Modules

The modules $\nabla(\lambda) = \operatorname{ind}_B^G \lambda$ satisfy:

- $\operatorname{Hom}_G(M, \nabla(\lambda)) \cong \operatorname{Hom}_B(M, \lambda)$.
- $\dim \nabla(\lambda) < \infty$ (since G/B projective variety).
- $\nabla(\lambda) \neq \{0\} \iff \lambda \in X_+(T)$
- $\operatorname{soc}_G(\nabla(\lambda))$ is simple, denoted $L(\lambda)$
- The set $\{L(\lambda),\ \lambda \in X_+(T)\}$ is complete listing of simple G-modules (up to isomorphism).

If $\mathrm{char}(k) = 0$, we have just described all indecomposable G-modules.

$\nabla(\lambda) \cong L(\lambda)$ for all $\lambda \in X_+(T)$

The set is then self-dual:

$$\nabla(\lambda)^* \cong L(\lambda)^* \cong L(-w_0\lambda) \cong \nabla(-w_0\lambda).$$

Standard/Weyl modules

We can define the **standard/Weyl module** $\Delta(\lambda)$ by

$$\Delta(\lambda) = \nabla(-w_0\lambda)^* = (\mathrm{ind}_{B^+}^{G}(-\lambda))^*$$

$\Delta(\lambda)/\mathrm{rad}_G\Delta(\lambda) \cong L(\lambda) \cong \mathrm{soc}_G\nabla(\lambda)$

$\Delta(\lambda)$ is only "new" in characteristic p.

Ext-vanishing properties

Standard and Costandards have no cohomology in one direction

$$\mathrm{Ext}^i_G(\Delta(\lambda), \nabla(\mu)) \cong \begin{cases} k & \text{if } i = 0 \text{ and } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

In $\mathcal{C}(\leq \lambda) =$ subcategory of G-Mod gen. by $L(\gamma)$, $\gamma \leq \lambda$,

$\Delta(\lambda)$ is a projective indecomposable object

$\nabla(\lambda)$ is an injective indecomposable object

G-Mod is a highest weight category

A curious observation

$\nabla(0) \cong k \cong \Delta(0)$

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From above, it follows that

$$\mathrm{Ext}^i_G(k, \nabla(\lambda)) = 0 \qquad \text{for all } \lambda \text{ and all } i > 0.$$

and specifically that

$$\mathrm{Ext}^i_G(k, k) = 0 \qquad \text{for all } i > 0.$$

Good filtrations

Let M be a G-module. A chain of G-submodules of M

$$\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$$

is a **good filtration** of M if every M_i/M_{i-1} is isomorphic to some $\nabla(\lambda_i)$.