Open problems in the representation theory of reductive groups

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Setting

Let $k = \bar{k}$, $\mathrm{char}(k) = p$.

Let G be reductive group over k.

Main examples: GL_n, SL_n

Representation theorists want to know all rational G-modules.

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Overarching Open Problem

Resolve these issues.

Goal of this talk

Discuss some primary tools, each related in some say to the contents of this meeting.

Cohomological Support Varieties

The Steinberg Module

More About G-Mod

Fix maximal torus in Borel subgroup $T \leq B \leq G$

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$\Delta(\lambda)/\mathrm{rad}_G \Delta(\lambda) \cong L(\lambda) \cong \mathrm{soc}_G \nabla(\lambda)$

Ext-vanishing properties Let $i > 0$.

$\mathrm{Ext}^i_G(L(\mu), \nabla(\lambda)) \neq 0 \;\Rightarrow\; \mu > \lambda.$

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As a consequence

In $\mathcal{C}(\leq \lambda) =$ subcategory of G-Mod gen. by $L(\gamma)$, $\gamma \leq \lambda$,

$\Delta(\lambda)$ is a projective indecomposable object
$\nabla(\lambda)$ is an injective indecomposable object

$\mathrm{Ext}^i_G(\Delta(\lambda), \nabla(\mu)) = 0 \qquad$ for all $\lambda, \mu \in \mathbb{X}^+,\; i > 0.$

A filtration of a G-module M is called a

- **good filtration** if the quotients are isomorphic to $\nabla(\lambda)$'s.

- **Weyl filtration** if the quotients are isomorphic to $\Delta(\lambda)$'s.

A module with each of these filtrations is called a **tilting module**.

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Theorem (Ringel, Donkin)

There is a unique indecomposable tilting module $T(\lambda)$ of highest weight λ for each $\lambda \in X_+(T)$.

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Theorem (Ringel, Donkin)

There is a unique indecomposable tilting module $T(\lambda)$ of highest weight λ for each $\lambda \in X_+(T)$.

(diagram of highest weight modules)

Many interesting questions about the modules $T(\lambda)$

Example: work by Riche-Williamson last few years.

Support Varieties

Let $\mathcal{G}$ be a finite group scheme over k.

By Friedlander-Suslin Theorem, $\mathrm{H}^*(\mathcal{G}, k)$ is finitely generated k-algebra.

The even part

$$\mathrm{H}^{ev}(\mathcal{G}, k) = \bigoplus_{i \geq 0} \mathrm{H}^{2i}(\mathcal{G}, k)$$

is commutative, so modulo nilpotents, is ring of functions on affine k-variety.

Let $\mathcal{V}_{\mathcal{G}}$ denote the corresponding variety.

Let M be finite dimensional $\mathcal{G}$-module. Then

$\mathrm{H}^*(\mathcal{G}, M^* \otimes M)$

is finitely generated $\mathrm{H}^{ev}(\mathcal{G}, k)$-module.

Let $I_M \subseteq \mathrm{H}^{ev}(\mathcal{G}, k)$ denote the annihilator of $\mathcal{V}_{\mathcal{G}}(M)$.

Let $\mathcal{V}_{\mathcal{G}}(M)$ denote subvariety corresponding to I_M.

$\mathcal{V}_{\mathcal{G}}(M)$ is **cohomological support variety** of M.

Key Points

(Thanks to: Carlson, Avrunin-Scott, Friedlander-Parshall, Suslin-Friedlander-Bendel, Friedlander-Pevtsova,...)

- Support varieties give an invariant that can be attached to $\mathcal{G}$-modules and sees much of module structure. (injectivity/projectivity, direct sums, tensor products)

- Support varieties have a non-cohomological description that leads to other invariants and establishes some of the properties above.

Recall r-th Frobenius morphism $F^r : G \to G$.

Gives rise to two families of subgroup schemes:

Finite Chevalley Subgroups

$$G(\mathbb{F}_q) = G^{F^r} \qquad (q = p^r).$$

Frobenius Kernels

G_r

These families are quite different, yet oddly similar too.

Just for fun

Can also play around with subgroup schemes of the form

$$G_r G(\mathbb{F}_q) \cong G_r \rtimes G(\mathbb{F}_q) \leq G$$

(since G_r is normal in G)

Given a G-module M, can consider, for example

$\mathcal{V}_{G(\mathbb{F}_q)}(M)$

$\mathcal{V}_{G_r}(M)$

Can ask:

- How do these relate?
- Can we compute these for $L(\lambda)$, $\nabla(\lambda)$, $T(\lambda)$?

Some partial and complete answers can be found in:

- Lin-Nakano, Carlson-Lin-Nakano, Friedlander
- Suslin-Friedlander-Bendel, Nakano-Parshall-Vella, Drupieski-Nakano-Parshall, Cooper, Hardesty, Achar-Hardesty-Riche,...

Still remains to know better, for example:

$\mathcal{V}_{G_1}(T(\lambda))$, $\mathcal{V}_{G_1}(L(\lambda))$

$\mathcal{V}_{G_r}(\nabla(\lambda))$, $\mathcal{V}_{G_r}(L(\lambda))$, $\mathcal{V}_{G_r}(T(\lambda))$

Steinberg Modules

Let $\rho = \frac{1}{2}\left(\sum_{\alpha \in \Phi^+} \alpha\right)$

$\mathrm{St}_r = \nabla((p^r - 1)\rho)$ **r-th Steinberg module**

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- $\mathsf{St}_r \cong \mathsf{St}_r^*$.
- Is simple, standard, costandard, tilting.
- Is simple and projective over G_r and $G(\mathbb{F}_q)$.

We can embed G-Mod inside G-Mod via

$M \to \mathrm{St}_r \otimes M^{(r)}$

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Kaneda-Gros have studied the Frobenius Contraction Functor from G-Mod to G-Mod

$M \to \mathrm{Hom}_{G_r}(\mathrm{St}_r, \mathrm{St}_r \otimes M)^{(-r)}$