Group Scheme Actions on Rings and the Cech Complex

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Context

Notation: k a field, of finite characteristic p to be interesting. G a finite group scheme. $S = k[x_1, \ldots, x_n]$ a polynomial ring, graded with the x_i in degree 1. G acts on S preserving grading.

Our aim is to prove:

Theorem $\text{hreg}(S^G) \leq -n$ for all groups G and certain types of group schemes.

and see how this gives us bounds on the degrees of the generators and relations of G.

The proof will also show that S, considered as a kG-module, only contains finitely many non-isomorphic indecomposable summands.

We say that S is of finite representation type.

Cech Complex

R is a commutative noetherian $\mathbb{Z}$-graded k-algebra in non-negative degrees, finite dimensional over k in each degree.

$x_1, \ldots, x_r \in R$, $\quad M$ an R-module.
The (augmented) Cech complex $C_{(\underline{x})}(M)$ is the cochain complex

$$M \to \bigoplus_i M_{x_i} \to \bigoplus_{i<j} M_{x_i x_j} \to \cdots \to M_{x_1 \cdots x_r},$$

where M_x denotes the localization obtained by inverting x. It can be obtained as follows. $C(x_i; R)$ is the complex

$$R \to R_{x_i},$$

with R in degree 0 and R_{x_i} in degree 1 and

$$C(\mathbf{x}; M) = \left(\bigotimes_{i=1}^{r}{}_R \; C(x_i; R) \right) \otimes_R M.$$

If M is an RG-module then $C_{(\underline{x})}(M)$ is a complex of RG-modules.

Local Cohomology

Local cohomology $H^i_{(\underline{x})}(M)$ is the homology of $C_{(\underline{x})}(M)$.

- $H^i_{(\underline{x})}(M)$ only depends on $\mathrm{rad}(\underline{x})$. We will always want $\mathrm{rad}(\underline{x}) = \mathfrak{m} = \mathrm{rad}\, R_{>0}$.
- If M is Noetherian then $H^i_{\mathfrak{m}}(M)$ is 0 in sufficiently high degrees.

Definition $\mathrm{hreg}(M)$ is the largest degree for which $H^*_{\mathfrak{m}}(M) \neq 0$.

We want to show that $\mathrm{hreg}(S^G) \leq -n$. (This is slightly stronger than $\mathrm{reg}(S^G) \leq 0$.)

Castelnuovo-Mumford Regularity

Let $R = k[d_1, \ldots, d_n]$ be a polynomial ring with $\deg(d_i) > 0$. For any finite R-module M, take the minimal projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

$P_i \cong \oplus R(-a_{i,j})$,

where $R(-a)$ denotes R shifted up in degree by a.

Set $\rho_i(M) = \max_j\{a_{i,j}\}$,

and $\mathrm{hPreg}(M) = \max\{\rho_i(M)\} - \Sigma_i \deg(d_i)$.

Notice that $\rho_0(M)$ is a bound on the degrees of the generators of M as an R-module and $\rho_1(M)$ is a bound on the degrees of the relations. If M is a k-algebra then $\max\{\rho_0(M), \deg(d_i)\}$ is a bound on the degrees of the generators as a k-algebra and $\max\{2\rho_0(M), \rho_1(M), \deg(d_i)\}$ is a bound for the relations.

In terms of hPreg, the two bounds for a k-algebra are $\max\{\mathrm{hPreg}(M) + \Sigma_i \deg(d_i), \deg(d_i)\}$ for the generators and $\max\{2(\mathrm{hPreg}(M) + \Sigma_i \deg(d_i)), \deg(d_i)\}$ for the relations.

Local Duality

Local Duality Over a polynomial ring R we have

$$\mathrm{Hom}_k(H^i_{\mathfrak{m}}(M), k) \cong \mathrm{Ext}^{n-i}_R(M, R(\Sigma \deg d_i)).$$

Using this we can show:

Corollary $\mathrm{hPreg}(M) = \mathrm{hreg}(M)$.

Returning to S^G, the aim of this talk is to show that:

Theorem $\mathrm{hreg}(S^G) \leq -n$.

Now take a Noether normalization $R = k[d_1, \ldots, d_n]$ of S^G, i.e. a finite map $R \to S^G$.

Since $\mathrm{hPreg}(S^G) \leq -n$, we can deduce that S^G is generated in degrees at most

$$\max\{\Sigma_i \deg(d_i) - n, \deg(d_i)\}$$

and the relations are in degrees at most twice this.

By Dade's Lemma we can take $d_i = \mathcal{N}_G X_i$ for the X_i in general position. This leads to the bound

$$n(|G| - 1)$$

for the degrees of the generators (provided $n, |G| \geq 2$), which was a conjecture of Kemper (for groups).

Strategy

We want to show that $C_{(\underline{x})}(S^G)_{>-n}$ is exact for S a polynomial ring and for some invariants $\underline{x}$ such that $\mathrm{rad}(\underline{x}) = \mathfrak{m}$.

Suppose we can show that $C_{(\underline{x})}(S)_{>-n}$ is split exact over kG.

It is exact because $\mathrm{hreg}\, S = -n$.

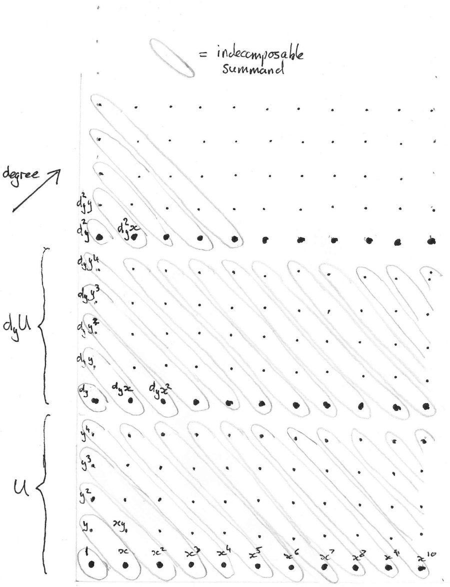
Then $C_{(\underline{x})}(S)^G_{>-n}$ must also be exact. But $C_{(\underline{x})}(S)^G_{>-n} = C_{(\underline{x})}(S^G)_{>-n}$, so the latter is exact, as required.

Example A cyclic group of order p acts on $S = k[x, y]$ by $y \mapsto y + x$, $x \mapsto x$. Invariants $k[x, d_y]$, $d_y = y^p - x^{p-1}y$

Let $T \subset S$ be the k-submodule spanned by the monomials with y-degree $< p$.

Then $S = T \otimes_k k[d_y] = T \otimes_k U$.

Example: Cyclic Group and Two Variables



$G = C_p$ cyclic order p

$S = k[x, y]$

$d_y = y^p - x^{p-1} y$

$S^G = k[x, d_y]$

Action: $y \mapsto y + x$, $x \mapsto x$

k-basis: $\{x^i y^j d_y^k : 0 \leq j \leq p-1\}$

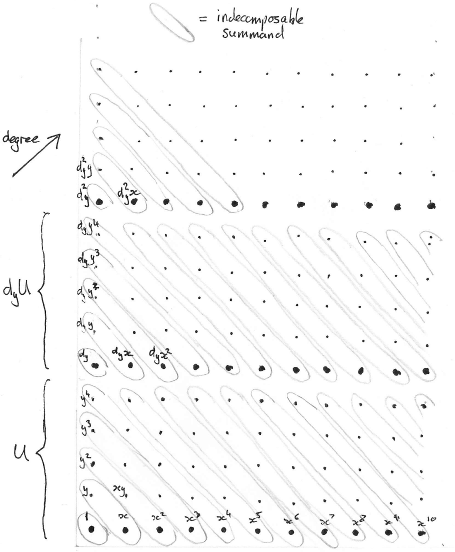
U = span of $\{x^i y^j : 0 \leq j \leq p-1\}$

U is a $k[x]G$-submodule of S

$S \cong k[d_y] \otimes U$ as a $k[x, d_y]G$-module

$S^G = \mathrm{soc}_{kG} S = k[x, d_y] =$

Example: α_p and Two Variables



$G = \alpha_p$

$S = k[x,y]$

$d_y = y^p$

$S^G = k[x, d_y]$

Action: $x\frac{\partial}{\partial y}$

k-basis: $\{ x^i y^j d_y^k : 0 \le j \le p-1 \}$

U = span of $\{ x^i y^j : 0 \le j \le p-1 \}$

U is a $k[x]G$-submodule of S

$S \cong k[d_y] \otimes U$ as a $k[x, d_y]G$-module

$S^G = \mathrm{soc}_{kG} S = k[x, d_y] =$

Relative Projectivity

How can we show that a complex is split?

Let $\mathcal{C}$ be a set of subgroup schemes of G.

Definition A kG-module M is *projective relative to* $\mathcal{C}$ if the following equivalent conditions hold.

a) M is a summand of some $\oplus_{H \in \mathcal{C}} kG \otimes_{kH} V_H$.

b) M is a summand of $\oplus_{H \in \mathcal{C}} kG \otimes_{kH} M$.

c) Any surjection $L \twoheadrightarrow M$ that splits on restriction to any $H \in \mathcal{C}$ is split over G.

d) $\mathrm{Id}_M \in \sum \mathrm{Im}\,\mathrm{Tr}_H^G$.

Transfer

For G a group, $H \leq G$, M a G-module, $\mathrm{Tr}_H^G \colon M^H \to M^G$ is defined by $m \mapsto \sum_{g \in G/H} gm$.

For group schemes this is a little trickier. Let Ind_H^G be the *left* adjoint to restriction and Coind_H^G the *right* adjoint. They are related by

$$\mathrm{Ind}_H^G(M) \cong \mathrm{Coind}_H^G(\mu_H M),$$

where μ_H is a certain 1-dimensional representation of H.

There is the adjunction map $\eta \colon k \to \mathrm{Coind}_H^G k$.

$$\mathrm{Hom}_H(k, \mu_H M) \cong \mathrm{Hom}_H(\mu_H^{-1}, M) \cong \mathrm{Hom}_G(\mathrm{Ind}_H^G \mu_H^{-1}, M) \cong \mathrm{Hom}_G(\mathrm{Coind}_H^G k, M) \xrightarrow{\eta^*} \mathrm{Hom}_G(k, M)$$

This gives us $\mathrm{Tr}_H^G \colon (\mu_H M)^H \to M^G$

and $\mathrm{Tr}_H^G \colon \mathrm{Hom}_H(M, \mu_H M) \to \mathrm{Hom}_G(M, M)$.

Geometry

Theorem Let S be a commutative ring on which G acts. If $\mathrm{Spec}(S \otimes_k \bar{k})^G = \emptyset$ then $S^G = \sum_{H<G, p|[G:H]} \mathrm{Tr}_H^G(S^H)$.

It follows that S is projective relative to proper subgroup schemes.

Background proved using a change of category.

1) If M is of finite decomposition type so is M_x.

2) $C_{(\underline{x})}(M)$, considered as a complex of kG-modules, is unique up to homotopy, depending only on $\mathrm{rad}(\underline{x})$ and M.

Splitting

$$(\text{split}) \otimes (\text{anything}) = (\text{split})$$

If A is split in degrees $> a$ and B is split in degrees $> b$ then $A \otimes B$ is split in degrees $> a + b$.

Suppose the action on A is trivial. If a and b are best possible so is $a + b$.

Back to the example.

$C_{x,d_y}(S) = C_x(T) \otimes C_{d_y}(U) = C_x(T) \otimes C_{d_y}(k[d_y])$

$k[d_y] \to k[d_y, d_y^{-1}]$ is split in degrees $> -p$.

$T \to T_x$ is split in degrees $> p - 2$.

So $C_{x,d_y}(S)$ is split in degrees > -2, as required.

Main Theorem

Theorem G acts on S and M is a finitely generated SG-module. Suppose that $\mathrm{hreg}(M) \leq N$ and for each p-subgroup P of G there exist $y_1, \ldots, y_r, z_1, \ldots, z_s \in S^P_{>0}$ such that:

a) y_i vanishes on $\mathrm{Spec}(S)^P$,

b) $\mathrm{rad}(\underline{y}, \underline{z}) = \mathfrak{m}_S$,

c) there is a $k[\mathbf{y}]P$-module T and a $k[\mathbf{z}]P$-module U such that $M_{>N} \cong (T \otimes_k U)_{>N}$ as $k[\mathbf{yz}]P$-modules,

d) P acts trivially on U or....

Then:

1) for any $\mathbf{x}$ in $S^G_{>0}$ such that $\mathrm{rad}_S(\mathbf{x}) = \mathfrak{m}_S$, the complex of kG-modules $C_{\mathbf{x}}(M)_{>N}$ is split exact;

2) $\mathrm{hreg}(M^G) \leq N$;

3) M is of finite decomposition type.

Polynomial Rings

$S = k[V]$. A p-group scheme P acts.

Suppose that we can choose a basis $x_1^*, \dots, x_n^*$ for V such that the matrices for the action of P are lower-triangular and $x_{r+1}^*, \dots, x_n^*$ span V^P. $x_1, \dots, x_n \in S_1$ the dual basis.

Let $d_{x_i} = \mathcal{N}_G x_i$. Set

$$y_1, \dots, y_r = d_{x_1}, \dots, d_{x_r}$$
$$z_1, \dots, z_s = d_{x_{r+1}}, \dots, d_{x_n}$$

Let $T \subset S$ be the subspace spanned by monomials with x_{r+i}-degree $< \deg z_i$. It is a kP-submodule.

$U = k[z_1, \dots, z_n]$.

Then $S \cong T \otimes U$ and the hypotheses of the theorem are satisfied with $M = S$ (for this subgroup P and $N = -n$).

We can always find such a basis if the p-group scheme (or, equivalently, its identity component) is trigonalizable, meaning that every simple module is 1-dimensional.

Thus the regularity result mentioned at the beginnining holds for such group schemes.

Proof

Reduce to p-groups P (transfer argument). Use induction on $|P|$.

$$\mathrm{Spec}(S_{y_i}) = \mathrm{Spec}(S) - L(y_i)$$

$$\begin{aligned} \mathrm{Spec}(S_{y_i})^P = \emptyset &\Rightarrow S_{y_i} \text{ projective rel. proper subgroups} \\ &\Rightarrow (T_{y_i} \otimes U)_{>N} \text{ projective rel. proper subgroups} \\ &\Rightarrow T_{y_i} \text{ projective rel. proper subgroups} \\ &\Rightarrow T_{y_i} \text{ of finite decomposition type.} \end{aligned}$$

For $Q < P$, $\quad C_{\mathbf{yz}}(S)_{>N}\downarrow^P_Q$ is split, by induction.

$$C_{\mathbf{yz}}(S) = C_{\mathbf{y}}(T) \otimes C_{\mathbf{z}}(U)$$

Let d be the top degree in which $C_{\mathbf{z}}(U)$ is not exact.
Then $C_{\mathbf{y}}(T)_{>N-d}\downarrow^P_Q$ is split.

We can now split $C_{\mathbf{y}}(T)_{>N-d}$ term by term starting from the right.
Thus $(C_{\mathbf{y}}(T) \otimes C_{\mathbf{z}}(U))_{>N}$ is split.

Also $S_{>N}$ is a summand of $\oplus S_{x_i}$, so of finite decomposition type, hence also S.

Other Group Schemes

If we consider sl_2 acting on $k[x, y]$ in the canonical way, the theorem does not apply.

$k[x, y]^{\mathrm{sl}_2} = k[x^p, y^p]$, so $\operatorname{hreg} k[x, y]^{\mathrm{sl}_2} = -2p \leq -2$,

so the regularity result does hold.

However, $k[x, y]$ is not of finite representation type over sl_2.

This seems to be related to the failure of the Normal Basis Theorem. Recall that if a (genuine) group acts on a field K then, as $K^G G$-modules, $K \cong K^G G$.

For the action of sl_2 on $K = k(x, y)$ we find that K is not even projective over $K^{\mathrm{sl}_2} \otimes_k \mathrm{sl}_2$ and $|K : K^{\mathrm{sl}_2}| = p^2 \neq |\,\mathrm{sl}_2\,|$.

Examples

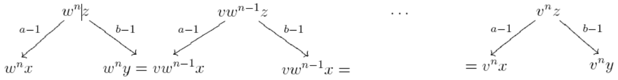
1 G the Klein four group, $\mathrm{char}(k) = 2$. $S = k[v, w, x, y, z]/(vx + wy)$, all generators in degree 1.

v, w, x, y fixed, $a(z) = z + x$, $b(z) = z + v$.

Set $y_1 = v$, $y_2 = w$, $z_1 = w + y$ and $z_2 = d_z$. Let T be the free $k[v, x]$-submodule of S spanned by $\{1, z, z^2, z^3, y, yz, yz^2, yz^3\}$ and $U = k[z_1, z_2]$. Then $S \cong T \otimes_k U$, verifying the hypotheses of the theorem for G.

Since the other subgroups are cyclic, so certainly of finite decomposition type, this shows that S is of finite decomposition type.

v-degree+w-degree. Let A_n be the part with total degree $n + 1$ and v-degree+w-degree=n. $\dim A_n = 2n + 3$.



This is known to be indecomposable (Conlon), so S is not of finite decomposition type.

$U_3(\mathbb{F}_3)$

degree	dimensions of indecomposable summands										
0	1										
1	3										
2	6										
3	10										
4	15										
5	21										
6	$3'$	9	16								
7	9	$9'$	18								
8	9	18	$18'$								
9	1	$3'$	9	18	24						
10	3	9	$9'$	18	27						
11	6	9	18	$18'$	27						
12	$3'$	9	10	18	24	27					
13	9	$9'$	15	18	27	27					
14	9	18	$18'$	21	27	27					
15	$3'$	$3'$	9	9	16	18	24	27	27		
16	9	9	$9'$	$9'$	18	18	27	27	27		
17	9	9	18	18	$18'$	$18'$	27	27	27		
18	1	$3'$	$3'$	9	9	18	18	24	24	27	27	27
19	3	9	9	$9'$	$9'$	18	18	27	27	27	27	27
20	6	9	9	18	18	$18'$	$18'$	27	27	27	27	27

$U_3(\mathbb{F}_3)$

degree | dimensions of indecomposable summands