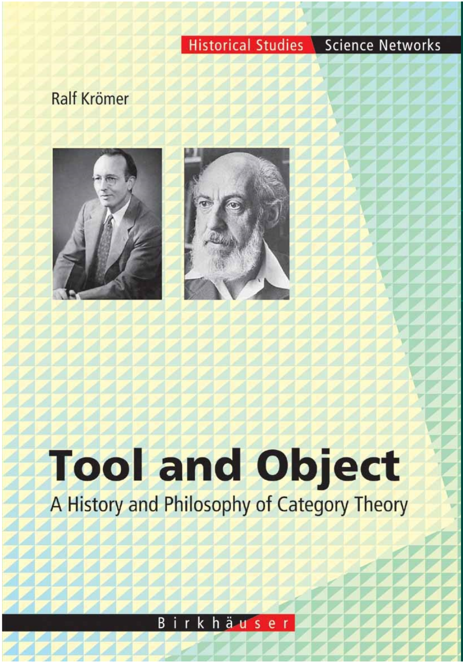
Abelian categories from Mac Lane to Gabriel

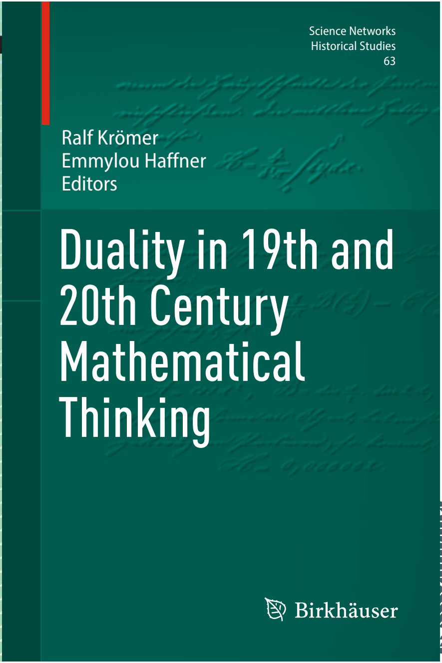
A historical approach

Ralf Krömer

Bergische Universität Wuppertal

August 28, 2025


Historical Studies Science Networks
Ralf Krömer
Tool and Object
A History and Philosophy of Category Theory
Birkhäuser


Science Networks
Historical Studies
63
Ralf Krömer
Emmylou Haffner
Editors
Duality in 19th and 20th Century Mathematical Thinking
Birkhäuser

- Eilenberg and Steenrod 1952: axiomatization of (co-)homology theories
- Mac Lane wants to express the latent duality in constructions for Abelian groups
- Cartan and Eilenberg 1956, homological algebra: construction of derived functors for modules with the help of projective and injective resolutions
- Buchsbaum again wants to express the latent duality in this theory. But dual categories of module categories aren't module categories again, therefore he introduces an abstract notion of exact category
- Grothendieck 1957, building on Serre: transfer the Cartan-Eilenberg theory to sheaves by proving that sheaves form an Abelian category having enough injectives.
- Gabriel builded on this

Some of Gabriel's purely category-theoretic innovations

- first clear definition of equivalence of categories
- first use of Kan's adjoints outside the context of simplicial sets, including their relation to representability of functors and to equivalence
- "first published document in which one finds the expression 'duality' expressing an equivalence between a category C and the dual of a category D" (p.395, see Marquis, An Historical Perspective on Duality and Category Theory: Hom is where the heart is, in Krömer and Haffner 2024, pp.759-862)
- first publication using Grothendieck universes, in particular in the construction of functor categories

Peter Gabriel: some biographical elements

- born 1933 Bitche, died 2015 St. Gallen
- first Ph.D. student of Serre (among 5 in MGP), thèse 1961
- MR Review 38 # 1144 (1969) by Tsit Yuen Lam (Ph.D. student of Hyman Bass, Columbia), reprinted in Zbl

  "In this definitive work on category theory, the author discusses abelian categories in the framework of a Grothendieck universe."

  "This rather belated review is still not a complete glossary of the many important results in the author's original treatise."

- Gabriel-Ulmer: Marquis discusses an "unpublished manuscript by Gabriel also dated from 1966, called 'Rétracts et catégories algébriques', which led to Gabriel and Ulmer" (1971)

Mac Lane 1950

- Mac Lane (1950) uses the terminology "abelian category" for a concept related but not completely equivalent to the concept now commonly bearing this name.
- Mac Lane, in his various historical accounts of his own work in category theory, repeatedly discusses his tentative but unsuccessful definition and its context; speaks about a *"clumsy prelude to the development of Abelian categories"*
- this first attempt at definition was not motivated by the task to transfer the derivation of functors to new contexts, but by considerations of duality

Buchsbaum 1955

- Aim: duality, but dual categories "not concretely defined". Therefore axiomatic approach
- the procedure of derivation can already be realized for objects of a type of categories that share some important properties with a category of modules.
- He does not yet call these categories "abelian", as later suggested by Grothendieck, but "exact"; however, this concept is more or less equivalent to Grothendieck's concept of abelian category.
- Just like category theory (or, more precisely, the language of categories) was conceived at first as a general linguistic framework for expressing facts about the commutativity of diagrams, the theory of exact (or, respectively, abelian) categories is such a framework for the expression of facts about the exactness of sequences.

Terminology

- Buchsbaum's exact cat. $\neq$ Mac Lane's abelian cat.
- thus, it is not astonishing that Buchsbaum who explicitly refers to Mac Lane's work develops a new terminology for his own concept—precisely to distinguish it from Mac Lane's.
- Grothendieck, on the contrary, developed his theory at first without knowledge of the work by Mac Lane and Buchsbaum. In fact, publication of the Tohoku paper even was delayed, among other reasons, because Eilenberg complained about lacking credit to Buchsbaum
- so it is again not astonishing that Grothendieck chose a terminology already used differently; originally, he spoke even about *"classes abéliennes"*, inspired by Serre (1953)
- Gabriel (1962) mentions Buchsbaum in the introduction, but not in the bibliography. But: "nous utilisons une méthode dont Buchsbaum se sert dans la construction des foncteurs satellites." (p.349)

"An equivalence *of a category* $\mathbf{C}$ *and a category* $\mathbf{C}'$ *is a system* (F, G, ϕ, ψ) *of covariant functors*

$$F : \mathbf{C} \to \mathbf{C}' \qquad G : \mathbf{C}' \to \mathbf{C}$$

and functorial homomorphisms

$$\phi : 1_{\mathbf{C}} \to GF \qquad \psi : 1_{\mathbf{C}'} \to FG$$

[. . .] such that for every $A \in \mathbf{C}, A' \in \mathbf{C}'$ *the compositions*

$$F(A) \xrightarrow{F(\phi(A))} FGF(A) \xrightarrow{\psi^{-1}(F(A))} F(A)$$
$$G(A') \xrightarrow{G(\psi(A'))} GFG(A') \xrightarrow{\phi^{-1}(G(A'))} G(A')$$

are the identity on $F(A)$ *resp.* $G(A')$*. [. . .] Two categories are called* equivalent *if there exists an equivalence between these categories." (Grothendieck, 1957, 125).*

- This presentation was not entirely satisfactory. In Grothendieck's own words (SGA 1, *exposé* VI p.3) *"la notion d'équivalence de catégories [. . .] n'est pas exposée de façon satisfaisante dans [Grothendieck (1957)]"*.
- First of all, ϕ and ψ should be isomorphisms and not simply functorial morphisms. (But he treats them as if they were invertible, and he later speaks explicitly about the *"isomorphismes"* $\phi(A)$ and $\phi(B)$ induced by ϕ).
- In principle, the equations concerning ϕ and ψ (written as sequences) are not needed in the definition, since they are automatically valid for *iso*morphisms ϕ and ψ (hence a *corollary* of the now usual definition of equivalence which would be complete just before the *"such that"*).
- Incidentally, Grothendieck's equivalence is given as a special case of what will be called adjunction later on; his ψ^{-1} and ϕ will later be called "units of adjunction" (which however do not need to be isomorphisms in the general case).
- The now usual definition, and the fact that an equivalence is in particular an adjunction, are contained in Gabriel (1962, 341).

Adjoints and representable functors

- the French community when starting to use the concept of adjoint functor relies on Kan.
- Gabriel (1962) cites Shih (1959) who, in a paper on simplicial sets, in turn cites Kan (1958) and Cartan (1958).
- Since Shih's contribution is a talk in the Cartan seminar, it is to be supposed that he was one of Cartan's students then. Cartan certainly knew about Kan's work through Eilenberg (much like in the case of Buchsbaum); this shows how important the Cartan–Eilenberg connection was for the development of category theory, since Gabriel, whose primary concern was not in simplicial sets, might very well have missed Kan's paper otherwise.
- Gabriel (1962, 332), when introducing the concept of representable functor, mentions that it goes back to Grothendieck (1960). He is the first to clarify the relation of the two concepts (p.340)

- The notion of Grothendieck universe first appeared in internal Bourbaki papers, namely in the rédaction *n°307* written by Grothendieck after July 58 and before March 59.
- More generally, there was an internal debate in the group on whether and how to include category theory in the *Eléments* (see Krömer (2006)). But this made it necessary to settle foundational issues. In this sense, this debate motivated the introduction of universes
- The first published definition is in Gabriel (1962).
- Later occurrences include SGA 1 and SGA 4 (*exposé* I 1-4 by Grothendieck and Verdier and *exposé* I 185-217 by Bourbaki).
- Add an axiom to ZFC asserting that every set is contained in a universe; this amounts to the postulation of an infinite sequence of universes (or equivalently of an infinite sequence of strongly inaccessible cardinals). One may now construe U-categories (which means, for example, the category U-$\mathbf{Grp}$ of all groups in U rather than $\mathbf{Grp}$) and the functor categories between them; this settles most of the foundational problems

Gabriel's use of universes

- Gabriel is the first to speak of "Grothendieck universes".
- *"We choose once and for all a universe $\mathfrak{U}$ which will never 'vary' in what follows"* (p.328).
- But this does not mean that one universe is enough; it simply means that all constructions are relative to $\mathfrak{U}$, and one could re-read the whole paper by replacing $\mathfrak{U}$ by another universe
- The choice of a universe does not stop some constructions from transcending the universe once chosen (otherwise, the whole talk about universes would be superfluous, after all).

"From now on, we say that a category **C** *is a* $\mathfrak{U}$-category *if* $\mathrm{Hom}_{\mathbf{C}}(M, N)$ *is an element of the universe* $\mathfrak{U}$ *for every couple* (M, N) *of objects of* **C**. *If not explicitly stated otherwise,* all categories considered in this paper are $\mathfrak{U}$-categories" *[p.330].*

Things become more complicated when he gives equivalent conditions for the existence of an equivalence between given categories **A** and **B**:

> *"If $\mathfrak{B}$ is a universe which has the universe $\mathfrak{U}$ as an element, we can construct a new category* **E***: the objects of* **E** *are the categories whose set of morphisms is element of $\mathfrak{B}$ (one identifies the objects with the identity morphisms); if* **A** *and* **B** *are two objects of* **E**, $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ *is the set of isomorphy classes of functors from* **A** *to* **B***, composition being carried out in the obvious manner. One observes that* **E** *is not a $\mathfrak{U}$-category. Assertion (c) [of proposition 12] affirms that the class of functors isomorphic to T [the functor from* **A** *to* **B** *establishing an equivalence of these categories according to this proposition] is an isomorphism of the category* **E***" [p.342].*

Hence, the category of categories **E** is constructed in a way that equivalent categories are isomorphic in the sense that in **E** there is an isomorphic arrow between them.

- On p.345f, he introduces a certain functor category **Funct(C,D)** where **C** is a category whose class of objects and whose class of arrows are *elements* of $\mathfrak{U}$; this is something smaller than a $\mathfrak{U}$-category in Gabriel's terminology, and he has to make this restriction if he wants **Funct(C,D)** to be a $\mathfrak{U}$-category.
- For an arrow of **Funct(C,D)** is a natural transformation between two functors, which means a class of arrows, one for every object of **C**, and the collection of all arrows for a pair of objects of **Funct(C,D)** is the collection of all these classes, hence belongs to $\mathfrak{U}$ only for categories **C** with the above property (since $\mathfrak{U}$ is closed under infinite union only for index sets in $\mathfrak{U}$).
- But he really wants to work with this **Funct(C,D)**!

Sehr geehrter Herr Gabriel,

in der Springer-Fassung von SGA 1 (1970) liest man zu einer geplanten Einführung in die Kategorientheorie auf S.146 die recht sarkastische Bemerkung ≪ Les auteurs définitifs sont C. Chevalley et P. Gabriel. Le livre doit sortir en l'an 2000 ≫.

Können Sie irgendetwas dazu sagen, was aus dem Projekt geworden ist? Ist ein Manuskript entstanden und gibt es dieses vielleicht sogar noch? [. . .]

Herzliche Grüße

Ralf Krömer

Lieber Herr Krömer,

schön zu erfahren, dass SGA1 etwa so zuverlässig ist wie Wikipedia. Ich weiß nicht, wer die Bemerkung schrieb, weiß nur, dass Grothendieck sich in den Sechzigern gerne mit Planung der Mathematik beschäftigte. Ich besitze noch einen Brief von ihm, in dem er mich zum Nachfolger von Dieudonné ernennt. Später sollte ich Chevalley bei der Verfassung eines Buches über Kategorien behilflich sein, weil die Bourbakisten anscheinend gerne Chevalleys Manuskripte fürchteten. Also war ich zweimal zu gemeinsamen Beratungen in Chevalleys Wohnung und beschloss alsdann, Chevalleys Pläne seien nichts für mich, genau so wenig wie diejenigen von "Sacha", den manche gern als meinen Doktorvater betrachten, obschon er vor der Publikation meiner Arbeit nicht die geringste Ahnung davon hatte. So laufen die Gerüchte.

Über Chevalleys Pläne könnte Jean Benabou sie vermutlich informieren.

Mit freundlichen Grüßen, Peter Gabriel

Henri Cartan. Sur la théorie de Kan. In *Séminaire Henri Cartan; 9e année: 1956/57. Quelques questions de topologie, Exposé no. 1*, page 19. Secrétariat mathématique, Paris, 1958.

Pierre Gabriel. Des catégories abéliennes. *Bull. Soc. Math. France*, 90:323–448, 1962.

Alexander Grothendieck. Sur quelques points d'algèbre homologique. *Tôhoku Math. J.*, 9:119–221, 1957.

Alexander Grothendieck. Technique de descente et theoremes d'existence en geometrie algebrique. II: Le theoreme d'existence en theorie formelle des modules. In *1959/60*, volume 12 of *Sem. Bourbaki*, pages No.195, 22 p. 1960.

Daniel M. Kan. Adjoint functors. *Trans. Amer. Math. Soc.*, 87: 294–329, 1958. MR24 #A1301.

Ralf Krömer. La 'machine de Grothendieck', se fonde-t-elle seulement sur des vocables métamathématiques? Bourbaki et les catégories au cours des années cinquante. *Revue d'Histoire des Mathématiques*, 12:111–154, 2006.

Saunders Mac Lane. Duality for groups. *Bull. Amer. Math. Soc.*, 56:485–516, 1950.

Jean-Pierre Serre. Groupes d'homotopie et classes de groupes abéliens. *Annals Math. (2)*, 58:258–294, 1953. MR15,548c.

Weishu Shih. Ensembles simpliciaux et opérations cohomologiques. *Séminaire Cartan*, 11 (1958/59):n?7, 1959.