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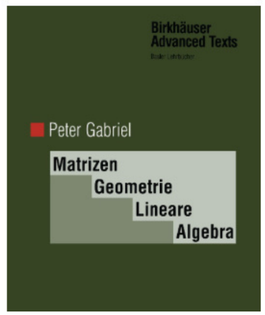
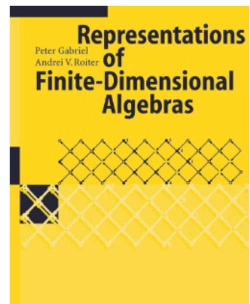
The Legacy of Peter Gabriel - Conference

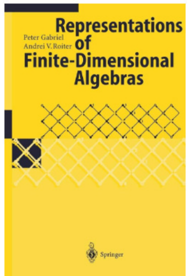




Matrix Problems

Thomas Brüstle

We present matrix problems which are introduced prominently as the opening chapter in the book Representations of Finite-Dimensional Algebras by Gabriel and Roiter. Building on this, we will focus on one-sided matrix problems and present key results by Gabriel and his collaborators concerning the classification into finite, tame, and wild types.



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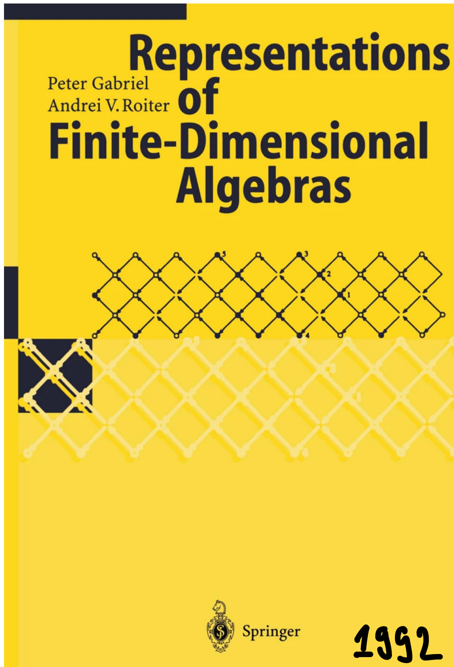


1. Matrix Problems

In the present section, k denotes an algebraically closed field.

1.1. Since (finite) matrices describe linear maps between the spaces k^n, $n \in \mathbb{N}$, we admit matrices with no row or no column.[1] By $k^{m \times n}$ we denote the space of

1.10. Remarks and References

1. In our terminology, a matrix A is defined by two numbers $m, n \in \mathbb{N}$ *and* a family of entries A_{ij}, $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.

0×1 - matrix : $\vdash\!\dashv$

1×0 - matrix : I

$GL_m \times GL_n$ acts on $K^{m \times n}$ (space of $m \times n$ - matrices)

$$\begin{matrix} GL_m \times GL_n & & K^{m\times n} & & \\ \Psi & & \Psi & & \\ (X, Y) & & A & \longmapsto & XAY^{-1} \underset{\text{same orbit}}{\sim} A \end{matrix}$$

$$A \sim \begin{array}{cc} \begin{bmatrix} \boxed{1_r} & 0 \\ 0 & 0 \end{bmatrix} & \begin{matrix} \\ t \end{matrix} \\ \quad\quad\quad s & \end{array} = \boxed{1}^{\,r} \oplus (\vdash\!\dashv)^{s} \oplus \mathrm{I}^{t}$$

$r = \operatorname{rk} A$ $\quad$ $s = n - r, \quad t = m - r$

$\ell = m+n \; : \; G \leq GL_\ell$

1.4 Def: Matrix problem of size $m \times n$ is

$(G, \mathcal{M})$ where $\underset{\text{subgroup}}{G \leq} GL_m \times GL_n$, $\mathcal{M} \underset{\text{subset}}{\subseteq} K^{m \times n}$

s.t. $XAY^{-1} \in \mathcal{M} \quad \forall (X,Y) \in G, \; A \in \mathcal{M}$

Problem: classify orbits!

Often: $G = \operatorname{Aut} R$, $R \underset{\text{subalgebra}}{\leq} K^{m \times m} \times K^{n \times n}$ (linear matrix problem)

$\underset{\text{affine subspace}}{\mathcal{M} = v + U}$ for $v \in K^{m \times n}$, $U \underset{\text{lin. subspace}}{\leq} K^{m \times n}$

Example 1: $\Lambda = KQ/I$ f.d. algebra, $M \in \mathrm{mod}\,\Lambda$

[Drozd '79]

$$P_1 \xrightarrow{A} P_0 \xrightarrow{f_0} M \longrightarrow 0 \quad \text{projective resolution}$$

$$A \in \mathrm{Hom}_\Lambda(P_1, P_0) =: \mathcal{M} \subseteq \mathrm{Hom}_K(P_1, P_0)$$

$\circlearrowleft$ action

linear subspace

$$\mathrm{Aut}_\Lambda(P_1) \times \mathrm{Aut}_\Lambda(P_0) =: G \quad \text{not reductive}$$

G-orbits on $\mathcal{M}$ = isoclasses of Λ-modules $\left[M = \mathrm{coker}(P_1 \xrightarrow{A} P_0) \right]$

$\underline{\dim}\, M = (d_1, \dots, d_t)$: $M \in \prod_{i \to j} K^{d_j \times d_i} / \langle I \rangle \subseteq \prod K^{d_j \times d_i}$

quiver representation

$\circlearrowleft$ action

algebraic var.

reductive group $\prod_{1 \le i \le t} GL_{d_i} = G'$ orbits = isoclasses of Λ-modules

$\leadsto$ use GIT

Example 2: category $\mathrm{Mat}\, B$

$\mathcal{C}$ k-category

$B : \mathcal{C} \times \mathcal{C} \longrightarrow \mathrm{mod}\, k$ k-linear bifunctor

$$\mathrm{Obj}\ \mathrm{Mat}\, B = \{ (C, m) \mid C \in \mathcal{C},\ m \in B(C,C) \}$$

$$\begin{array}{l} (C, m) \\ \Big\downarrow \gamma : C \to C' \text{ s.t. } \gamma m = m' \gamma \text{ in } B(C, C') \\ (C', m') \end{array}$$

$(C,m) \cong (C, m') \iff m' = \gamma m \gamma^{-1}$ some $\gamma \in \mathrm{Aut}(\mathcal{C}(C,C)) =: G$ orbits = isoclasses

Exact structure on $\mathrm{Mat}\, B$:

$$\{ 0 \to (C', m') \to (C, m) \to (C'', m'') \to 0 \mid 0 \to C' \to C \to C'' \to 0 \text{ splits} \}$$

Example: $B = \mathrm{Hom}_\Lambda(-,-)$ for $\mathcal{C} = \mathrm{proj}\, \Lambda$ $\to$ (Ex. 1)

Example 3:

$P =$ poset $1 \to 2$ $\qquad \mathcal{M} = K^{m \times n}$ $\qquad n = n_1 + n_2$

$$\operatorname{rep} P = \left\{ \begin{array}{c|cc} & \bullet \dashrightarrow & \bullet \\ m & A_1 & A_2 \\ & n_1 & n_2 \end{array} \right\} \qquad G = GL_m \times \left\{ \begin{array}{cc|c} n_1 & n_2 & \\ \hline Y_1 & Z & n_1 \\ 0 & Y_2 & n_2 \end{array} \;\; Y_1, Y_2 \text{ invertible} \right\}$$

any poset $\rightsquigarrow$ one-sided matrix problem

Matrix Reduction:

$$\begin{array}{c|c|c|} & \bullet \dashrightarrow & \bullet \\ m & A_1 & A_2 \\ & n_1 & n_2 \end{array} \sim \begin{array}{|cc|c|} \multicolumn{2}{c}{\overset{1}{\bullet} \dashrightarrow} & \overset{2}{\bullet} \\ \mathbb{1} & 0 & 0 \\ \hline 0 & 0 & B_2 \\ \multicolumn{2}{c}{n_1} & n_2 \end{array} \sim \begin{array}{|cc|cc|} \multicolumn{2}{c}{\overset{1}{\bullet} \dashrightarrow} & \multicolumn{2}{c}{\overset{2}{\bullet}} \\ \mathbb{1} & 0 & \multicolumn{2}{c|}{0} \\ \hline 0 & 0 & \begin{smallmatrix}\mathbb{1} & 0 \\ 0 & 0\end{smallmatrix} & \\ \multicolumn{2}{c}{n_1} & \multicolumn{2}{c}{n_2} \end{array} = \boxed{\overset{1}{1}}^{\,r_1} \oplus \boxed{\overset{2}{1}}^{\,r_2} \oplus I^{s} \oplus \overset{1}{\vdash\!\dashv}{}^{t_1} \oplus \overset{2}{\vdash\!\dashv}{}^{t_2}$$

Reduced Matrix Problem:

$$\mathcal{M} = \mathcal{U} + \mathcal{V}, \quad \mathcal{U} = \begin{array}{|cc|c|} \multicolumn{2}{c}{\overset{1}{\bullet} \dashrightarrow} & \overset{2}{\bullet} \\ \mathbb{1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \multicolumn{2}{c}{n_1} & n_2 \end{array}, \quad \mathcal{V} = \left\{ \begin{array}{|cc|c|} \multicolumn{2}{c}{\overset{1}{\bullet} \dashrightarrow} & \overset{2}{\bullet} \\ 0 & 0 & 0 \\ \hline 0 & 0 & B_2 \\ \multicolumn{2}{c}{n_1} & n_2 \end{array} \right\}$$

Example 3: $P =$ poset $1 \xrightarrow{\alpha} 2$

$\operatorname{rep} P = \operatorname{Mat} \mathcal{B}$, $\mathcal{B}$ bimodule over $KP \times K$

$\mathcal{B}(e_1, e_3) = a_1$

$\mathcal{B}(e_2, e_3) = a_2$ $\qquad a_1 \alpha = a_2$

Matrix Reduction:

$$\begin{array}{cccc} & 1 & \dashrightarrow & 2 \\ m & \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right] \\ & u_1 & & u_2 \end{array} \sim \begin{array}{ccc} 1 & \dashrightarrow & 2 \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & B_2 \end{array}\right] \\ u_1 & & u_2 \end{array}$$

Reduced Matrix Problem:

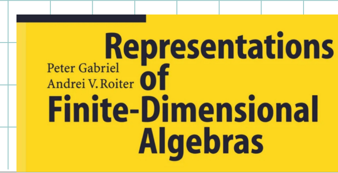
$$\mathcal{M}' = u + V, \quad u = \begin{array}{ccc} 1 & \dashrightarrow & 2 \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \\ u_1 & & u_2 \end{array}, \quad V = \left\{ \begin{array}{ccc} 1 & \dashrightarrow & 2 \\ \left[\begin{array}{cc|c} 0 & 0 & 0 \\ \hline 0 & 0 & B_2 \end{array}\right] \\ u_1 & & u_2 \end{array} \right\}$$

$$G' = \operatorname{Stab} u \subseteq G$$

CompSci: Reduction Algorithm (store u, compute $\operatorname{Stab} u$, continue)

Math (category): need more input than $\operatorname{Mat} \mathcal{B}$

$\rightsquigarrow$ BOCS (cf.



)

1.2 Example 4:

Q = quiver $1 \to 3 \leftarrow 2$

$$\mathcal{M} = K^{m \times n} \qquad n = n_1 + n_2$$

$$\operatorname{rep} Q = \left\{ \begin{array}{cccc} & 1 & 2 & \\ m & \begin{array}{|c|c|} \hline A_1 & A_2 \\ \hline \end{array} & & 3 \\ & n_1 & n_2 & \end{array} \right\}$$

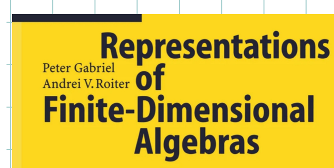
$$G = GL_m \times \left\{ \begin{array}{ccc} n_1 & n_2 & \\ \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix} & & \begin{array}{c} n_1 \\ n_2 \end{array} \end{array} \quad Y_1, Y_2 \text{ invertible} \right\}$$

one-sided matrix problem

Reduction:

$$m \begin{array}{|c|c|} \hline A_1 & A_2 \\ \hline \end{array}_{\,n_1 \quad n_2} \sim \qquad \bar{A} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Fig. 3



$\bar{A} \in K^{4 \times 6} \implies$ $\operatorname{Stab} \bar{A}$ = Auslander algebra of KQ

1.8 Example 5: [L. Kronecker, 1890]

Q = quiver $1 \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} 2$

$$\mathcal{M} = K^{m\times 2n}$$

$$\operatorname{rep} Q = \left\{ \begin{array}{c|cc} & & \\ m & A_1 & A_2 \\ & n & n \end{array} \right\}$$

$$G = GL_m \times \left\{ \begin{array}{cc|c} n & n & \\ \hline Y & 0 & n \\ 0 & Y & n \end{array} \quad Y \text{ invertible} \right\}$$

Reduction:

$$m\ \boxed{A_1}\ \boxed{A_2} \quad (n,\ n) \quad \sim$$

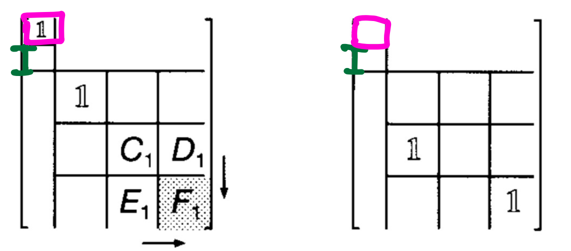


Fig. 13

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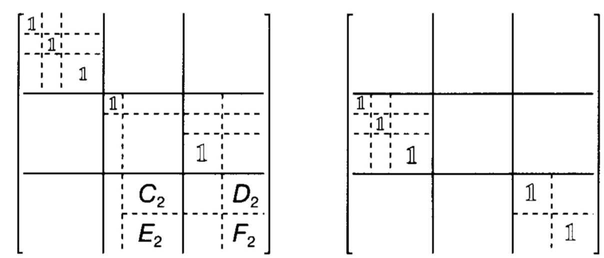
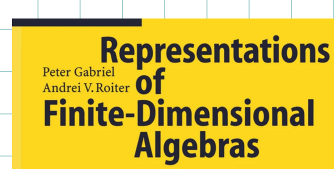


Fig. 14

$\cdots$



1.8 Example 5: [L. Kronecker, 1890] Q = quiver $1 \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} 2$

(generic) Reduction Algorithm never stops,

splitting off summands :

$K \overset{1}{\underset{0}{\rightrightarrows}} K$, $0 \overset{I}{\underset{I}{\rightrightarrows}} K$, $K \overset{\begin{bmatrix}1\\0\end{bmatrix}}{\underset{\begin{bmatrix}0\\1\end{bmatrix}}{\rightrightarrows}} K^2$, $K^2 \overset{\begin{bmatrix}1&0\\0&1\end{bmatrix}}{\underset{\begin{bmatrix}0&0\\1&0\end{bmatrix}}{\rightrightarrows}} K^2$, ...

(preprojective and some regular indecomposables)

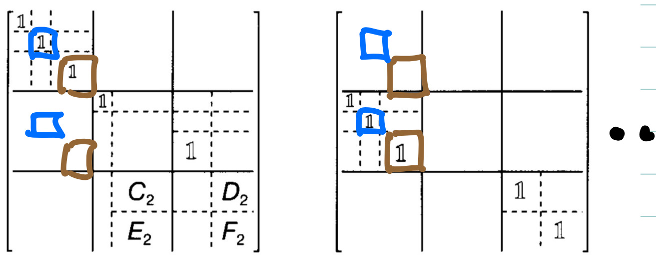


Fig. 14

generic reduction algorithm:

keep matrix sizes variable, work within category Mat $\mathcal{B}$

If Mat $\mathcal{B}$ is of finite type, algorithm stops with output:

- all indecomposables (given by 0,1-matrices)
- $\mathrm{End}\left(\bigoplus_{U \text{ indec}} U\right)$

If Mat $\mathcal{B}$ is of infinite type, algorithm never stops!

Apply <u>Bongartz' criterion</u> when Mat $\mathcal{B} \cong \mathrm{mod}\, KQ/\underline{I}$:

Stop when pointwise dimension of an indecomposable is > 12

$\boxed{1.8}$ Example 5: [L. Kronecker, 1890] $Q =$ quiver $1 \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} 2$

Reduction Algorithm applied to one $A \in \mathcal{M}$ always stops, eventually leads to:

$$m\underbrace{\boxed{A_1}}_{n}\ \underbrace{\boxed{A_2}}_{n} \sim \underbrace{\boxed{\mathbb{1}_n}}_{n}\ \underbrace{\boxed{A_2'}}_{n} \quad \text{with } \operatorname{Stab}\boxed{\mathbb{1}_n} = \left\{ [x], \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \right\},$$

so the reduced problem on A_2' is $A_2' \sim X A_2' X^{-1}$ matrix conjugation

$\rightsquigarrow$ ∞ family of indecomposables

$\boxed{/\!/\!/}$ is a minimal ∞ box, reduce it to $\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$

A matrix problem is *tame* if after appearance of the first minimal ∞ box, everything reduces to $0,1$-matrices, for all $\lambda \in K \setminus \{\lambda_1, \dots, \lambda_r\}$. "punched line"

$\{\text{orbits}\} = \bigcup$ discrete $0,1$-matrices $\bigcup$ punched lines.

Theorem [Drozd '79; Crawley-Boevey '88]

Every finite-dimensional algebra is either tame or wild

Proof uses bocses.

"if a second ∞ box appears in the reduction, have to show that leads to a functor $\mathrm{mod}\, k\langle s,t\rangle \longrightarrow \mathrm{Mat}\,\mathcal{B}$ preserving indecomposables"

This is highly non-trivial!

Example: Matrices $A_{s,t} = \boxed{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & s & t \end{matrix}}$ acted upon by $\mathrm{Aut}\, R$ where $s,t \in k$

R generated by $\mathbb{1}$, $\begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & & 0 & \\ & & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & & & \\ & 0 & & \\ 1 & & 0 & \\ & 1 & & 0 \end{bmatrix}$

Then $A_{s,t} \sim A_{s',t'} \iff s = s'$ and $t = t'$,

but replacing scalars s,t by matrices S,T fails to produce the desired functor $\mathrm{mod}\, k\langle S,T\rangle \to \mathrm{Mat}\,\mathcal{B}$

Theorem [B, Sergeichuk '02]

If $kQ/_I$ is a tame fin.dim'l algebra

($\forall d \geq 0 \ \exists f(d) \in \mathbb{N}$:
all indecomposable Λ-modules of dimension d appear in at most $f(d)$ 0 or 1-parameter families)

Then $f(d) \leq \binom{d+r}{r} 4^{d^2 \sum \delta_i^2}$

where $r = \#$ vertices of Q, $\delta_i = \dim P(i)$

Proof is based on:

[Gabriel, Nazarova, Roiter, Sergeichuk, Vossieck '93]

4.1 Def: Subspace Problem

$M : \mathcal{A} \to \operatorname{mod} k$ functor (pfd module)

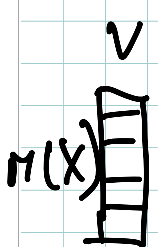
$\mathcal{A}$ Krull-Schmidt (aggregate)

$$\operatorname{Sub} M = \{ (V, f, X) \mid V \in \operatorname{mod} k,\ X \in \mathcal{A},\ \underset{k\text{-linear}}{f : V \to M(X)} \}$$

$$\begin{array}{c} (V, f, X) \\ \varphi \downarrow \quad \downarrow \psi \\ (V', f', X') \end{array} \quad \text{s.t. } f' \circ \varphi = M(\psi) \circ f$$

$\operatorname{Sub} M = \operatorname{Mat} B$ where $B : \operatorname{mod} k \times \mathcal{A} \to \operatorname{mod} k$

$$(V, X) \longmapsto \operatorname{Hom}_k(V, M(X))$$



one-sided matrix problem

One-point extension: $\Lambda = \begin{bmatrix}\Lambda_0 & R \\ 0 & K\end{bmatrix}$

$$\operatorname{mod}\Lambda \cong \left\{ (V_*, f, X_0) \;\middle|\; V_* \in \operatorname{mod} K,\ X_0 \in \operatorname{mod}\Lambda_0,\ f : V_* \to \operatorname{Hom}_{\Lambda_0}(R, X_0) \right\}$$

$= \operatorname{Sub} M$, $M = \operatorname{Hom}_{\Lambda_0}(R, -)$

Construct Λ-modules inductively from subspace categories over smaller algebra Λ_0

4.3 $\forall \Lambda$ fin.dim'l, $\exists$ functor $\operatorname{mod}\Lambda \to \operatorname{sub} M$ inducing bijection on iso classes,

where $M = \operatorname{Ext}^1_\Lambda(S, -) : \operatorname{mod}\Lambda_0 \to \operatorname{mod} K$,

$\Lambda_0 = \Lambda / J$, S some simple Λ-module.

4.4 Reduction Algorithm for Subspace Problems:

Given submodule $N \leq M : \mathcal{A} \to \operatorname{mod} K$,

define $\hat{\mathcal{A}} = \operatorname{sub} N \supseteq \mathcal{A} = \{(0,0,X) \mid X \in A\}$

$$\hat{M} : \hat{\mathcal{A}} \to \operatorname{mod} K, \quad (V,f,X) \mapsto M(X)/f(V)$$

Proposition: The reduction functor

$$F : \operatorname{sub} M \to \operatorname{sub} \hat{M}_{\hat{N}} \quad \leftarrow \text{assume } f''(N(X)) = 0$$

$$(V,f,X) \mapsto (V/V', f'', X) \text{ with } V' = f^{-1}(N(X))$$

induces bijection on iso classes

4.4 Reduction Algorithm for Subspace Problems:

Example:

$\boxed{B_1 \mid B_2}$ row-linear independent

$$A = \left[\, A_1 \mid A_2 \mid \overbrace{A_3 \mid A_4}^{N} \,\right] \sim$$

sub M

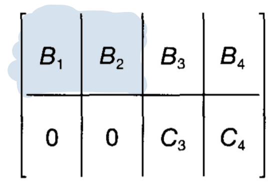


Fig. 1

$\sim$

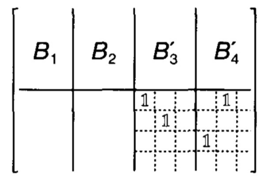
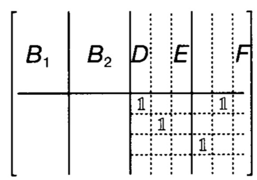


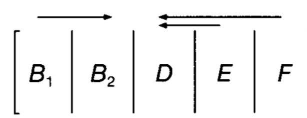
Fig. 2

$\sim$

4. Finitely Spaced Modules

$\sim$





sub $\hat{M}$, $\hat{N}$

Sub N "known" by induction,

reduction $\rightsquigarrow$ new subspace problem

(stay within same class, no need for boxes)

4.5

Theorem. *Let M be a pointwise finite left module over an aggregate $\mathscr{A}$ and $b \in \mathbb{N}$ a number such that, for each $X \in \mathscr{A}$ satisfying* $\dim M(X) \geqslant b$, *there are only finitely many isoclasses of indecomposable M-spaces avoiding $\mathscr{L}$ of the form (V, f, X). Then the same conclusion holds if* $\dim M(X) < b$.

Corollary.[5] *Let A be a finite-dimensional algebra such that the dimensions of the indecomposable A-modules are bounded. Then A admits only finitely many isoclasses of indecomposable modules.*

(Brauer – Thrall I)

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TAME AND WILD SUBSPACE PROBLEMS

РУЧНІ ТА ДИКІ ЗАДАЧІ ПРО ПІДПРОСТОРИ

Second main theorem. *If M is not $\mathcal{L}$-wild, there is a locally finite set $\mathcal{P}$ of $\mathcal{L}$-reliable punched lines such that:*

a) *for each $X \in \mathcal{A}$, the set of isoclasses of indecomposable M-spaces (V, f, X) which avoid $\mathcal{L}$ and are not produced by a punched line of $\mathcal{P}$ is finite;*

b) *distinct punched lines of $\mathcal{P}$ produce non-isomorphic M-spaces.*

(tame subspace problem)

Third main theorem. *If B is a finite-dimensional k-algebra, one and only one of the following two statements holds:*

a) *B is wild, i. e. there exists a B-reliable plane;*

b) *There exists a family of B-reliable punched lines $S_i \setminus E_i \subset \mathrm{Hom}_k(V_i \otimes_k B, V_i)$, $i \in I$, with the following properties: For each $d \in \mathbb{N}$, the number of $i \in I$ satisfying $d = \dim V_i$ is finite, and almost all isoclasses of indecomposable B-modules of dimension d consist of modules produced by the $S_i \setminus E_i$; furthermore, if $i \neq j$, no indecomposable produced by $S_i \setminus E_i$ can be produced by $S_j \setminus E_j$.*