UNIVERSITÄT PADERBORN
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Poisson Transforms and Homological Algebra

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Classical Poisson integrals

- **Complex analysis**: $B_R(0) \subseteq \mathbb{C}$ open disk, $f : \overline{B_R(0)} \to \mathbb{C}$ continuous and harmonic on $B_R(0)$. Then

$$f(z) = \int_0^{2\pi} h(\zeta) P_R(z,\zeta)\, \mathrm{d}\vartheta$$

for $z = re^{it}, \zeta = Re^{i\vartheta}$, $h = f\big|_{\partial B_R(0)}$, and
$P_R(z,\zeta) = \frac{1}{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(\vartheta-t)+r^2}$ (Poisson kernel).

- **Boundary value problems**: $\Omega \subseteq \mathbb{R}^n$ bounded domain with smooth boundary $\partial\Omega$. Then the PDE

$$\Delta w = 0 \text{ on } \Omega, \quad w = g \quad \text{ on } \partial\Omega$$

is solvable via a Green's function $G : \Omega \times \overline{\Omega} \to \mathbb{C}$ via

$$w(x) = \int_{\partial\Omega} g(y) \partial_{\nu_y} G(x,y)\, \mathrm{d}\sigma(y).$$

Abstract Poisson transforms

X, B manifolds (e.g. $B \subset \partial X$ in some compactification of X),
$P : X \times B \to \mathbb{C}$ (abstract Poisson kernel),
$\mathcal{F}(X), \mathcal{G}(B)$ spaces of (generalized) functions on X, resp. B,
σ a measure on B.
Poisson transform:

$$\begin{array}{rcl} \mathcal{P} : \mathcal{G}(B) & \longrightarrow & \mathcal{F}(X) \\ f & \mapsto & \int_B f(y) P(\bullet, y) \, \mathrm{d}\sigma(y) \end{array}$$

Question: mapping properties of $\mathcal{P}$?
Generalization: sections of vector bundles instead of scalar valued functions.

Example: Poisson transforms for symmetric spaces

G/K (Riemannian symmetric space of non-compact type)

$K/M = G/P = G/MAN$
(Furstenberg boundary of G/K)

$H : KAN \to \mathfrak{a},\ kan \mapsto \log a$
(Iwasawa projection)

$\langle\ ,\ \rangle : G/K \times K/M \to \mathfrak{a},\ (gK, kM) \mapsto -H(g^{-1}k)$
(horocycle bracket)

$p_\lambda(x, b) := e^{(\lambda+\rho)\langle x,b\rangle}$
(Poisson kernel for $\mu \in \mathfrak{a}^*_{\mathbb{C}}$)

$\mathcal{P}_\lambda : \mathcal{C}^{-\infty}(K/M) \to \mathcal{E}^*_\lambda(G/K) \subset C^\infty(G/K)$
(Poisson transform; values in joint eigenfunctions)

Helgason "conjecture"

- T be a function, distribution, hyperfunction on $B = K/M$
- Poisson transform of T:

$$\mathcal{P}_\lambda(T)(x) := \int_B p_\lambda(x, b)\, T(db)$$

Range: growth conditions depending on the regularity of T.
- For generic λ: $\mathcal{P}_\lambda$ is invertible by taking "boundary values" (Helgason conjecture [H76], proved by Kashiwara et al. [K+78])
- [K+78] provides a complicated construction of a "boundary value map" giving the inverse.

Schmid's Theorem

Fix G semisimple with finite center and a maximal compact subgroup $K \subseteq G$. Write $\mathfrak{g} := \mathrm{Lie}(G)$ for the Lie algebra.

Given a Harish-Chandra module V one can construct a canonical minimal globalization $V_{\min}$ and a canonical maximal globalization $V_{\max}$.

Theorem (Schmid '85)

Let (π, V_π) be a Banach-globalization.

(i) *$V_{\min} \hookrightarrow V_\pi^\omega$ (analytic vectors) is an isomorphism of topological vector spaces.*

(ii) *If V_π is reflexive, $V_{\max}$ is topologically isomorphic to $V_\pi^{-\omega}$ (hyperfunction vectors)*

Globalizations of Harish-Chandra modules

Definition (Harish-Chandra module)

A $(\mathfrak{g}_{\mathbb{C}}, K)$-module V is a Harish-Chandra module if

(a) V is finitely generated over $U(\mathfrak{g})$ (universal enveloping algebra),

(b) V is K-semisimple with finite multiplicities,

(c) the actions of K and $\mathfrak{g}_{\mathbb{C}}$ are compatible.

Definition (Globalization of a Harish-Chandra module)

A represention π of G on a complete locally convex Hausdorff space V_π is called a globalization of the Harish-Chandra module V if

(a) V_π is admissible (K-types occur with finite multiplicity) of finite length,

(b) $V \cong V_\pi^{K-\mathrm{fin}}$ (K-finite vectors).

Analytic and hyperfunction vectors

Definition

Let (π, V_π) be a G-representation.

(i) $v \in V_\pi$ is an analytic vector, if $G \to V_\pi, g \mapsto \pi(g)v$ is analytic.

(ii) The space $V_\pi^\omega \hookrightarrow C^\omega(G, V_\pi)$ of analytic vectors in V_π is equipped with the topology induced from the compact-open topology.

(iii) If V_π is a reflexive Banach space, the space $V_\pi^{-\omega}$ of hyperfunction vectors is the strong dual of the space $(V'_\pi)^\omega$, where V'_π is the topological dual of V_π.

Recall Schmid's Theorem

Fix G semisimple with finite center and a maximal compact subgroup $K \subseteq G$. Write $\mathfrak{g} := \mathrm{Lie}(G)$ for the Lie algebra.

Given a Harish-Chandra module V one can construct a canonical minimal globalization $V_{\min} = \mathrm{mg}(V)$ and a canonical maximal globalization $V_{\max} = \mathrm{MG}(V)$.

Theorem (Schmid '85)

Let (π, V_π) be a Banach-globalization.

(i) *$V_{\min} \hookrightarrow V_\pi^\omega$ (analytic vectors) is an isomorphism of topological vector spaces.*

(ii) *If V_π is reflexive, $V_{\max}$ is topologically isomorphic to $V_\pi^{-\omega}$ (hyperfunction vectors)*

Corollaries to Schmid's Theorem

Corollary

(i) *Any two Banach globalizations of a Harish-Chandra module have topologically isomorphic spaces of analytic vectors.*

(ii) *The functors $V \to V_{\min}$ and $V \to V_{\max}$ are exact in the topological sense.*

Schmid's Theorem reduces the Helgason conjecture to the algebraic problem (solved already by Helgason) of showing that

$$\mathcal{P}_\lambda : \mathcal{C}^\infty(K/M)^{K-\mathrm{fin}} \to \mathcal{E}_\lambda(G/K)^{K-\mathrm{fin}}$$

is an isomorphism.

[S85] contains no proofs!

Z: flag manifold of $G_{\mathbb{C}}$ (space of Borel subgroups).
$D^b_{G,\lambda}(Z)$: bounded equivariant derived category of constructible sheaves of $\mathbb{C}$-vectorspace on Z with twist λ
$\mathcal{O}_Z(\lambda)$: twisted sheaf of holomorphic functions

Conjecture (Kashiwara '87)

Fix $\mathcal{S} \in D^b_{G,\lambda}(Z)$.

(i) $\mathrm{Ext}^p(\mathcal{S}, \mathcal{O}_Z(\lambda))$ *and* $H^q(Z, \mathcal{S} \otimes \mathcal{O}_Z(-\lambda))$ *carry natural topologies and continuous linear* G*-actions which are admissible of finite length.*

(ii) $\mathrm{Ext}^p(\mathcal{S}, \mathcal{O}_Z(\lambda))$ *and* $H^{\dim Z - p}(Z, \mathcal{S} \otimes \mathcal{O}_Z(-\lambda))$ *are each others strong duals.*

(iii) *If* $\mathcal{M} \in D^b_{K_{\mathbb{C}},\lambda}(Z) \cong D^b_{G,\lambda}(Z)$ *corresponds to a holonomic* $(\mathcal{D}_{-\lambda}, K_{\mathbb{C}})$*-module* $\mathfrak{M}$ *under the Riemann-Hilbert correspondence, then* $H^p(Z, \mathfrak{M})$ *is the dual Harish-Chandra module of* $\mathrm{Ext}^{\dim Z - p}(\mathcal{M}, \mathcal{O}_Z(\lambda))^{K-\mathrm{fin}}$.

The Kashiwara-Schmid paper [KS94]

- [KS94] contains a sketch of proof for Kashiwara's Conjectures.
- $\mathrm{Ext}^p(\mathcal{S}, \mathcal{O}_Z(\lambda))$ and $H^{\dim Z - p}(Z, \mathcal{S} \otimes \mathcal{O}_Z(-\lambda))$ turn out to be the maximal resp. minimal globalization of their underlying Harish-Chandra module.
- The proof shows that Kashiwara's Conjectures are equivalent to Schmid's Theorem.

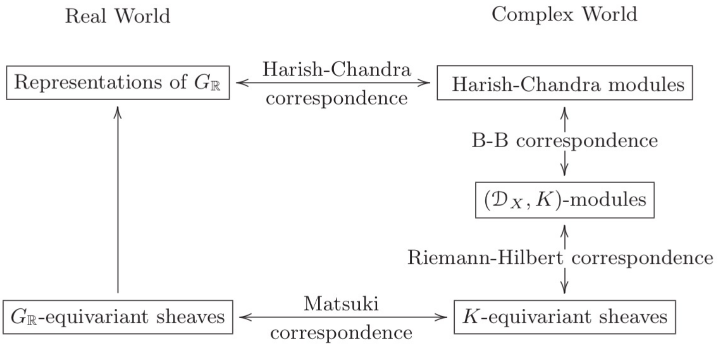
$$\phi : \underbrace{\mathrm{Mod}_{G_{\mathbb{C}}}(\mathcal{D}_{G_{\mathbb{C}}/K_{\mathbb{C}}})}_{\text{quasi-eq. } D\text{-modules}} \underset{\cong}{\longrightarrow} \mathrm{Mod}(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}}), \quad \mathfrak{M} \mapsto \mathfrak{M}/\mathfrak{J}_{eK_{\mathbb{C}}}\mathfrak{M}.$$

Theorem (Kashiwara-Schmid Vanishing Thm.)

For every Harish-Chandra module V and every $n \neq 0$,
$i : G/K \hookrightarrow G_{\mathbb{C}}/K_{\mathbb{C}}$

$$\begin{aligned} 0 &= \mathrm{Ext}^n_{\mathcal{D}_{G_{\mathbb{C}}/K_{\mathbb{C}}}}(\phi^{-1}(V) \otimes i_* i^! \mathbb{C}_{G_{\mathbb{C}}/K_{\mathbb{C}}}, \mathcal{O}_{(G_{\mathbb{C}}/K_{\mathbb{C}})^{\mathrm{an}}}) \\ &\cong \mathrm{Ext}^n_{(\mathfrak{g}_{\mathbb{C}}, K)}(V, C^\infty(G)^{K-\mathrm{fin}}). \end{aligned}$$

In [K08] Kashiwara explains the material from [KS94] in more detail. In particular, he explains how D-modules get into the picture (see picture below).



Kashiwara's Theorem

$\mathrm{HC}(\mathfrak{g}, K)$: category of Harish-Chandra $(\mathfrak{g}, K)$-modules
$\mathsf{FN}_{G_{\mathbb{R}}}$: category of Fréchet nuclear $G_{\mathbb{R}}$-modules
$\mathsf{DFN}_{G_{\mathbb{R}}}$: category of dual Fréchet nuclear $G_{\mathbb{R}}$-modules
$\mathrm{MG} : \mathrm{HC}(\mathfrak{g}, K) \to \mathsf{FN}_{G_{\mathbb{R}}}$ max. glob. functor
$\mathrm{mg} : \mathrm{HC}(\mathfrak{g}, K) \to \mathsf{DFN}_{G_{\mathbb{R}}}$ min. glob. functor
$\mathrm{MG}_{G_{\mathbb{R}}} \leq \mathsf{FN}_{G_{\mathbb{R}}}$ subcat.; objects: $\mathrm{MG}(V)$
$\mathrm{mg}_{G_{\mathbb{R}}} \leq \mathsf{DFN}_{G_{\mathbb{R}}}$ subcat.; objects: $\mathrm{mg}(V)$

Theorem ([K08])

(i) MG *and* mg *are exact functors.*

(ii) *All morphisms in* $\mathrm{MG}_{G_{\mathbb{R}}}$ *and* $\mathrm{mg}_{G_{\mathbb{R}}}$ *have closed range.*

(iii) $\mathrm{MG}_{G_{\mathbb{R}}}$ *and* $\mathrm{mg}_{G_{\mathbb{R}}}$ *are closed under extensions and taking* $G_{\mathbb{R}}$*-invariant closed subspaces.*

Spectral applications: Bunke & Olbrich

- Bunke and Olbrich studied Γ- and $\mathfrak{n}$-cohomologies in the context of dynamical zeta functions. In particular, in [BO97] and [O02, §8] one finds applications of the Kashiwara-Schmid results in this direction.
- Bunke and Olbrich mostly considered rank one groups. Even in that case the relation to the work of microlocal analysts (Guillarmou, Weich et al.) still has to be clarified.
- In higher rank, recent progress by Weich and coauthors make it plausible that more applications can be found.

Back to Poisson transforms: open problems

- Can one extend Kashiwara's theorem to other canonical globalizations such as smooth and distribution globalizations.
- If one starts with vector bundle on K/M rather than line bundles, one speaks about vector valued Poisson tranforms, see [O95]. Apart from the rank one case the analog of the Helgason conjecture is largely unclear.
- [K+78] provides a boundary value map inverting the Poisson transform for generic parameters. So far such a map has not been extracted from the Kashiwara-Schmid method.
- There are also Poisson transforms for non-Riemannian symmetric spaces. Can one prove analogs for the Helgason conjecture also in that case?

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