Arithmetic Representation Growth of Virtually Free Groups (arXiv: 2201.12319)

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Structure of the talk

§1 Introduction

§2 Virtually free groups & character varieties

§3 Results

- Let $\mathbb{F}_q$ be a finite field,
- $\mathcal{A}$ a finitely generated $\mathbb{F}_q$-algebra
- $\forall\, d \in \mathbb{N}_0 : \{\rho : \mathcal{A} \to \mathbf{M}_d(\mathbb{F}_q) \text{ representation}\}$ is a finite set
- Let xyz be a property of representations, e.g. one of $\{\text{simple},\ \text{absolutely sim.},\ \text{semisim.},\ \text{indecomposable},\ \text{abs. ind.}, \text{any}\}$
- define

$$r_d^{\text{xyz}}(q^{\alpha}) := \#\left(\{\rho{:}\mathcal{A}\otimes_{\mathbb{F}_q}\mathbb{F}_{q^\alpha}\to\mathbf{M}_d(\mathbb{F}_{q^\alpha}) \text{ xyz repr.}\}/_{\cong}\right)$$

Main theorem (weak version)

Theorem: (K., 2021)

If $\mathcal{A} = \mathbb{F}_q[\mathcal{G}]$, $\mathcal{G}$ a finitely generated virtually free group and $\mathbb{F}_q$ suitable for $\mathcal{G}$, then there are counting polynomials $R_d^{\text{absim}}, R_d^{\text{ss}} \in \mathbb{Z}[s], R_d^{\text{sim}} \in \mathbb{Q}[s]$ s.t.

$$\forall\, \alpha \geq 1 : R_d^{\text{xyz}}(q^\alpha) = r_d^{\text{xyz}}(q^\alpha)$$

Similar results:

- Kac '82: $\mathcal{A} = \mathbb{F}_q\vec{Q}$, xyz = abind, ind, any
- Reineke '06 & Mozgovoy-Reineke '09: $\mathcal{A} = \mathbb{F}_q\vec{Q}$, xyz = abs. stable, stable, polystable
- Mozgovoy-Reineke '15: $\mathcal{A} = \mathbb{F}_q[F_a]$, F_a free group on a gen.
- Hennecart-K. (ongoing): $\mathcal{A} = \mathbb{F}_q[\mathcal{G}]$, $\mathcal{G}$ fin. gen. virt. free, xyz = abind, ind, any

Examples of counting polynomials

- $\mathcal{G} = \mathbb{D}_\infty = C_2 * C_2$:

$$R_1^{\text{absim}} = 4 \quad , \quad R_2^{\text{absim}} = s - 2 \quad \& \quad R_d^{\text{absim}} = 0 \ \forall\, d \geq 3$$

- $\mathcal{G} = \mathbf{PSL}_2(\mathbb{Z}) = C_2 * C_3$:

$$\begin{aligned}
R_1^{\text{absim}} &= 6 \cdot 1 \\
R_2^{\text{absim}} &= 3 \cdot (s - 2) \\
R_3^{\text{absim}} &= 2 \cdot (s^2 - 3s + 3) \\
R_4^{\text{absim}} &= 3 \cdot (s^3 - 3s^2 + 5s - 4) \\
R_5^{\text{absim}} &= 6 \cdot (s^4 - 3s^3 + 5s^2 - 7s + 5) \\
R_6^{\text{absim}} &= s^7 + 3s^6 - 2s^5 - 23s^4 + 56s^3 - 83s^2 + 107s - 71
\end{aligned}$$

Examples of counting polynomials (continued)

- $\mathcal{G} = \mathbf{PSL}_2(\mathbb{Z}) = C_2 * C_3$:

$$
\begin{aligned}
R_{12}^{\text{absim}} &= s^{25} + 3s^{24} + 18s^{23} + 38s^{22} + 67s^{21} + 12s^{20} - 157s^{19} \\
&-471s^{18} - 366s^{17} + 613s^{16} + 2106s^{15} + 117s^{14} - 3906s^{13} \\
&-943s^{12} + 2976s^{11} + 3646s^{10} - 2399s^{9} - 5472s^{8} + 2048s^{7} \\
&+6199s^{6} + 907s^{5} - 21155s^{4} + 36300s^{3} - 37538s^{2} + 30194s \\
&-14372
\end{aligned}
$$