Gentle algebras arising from triangulations of surfaces with orbifold points

Joint work in progress with Lang Mou

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Generalized cluster algebras

Generalized cluster algebras

Definition

A matrix $B \in \mathbb{Z}^{n\times n}$ is **skew-symmetrizable** if there exists a diagonal matrix $D = \mathrm{diag}(d_1, \dots, d_n) \in \mathbb{Z}_{\geq 0}$ with positive diagonal entries, such that $DB = -(DB)^{\mathrm{T}}$.

Examples

$$D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

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$$D = \begin{bmatrix} 2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Fix positive integers $\rho = (r_1, \dots, r_n)$ such that r_j divides the j^{th} column of B, as well as monic palindromic polynomials $\theta_1, \dots, \theta_n \in \mathbb{C}[u, v]$.

$$\theta_j = \sum_{l=1}^{r_j} c_l u^l v^{r_j - l}$$

Generalized cluster algebras

Definition (Chekhov-Shapiro)

Let $\mathcal{F}$ be the field of rational functions in n indeterminates with complex coefficients. Suppose we have a skew-symmetrizable seed $(B, \mathbf{x})$ in $\mathcal{F}$.

1. For each $k \in \{1, \dots, n\}$, define the generalized seed mutation
$$\mu_k^{\rho,\theta}(B, \mathbf{x}) := (\mu_k(B), \mathbf{x}'), \quad \text{where}$$
$$\mathbf{x}' := \left(x_1, \dots, x_{k-1}, \frac{\theta_k\left(\prod_{i:b_{ik}>0} x_i^{\frac{b_{ik}}{r_k}}, \prod_{i:b_{ik}<0} x_i^{-\frac{b_{ik}}{r_k}}\right)}{x_k}, x_{k+1}, \dots, x_n \right).$$
2. The (coefficient-free) **generalized cluster algebra** $\mathcal{A}^{\rho,\theta}(B, \mathbf{x})$ is the $\mathbb{Q}$-subalgebra of $\mathcal{F}$ generated by the union of all clusters produced from $(B, \mathbf{x})$ by finite sequences of generalized seed mutations.

For $r_1 = \cdots = r_n = 1$ and $\theta_1 = \cdots = \theta_n = u + v$, we obtain Fomin-Zelevinsky's cluster algebra.

Generalized cluster algebras

Example

Let $B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix}$, $\mathbf{x} = (x_1, x_2, x_3)$, $\rho = (1, 2, 1)$,

$\theta_1 = u + v, \quad \theta_2 = u^2 + \omega uv + v^2, \quad \theta_3 = u + v.$ *Then:*

1. $\mu_1^{\rho,\theta}(B, \mathbf{x}) = \left(\begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \left(\frac{x_2+x_3}{x_1}, x_2, x_3 \right) \right)$
2. $\mu_2^{\rho,\theta}(B, \mathbf{x}) = \left(\begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}, \left(x_1, \frac{x_1^2+\omega x_1 x_3+x_3^2}{x_2}, x_3 \right) \right)$
3. $\mu_3^{\rho,\theta}(B, \mathbf{x}) = \left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -2 & 0 \end{bmatrix}, \left(x_1, x_2, \frac{x_1+x_2}{x_3} \right) \right)$

Theorem (Chekhov-Shapiro)

Generalized cluster algebras have the Laurent phenomenon.

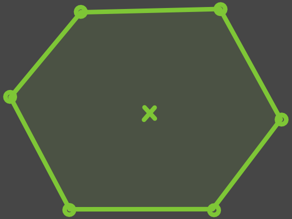
Surfaces with orbifold points

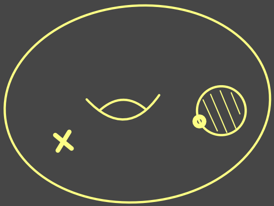
Surfaces with orbifold points

Definition

An **unpunctured surface with orbifold points** is a quadruple $(\Sigma, \mathbb{M}, \mathbb{O}, o)$ consisting of:

1. a compact, connected, oriented, two-dimensional real manifold Σ with non-empty boundary;
2. a finite subset $\mathbb{M} \subseteq \partial\Sigma$ with at least one point from each boundary component;
3. a finite subset $\mathbb{O} \subseteq \Sigma \setminus \partial\Sigma$;
4. a function $o : \mathbb{O} \to \mathbb{Z}_{\geq 2}$.

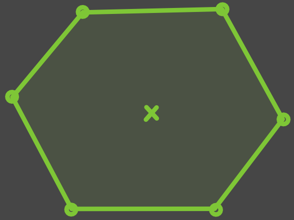




Surfaces with orbifold points

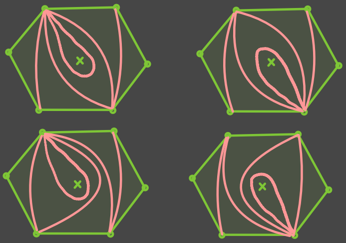
Definition

An **arc** on $(\Sigma, \mathbb{M}, \mathbb{O}, o)$ is a curve that connects points of $\mathbb{M}$, is not homotopic in $\Sigma \setminus \mathbb{O}$ to a point or a boundary segment, and does not cross itself.



Definition

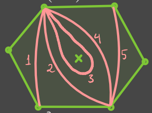
A **triangulation** of $(\Sigma, \mathbb{M}, \mathbb{O}, o)$ is a maximal collection (up to isotopy rel $\mathbb{M} \cup \mathbb{O}$) of arcs that do not cross each other.



Surfaces with orbifold points

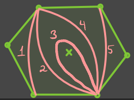
Definition (Chekhov-Shapiro, Felikson-Shapiro-Tumarkin)

Each triangulation T of $(\Sigma, \mathbb{M}, \mathbb{O}, o)$ gives rise to a skew-symmetrizable matrix $B(T)$:

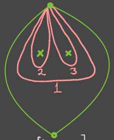


$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 2 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}$$

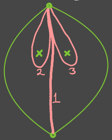


$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$



$$\begin{bmatrix} 0 & -2 & 2 \\ 1 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 2 \end{bmatrix}$$



$$\begin{bmatrix} 0 & 2 & -2 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Observation

We can take $r_1, \ldots, r_n$, to be any choice of positive divisors of $d_1, \ldots, d_n$.

Surfaces with orbifold points

Taking $r_1 =: d_1, \dots, r_n := d_n$, $\omega_q := 2\cos(\pi/o(q))$, and

$$\theta_j := \begin{cases} u+v & j \text{ not pending} \\ u^2 + \omega_q uv + v^2 & j \text{ pending around } q \in \mathbb{O} \end{cases} \qquad \text{we have:}$$

Theorem (Chekhov-Shapiro)

The ring of Penner lambda lengths on the decorated Teichmüller space of any surface with marked points and orbifold points is a generalized cluster algebra (so-called boundary coefficients have to be chosen). Moreover, there is a bijection

$$\{\textit{arcs on } (\Sigma, \mathbb{M}, \mathbb{O}, o)\} \quad \longleftrightarrow \quad \{\textit{cluster variables of } \mathcal{A}^{\rho,\theta}(B(T), \underline{\lambda}_T)\}$$

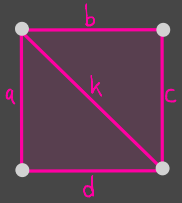
which in turn induces a bijection

$$\{\textit{triangulations of } (\Sigma, \mathbb{M}, \mathbb{O}, o)\} \quad \longleftrightarrow \quad \{\textit{clusters of } \mathcal{A}^{\rho,\theta}(B(T), \underline{\lambda}_T)\}$$

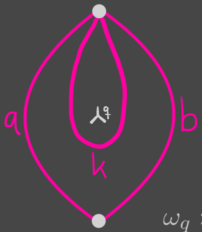
making flips correspond to generalized cluster mutations.

Surfaces with orbifold points

Concretely, the generalized cluster mutation corresponding to a flip takes one of the following forms:



$$X_k' = \frac{X_a X_c + X_b X_d}{X_k}$$

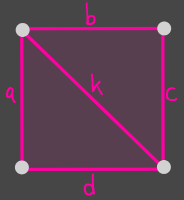


$$\omega_q := 2\cos(\pi/o(q))$$

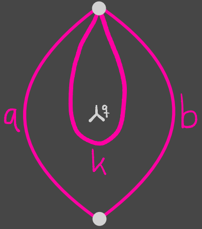
$$X_k' = \frac{X_a^2 + \omega_q X_a X_b + X_b^2}{X_k}$$

Surfaces with orbifold points

From now on, we assume that $(\Sigma, \mathbb{M}, \mathbb{O}, o) = (\Sigma, \mathbb{M}, \mathbb{O}, c_3)$ is an unpunctured surface with **orbifold points of order** 3. This implies $\omega_q = 1$ for all $q \in \mathbb{O}$, hence the generalized cluster mutation corresponding to a flip takes one of the following forms:



$$x_k' = \frac{x_a x_c + x_b x_d}{x_k}$$



$\omega_q := 2\cos(\pi/o(q))$

$\cos(\pi/3) = \frac{1}{2}$

$$x_k' = \frac{x_a^2 + x_a x_b + x_b^2}{x_k}$$

Gentle algebras associated to triangulations

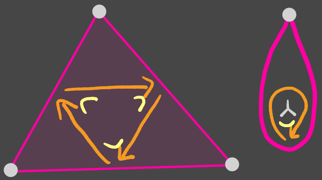
Gentle algebras associated to triangulations

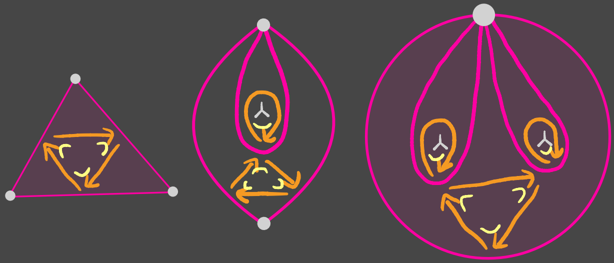
Definition (LF-Mou)

For each triangulation T of $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$, let $(Q(T), S(T))$ be the following quiver with potential:

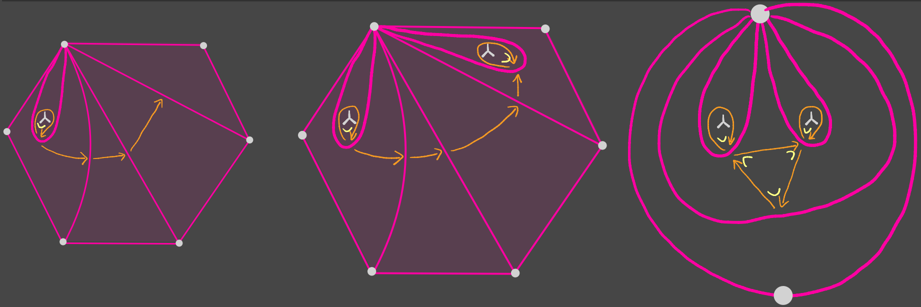
$Q_0(T) :=$ {*arcs belonging to* T}
$Q_1(T) :=$ *clockwisely drawn within triangles of* T

$$S(T) := \sum_{\triangle} \alpha^{\triangle}\beta^{\triangle}\gamma^{\triangle} + \sum_{j \text{ pending}} \varepsilon_j^3$$





Examples



Remark

For $\mathbb{O} = \varnothing$, *flip/DWZ-mutation behavior of* $(Q(T), S(T))$ *studied by LF (2008), representation theory of its* **Jacobian algebra** $A(T)$ *studied by Assem-Brüstle-Charbonneau-Plamondon (2009).*

Gentle algebras associated to triangulations

The **Jacobian algebra** $A(T)$ of $(Q(T), S(T))$ is finite-dimensional gentle. Thus, indecomposable $A(T)$-modules $\longleftrightarrow$ curves on $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$ not in T.

Theorem (Brüstle-Zhang, 2010)

Suppose $\mathbb{O} = \varnothing$. *Let* M, N, *be string modules over* $A(T)$ *and* γ_M, γ_N, *their corresponding arcs on* $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$. *The following are equivalent:*

1. $\mathrm{Hom}_A(N, \tau(M)) = 0 = \mathrm{Hom}_A(M, \tau(N))$;
2. γ_M *and* γ_N *do not cross in* $\Sigma \setminus \partial\Sigma$.

Theorem (Geiss-LF-Schröer, 2020)

Suppose $\mathbb{O} = \varnothing$. *Let* M, N, *be indecomposable* $A(T)$*-modules and* γ_M, γ_N, *their corresponding curves on* $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$. *The following are equivalent:*

1. $\mathrm{Hom}_A(N, \tau(M)) = 0 = \mathrm{Hom}_A(M, \tau(N))$;
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Gentle algebras associated to triangulations

Theorem (Geiss-LF-Schröer, 2020)

Suppose $\mathbb{O} = \varnothing$. For any triangulation T of $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$,

1. *there is a bijection between the set of laminations of $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$ and the set of τ-reduced irreducible components of $A(T)$;*
2. *the generic values of the Caldero-Chapoton map on the τ-reduced components of $A(T)$ coincide with Musiker-Schiffler-Williams' expansions in terms of perfect matchings of bipartite graphs.*

Main result

Main result

Theorem (LF-Mou)

For each triangulation T of $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$ there are commutative diagrams of bijections

$\{$*arcs on* $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)\}$

Chekhov-Shapiro

Palu-Pilaud-Plamondon

cluster vars. in $\mathcal{A}^{\rho,\theta}(B(T))$

LF-Mou

$CC_{A(T)}$

$\{\tau$*-rigid indec. pairs over* $A(T)\}$

$\{$*triangulations of* $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)\}$

Chekhov-Shapiro

Palu-Pilaud-Plamondon

unlabeled seeds in $\mathcal{A}^{\rho,\theta}(B(T))$

LF-Mou

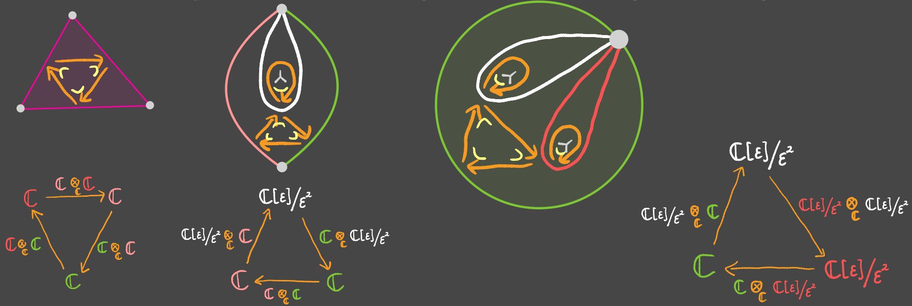
$CC_{A(T)}$

$\{$*support* τ*-tilting pairs over* $A(T)\}$

Mutations of representations

Key observation

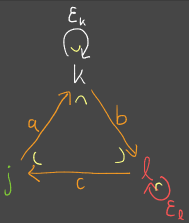
Recall the three types of basic configurations of triangles making up T:

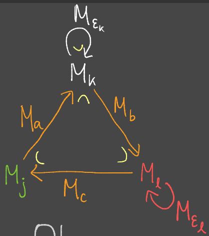


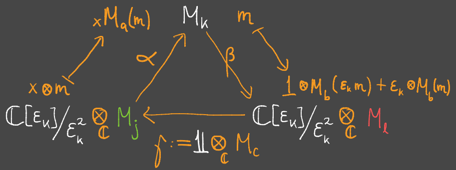
Observation

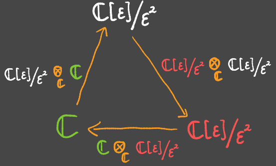
Whenever the third configuration appears somewhere in T, the bimodule we attach to $a : j \to k$ is free as a left module and as a right module.

Mutating a representation at a pending arc









Obs.

(i) $\alpha\gamma = 0 = \gamma\beta$

(ii) $\ker(\gamma) = \mathbb{C}[\varepsilon_k]/\varepsilon_k^2 \otimes_{\mathbb{C}} \mathrm{Ker}(M_c)$ is free over $\mathbb{C}[\varepsilon_k]/\varepsilon_k^2$

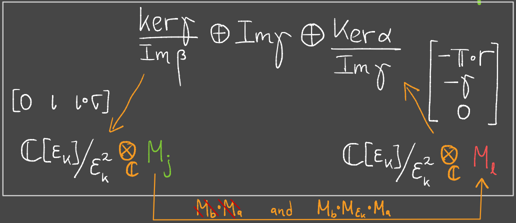
(iii) $\mathrm{Im}\gamma = \mathbb{C}[\varepsilon_k]/\varepsilon_k^2 \otimes_{\mathbb{C}} \mathrm{Im}(M_c)$ is injective over $\mathbb{C}[\varepsilon_k]/\varepsilon_k^2$,

so $0 \to \mathrm{Im}\gamma \to \mathrm{Ker}\alpha \rightleftarrows \frac{\mathrm{Ker}\,\alpha}{\mathrm{Im}\gamma} \to 0$ splits.

Choose $\mathbb{C}[\varepsilon_k]/\varepsilon_k^2$-module homomorphisms $r: \mathbb{C}[\varepsilon_k]/\varepsilon_k^2 \otimes_{\mathbb{C}} M_\ell \longrightarrow \ker\gamma$

$s: \frac{\ker\alpha}{\mathrm{Im}(\gamma)} \longrightarrow \ker\alpha$ such that $r\circ i = 1\!\!1_{\ker\gamma}$ and $\pi\circ s = 1\!\!1_{\frac{\ker\alpha}{\mathrm{Im}\gamma}}$

Def (LF-Mou) The **pre-mutation** $\tilde{\mu}_k(M)$



and use the natural isomorphisms

$$\mathrm{Hom}_{\mathbb{C}[\varepsilon_k]/\varepsilon_k^2}\left(\mathbb{C}[\varepsilon_k]/\varepsilon_k^2 \otimes_{\mathbb{C}} A, B\right) \cong \mathrm{Hom}_{\mathbb{C}}(A, B) \cong \mathrm{Hom}_{\mathbb{C}[\varepsilon_k]/\varepsilon_k^2}\left(A, \mathbb{C}[\varepsilon_k]/\varepsilon_k^2 \otimes_{\mathbb{C}} B\right)$$

Thm (LF-Mou) (i) 2-cycles deleted through reduction process

(ii) **mutation** $\mu_k(M)$ is module over $A(f_k(T))$

Generic bases and bangle bases

Generic bases and bangle bases

Theorem (LF-Mou)

Let $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$ be an unpunctured surface with orbifold points of order 3. If at least one boundary component of Σ has an odd number of marked points, then for any triangulation T, the set of generic values of the Caldero-Chapoton map on the τ-reduced irreducible components of $A(T)$ is linearly independent. This set is invariant under mutations of representations.

Conjecture

The aforementioned generic values of the Caldero-Chapoton map on the τ-reduced components coincide with Banaian-Kelley's expansions in terms of perfect matchings.

A proof would follow from a combination

(LF-Mou) + (ongoing work of Banaian-Valdivieso).

Some questions

Some questions

1. Is Geiss-Leclerc-Schröer's generic set always linearly independent?
2. does GLS's generic set span the Caldero-Chapoton algebra of $A(T)$?
3. is the Caldero-Chapoton algebra of $A(T)$ equal to the generalized cluster algebra of $(\Sigma, \mathbb{M}, \mathbb{O}, c_3)$?
4. what is the relation to Paquette-Schiffler's approach?
5. is there a way to tackle orbifold points of higher order?

Thank you!