Noncrossing Parking Functions

Drew Armstrong (with B. Rhoades and V. Reiner)

University of Miami
www.math.miami.edu/∼armstrong

"Non-crossing partitions in representation theory"
Bielefeld, June 2014

Plan

1. Parking Functions
2. Noncrossing Partitions
3. Noncrossing Parking Functions

What is a Parking Function?

What is a Parking Function?

Definition

A parking function is a vector $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ whose **increasing rearrangement** $b_1 \leq b_2 \leq \cdots \leq b_n$ satisfies:

$$\boxed{\forall\, i,\ \ b_i \leq i}$$

Imagine a one-way street with n parking spaces.

- There are n cars.
- Car i wants to park in space a_i.
- If space a_i is full, she parks in first available space.
- Car 1 parks first, then car 2, etc.
- "$\vec{a}$ is a parking function" $\equiv$ "everyone is able to park".

What is a Parking Function?

Example ($n = 3$)

111					
112	121	211			
113	131	311			
122	212	221			
123	132	213	231	312	321

Note that $\#\mathrm{PF}_3 = 16$ and $\mathfrak{S}_3$ acts on PF_3 with 5 orbits.

In General We Have

$\#\mathrm{PF}_n = (n+1)^{n-1}$	$\#\,\text{orbits} = \frac{1}{n+1}\binom{2n}{n}$
"Cayley"	"Catalan"

Structure of Parking Functions

Idea (Pollack, $\sim$ 1974)

Now imagine a circular street with $n+1$ parking spaces.

- Choice functions $= (\mathbb{Z}/(n+1)\mathbb{Z})^n$.
- Everyone can park. One empty spot remains.
- Choice is a parking function $\iff$ space $n+1$ remains empty.
- One parking function per rotation class.

Conclusion:

- $\mathrm{PF}_n =$ choice functions / rotation
- $\mathrm{PF}_n \approx_{\mathfrak{S}_n} (\mathbb{Z}/(n+1)\mathbb{Z})^n / (1, 1, \ldots, 1)$
- $\#\mathrm{PF}_n = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}$

Why do We Care?

Culture

The symmetric group $\mathfrak{S}_n$ acts diagonally on the algebra of polynomials in two commuting sets of variables:

$$\mathfrak{S}_n \curvearrowright \mathbb{Q}[\mathbf{x}, \mathbf{y}] := \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

After many years of work, Mark Haiman (2001) proved that the algebra of diagonal coinvariants carries the same $\mathfrak{S}_n$-action as parking functions:

$$\omega \cdot \mathrm{PF}_n \approx_{\mathfrak{S}_n} \mathbb{Q}[\mathbf{x}, \mathbf{y}] / \mathbb{Q}[\mathbf{x}, \mathbf{y}]^{\mathfrak{S}_n}$$

The proof was hard. It comes down to this theorem:

> *The isospectral Hilbert scheme of n points in $\mathbb{C}^2$ is Cohen-Macaulay and Gorenstein.*

Pollack's Idea $\Rightarrow$ Weyl Groups

Haiman, *Conjectures on the quotient ring...*, Section 7

Let W be a Weyl group with rank r and Coxeter number h. That is, $W \curvearrowright \mathbb{R}^r$ by reflections and stabilizes a "root lattice" $Q \leq \mathbb{R}^r$. We define the W-parking functions as

$$\boxed{\mathrm{PF}_W := Q/(h+1)Q}$$

This generalizes Pollack because we have

$$(\mathbb{Z}/(n+1)\mathbb{Z})^n/(1,1,\ldots,1) = Q/(n+1)Q.$$

Recall that $W = \mathfrak{S}_n$ has Coxeter number $h = n$, and root lattice

$$Q = \mathbb{Z}^n/(1,1,\ldots,1) = \{(r_1,\ldots,r_n) \in \mathbb{Z}^n : \sum_i r_i = 0\}.$$

Pollack's Idea ⇒ Weyl Groups

Haiman, *Conjectures on the quotient ring...*, Section 7

The W-parking space has dimension generalizing the Cayley numbers

$$\dim \mathrm{PF}_W = (h+1)^r \left(= (n+1)^{n-1}\right)$$

More generally: Given $w \in W$, the character of PF_W is

$$\begin{aligned}\chi(w) &= \#\{\vec{a} \in \mathrm{PF}_W : w(\vec{a}) = w\} \\ &= (h+1)^{r-\mathrm{rank}(1-w)} \left(= (n+1)^{\#\mathrm{cycles}(w)-1}\right)\end{aligned}$$

and the number of W-orbits generalizes the Catalan numbers

$$\#\mathrm{orbits} = \frac{1}{|W|}\prod_{i=1}^{r}(h+d_i) \left(= \frac{1}{n+1}\binom{2n}{n}\right)$$

Parking Functions $\Leftrightarrow$ Shi Arrangement

Parking Functions $\Leftrightarrow$ Shi Arrangement

Another Language

The W-parking space is the same as the Shi arrangement of hyperplanes. Given positive root $\alpha \in \Phi^+ \subseteq Q$ and integer $k \in \mathbb{Z}$ consider the hyperplane $H_{\alpha,k} := \{\mathbf{x} : (\alpha, \mathbf{x}) = k\}$. Then we define

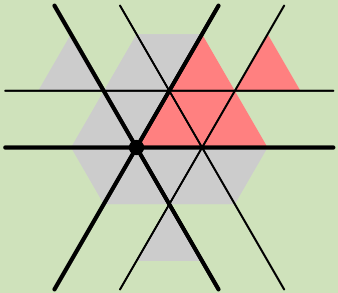
$$\mathrm{Shi}_W := \{H_{\alpha,\pm 1} : \alpha \in \Phi^+\}.$$

Cellini-Papi and Shi give an explicit bijection:

$$\text{elements of } Q/(h+1)Q \quad \longleftrightarrow \quad \text{chambers of } \mathrm{Shi}_W$$

Parking Functions ⇔ Shi Arrangement

Example ($W = \mathfrak{S}_3$)



There are $16 = (3+1)^{3-1}$ chambers and $5 = \frac{1}{4}\binom{6}{3}$ orbits.

Parking Functions $\Leftrightarrow$ Shi Arrangement

"Ceiling Diagrams"

I like to think of Shi chambers as elements of the set

$$\left\{(w, A) : w \in W, \text{ antichain } A \subseteq \Phi^+, A \cap \text{inv}(w) = \emptyset\right\}.$$

The Shi chamber with "ceiling diagram" (w, A)

- is in the cone determined by w
- and has ceilings given by A.

I.O.U.

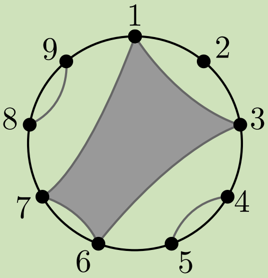
How to describe the W-action on ceiling diagrams?

Pause

What is a Noncrossing Partition?

What is a Noncrossing Partition?

Definition by Example



We encode this partition by the permutation $(1367)(45)(89) \in \mathfrak{S}_9$.

What is a Noncrossing Partition?

Theorem (Biane, and probably others)

Let $T \subseteq \mathfrak{S}_n$ be the generating set of all transpositions and consider the Cayley metric $d_T : \mathfrak{S}_n \times \mathfrak{S}_n \to \mathbb{N}$ defined by

$$d_T(\pi, \mu) := \min\{k : \pi^{-1}\mu \text{ is a product of } k \text{ transpositions}\}.$$

Let $c = (123 \cdots n)$ be the standard n-cycle. Then the permutation $\pi \in \mathfrak{S}_n$ corresponds to a noncrossing partition if and only if

$$d_T(1, \pi) + d_T(\pi, c) = d_T(1, c).$$

"π is on a geodesic between 1 and c"

What is Noncrossing Partition?

Definition (Brady-Watt, Bessis)

Let W be any finite Coxeter group with reflections $T \subseteq W$. Let $c \in W$ be any Coxeter element. We say $w \in W$ is a "noncrossing partition" if

$$d_T(1, w) + d_T(w, c) = d_T(1, c)$$

"w is on a geodesic between 1 and c"

The Mystery of NC and NN

Mystery

Let W be a Weyl group (crystallographic finite Coxeter group). Let $\mathrm{NC}(W)$ be the set of noncrossing partitions and let $\mathrm{NN}(W)$ be the set of antichains in Φ^+ (called "nonnesting partitions"). Then we have

$$\#\mathrm{NC}(W) = \frac{1}{|W|}\prod_{i=1}^{r}(h+d_i) = \#\mathrm{NN}(W)$$

- The right equality has at least two uniform proofs.
- The left equality is only known case-by-case.
- What is going on here?

The Mystery of NC and NN

Idea and an Anecdote

Idea: Since the parking functions can be though of as

$$\{(w, A) : w \in W, A \in \mathrm{NN}(W), A \cap \mathrm{inv}(w) = \emptyset\}$$

maybe we should also consider the set

$$\{(w, \sigma) : w \in W, \sigma \in \mathrm{NC}(W), \sigma \cap \mathrm{inv}(w) = \emptyset\}$$

where "$\sigma \cap \mathrm{inv}(w)$" means something sensible.

Anecdote: Where did the idea come from?

Pause

Now we define the W-action on Shi chambers

Definition of $\mathcal{F}$-parking functions

Recall the definition of the lattice of flats for W

$$\mathcal{L}(W) := \{\cap_{\alpha \in J} H_{\alpha,0} : J \subseteq \Phi^+\},$$

and for any flat $X \in \mathcal{L}(W)$ recall the definition of the parabolic subgroup

$$W_X := \{w \in W : w(\mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{x} \in X\}.$$

Now we define the W-action on Shi chambers

Definition of $\mathcal{F}$-parking functions

For any set of flats $\mathcal{F} \subseteq \mathcal{L}(W)$ we define the $\mathcal{F}$-parking functions

$$\mathrm{PF}_{\mathcal{F}} := \{[w, X] : w \in W, X \in \mathcal{F}, w(X) \in \mathcal{F}\} / \sim$$

where $\boxed{[w, X] \sim [w', X'] \Longleftrightarrow X = X' \text{ and } wW_X = w'W_{X'}}$

This set carries a natural W-action. For all $u \in W$ we define

$$u \cdot [w, X] := [wu^{-1}, u(X)]$$

Now we define the W-action on Shi chambers

The Prototypical Example of $\mathcal{F}$-Parking Functions

If we consider the set of nonnesting flats

$$\mathcal{F} = \mathcal{NN} := \{\cap_{\alpha \in A} H_{\alpha,0} : \text{ antichain } A \subseteq \Phi^+\}$$

then $\mathrm{PF}_{\mathcal{NN}}$ is just the set of ceiling diagrams of Shi chambers with the natural action corresponding to $W \curvearrowright Q/(h+1)Q$.

Now we define the W-action on Shi chambers

But There is **Another** Example

Noncrossing Parking Functions

But There is **Another** Example

Noncrossing Parking Functions

But There is **Another** Example

Given any $w \in W$ there is a corresponding flat

$$\ker(1-w) = \{\mathbf{x} : w(\mathbf{x}) = \mathbf{x}\} \in \mathcal{L}(W).$$

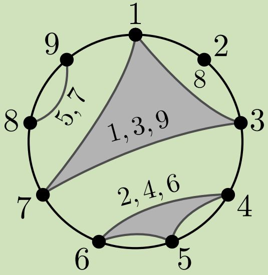
If we consider the set of noncrossing flats

$$\mathcal{F} = \mathcal{NC} := \{\ker(1-w) : w \in \mathrm{NC}(W)\}$$

then $\mathrm{PF}_{\mathcal{NC}}$ is something new and possibly interesting. We call $\mathrm{PF}_{\mathcal{NC}}$ the set of noncrossing parking functions.

Noncrossing Parking Functions

Example ($W = \mathfrak{S}_9$)



Type A NC parking functions are just NC partitions with labeled blocks.

Noncrossing Parking Functions

Theorem

If W is a Weyl group then we have an isomorphism of W-actions:

$$\mathsf{PF}_{\mathcal{NC}} \approx_W \mathsf{PF}_{\mathcal{NN}}$$

This is just a fancy restatement of a theorem of Athanasiadis, Chapoton, and Reiner. Unfortunately the proof is case-by-case using a computer.

Noncrossing Parking Functions

However

The noncrossing parking functions have two advantages over the nonnesting parking functions.

1. $\mathrm{PF}_{\mathcal{NN}}$ is defined only for Weyl groups but $\mathrm{PF}_{\mathcal{NC}}$ is defined also for noncrystallographic Coxeter groups.
2. $\mathrm{PF}_{\mathcal{NC}}$ carries an exta cyclic action. Let $C = \langle c \rangle \leq W$ where $c \in W$ is a Coxeter element. Then the group $W \times C$ acts on $\mathrm{PF}_{\mathcal{NC}}$ by

$$(u, c^d) \cdot [w, X] := [c^d w u^{-1}, u(X)].$$

Noncrossing Parking Functions

Cyclic Sieving "Theorem"

Let $h := |\langle c \rangle|$ be the Coxeter number and let $\zeta := e^{2\pi i/h}$. Then for all $u \in W$ and $c^d \in C$ we have

$$\begin{aligned}\chi(u, c^d) &= \#\left\{[w, X] \in \mathsf{PF}_{\mathcal{NC}} : (u, c^d) \cdot [w, X] = [w, X]\right\} \\ &= \lim_{q \to \zeta^d} \frac{\det(1 - q^{h+1} u)}{\det(1 - qu)} \\ &= (h+1)^{\mathrm{mult}_u(\zeta^d)},\end{aligned}$$

where $\mathrm{mult}_u(\zeta^d)$ is the multiplicity of the eigenvalue ζ^d in $u \in W$.

Unfortunately the proof is case-by-case. (And it is not yet checked for all exceptional types.)

Noncrossing Parking Functions

For more on noncrossing parking functions see my paper with Brendon Rhoades and Vic Reiner:

Parking Spaces (2012), `http://arxiv.org/abs/1204.1760`

Vielen Dank!

picture by +Drew Armstrong and +David Roberts