# Noncrossing partitions and reflection discriminants

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Let $V$ be complex vector space of finite dimension $n$.

**Definition**

A *reflection* in $V$ is a finite-order element $r \in \mathrm{GL}(V)$ such that $\operatorname{codim} \ker(r-1) = 1$.
A *(finite) reflection group* in $V$ is a (finite) subgroup $W \subseteq \mathrm{GL}(V)$ that is generated by reflections. For a reflection group $W$:

- we denote by $R$ the set of all reflections in $W$
- if $W$ can be generated by a subset of $R$ of cardinal $\dim V/V^W$, we say that $W$ is *well-generated*.

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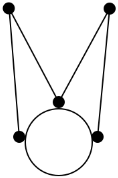
- Basic example of a well-generated complex reflection: $\mathfrak{S}_n$ in its permutation representation ("type $A_{n-1}$").
- By Coxeter theory, all real reflection groups are well-generated.
- Many interesting non-real complex reflection groups are well-generated.
- Non-well-generated example: $G_{31}$, a reflection group of rank 4 that cannot be generated by 5 reflections.



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Besides its use to classify real reflection groups, Coxeter theory provides:

- canonical generator sets and canonical presentations for real reflection groups,
- associated geometric objects (walls, chambers, galleries,...),
- fantastic homotopy-theoretic properties of the space of reduced decompositions, with corresponding homotopy-theoretic results about

$$V - \bigcup_{r \in R} \ker(r - 1).$$

**Conjecture (Brieskorn and ???, early 1970s)**

*Let* $W \subseteq \mathrm{GL}(V)$ *be a finite complex reflection group. Then* $V - \bigcup_{r \in R} \ker(r-1)$ *is a* $K(\pi, 1)$ *space.*

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**Theorem (D.B., 2006, to appear in Annals of Math.)**

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Understanding the well-generated case is at the core of proof.

Key ingredient: replace the "walls & chambers" geometry by new objects, whose combinatorics are controlled by noncrossing partitions.

## Definition

Let $W \subseteq \mathrm{GL}(V)$ be a finite complex reflection group.
The *discriminant* of $W$ is the algebraic hypersurface

$$\mathcal{H} \subseteq W \backslash V$$

defined as the image of $\bigcup_{r \in R} \ker(r-1)$ under the quotient map $V \mapsto W \backslash V$.

$$\begin{array}{ccc}
\bigcup_{r \in R} \ker(r-1) & \hookrightarrow & V \\
\big\downarrow & & \big\downarrow \\
\mathcal{H} & \hookrightarrow & W \backslash V
\end{array}$$

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**Theorem (Shephard-Todd)**

*As an algebraic variety, $W\backslash V$ is an affine space of dimension $n$.*

**Theorem (Steinberg)**

*The restriction of $V \mapsto W\backslash V$ to $V^{\mathrm{reg}} := V - \bigcup_{r \in R} \ker(r-1)$ is an unramified covering.*

In other words, $\mathcal{H}$ is the branch locus of the quotient map $V \mapsto W\backslash V$.

Invariant theory is the algebraic way to study the quotient space. As an algebraic variety, $V$ can be recovered from

$$\mathbb{C}[V] \simeq \mathbb{C}[X_1, \ldots, X_n]$$

(algebra of polynomial functions on $V$):

$$V = \operatorname{Spec} \mathbb{C}[V]$$

Similarly,

$$W \backslash V := \operatorname{Spec} \mathbb{C}[V]^W.$$

Shephard-Todd's theorem implies that, for any complex reflection group,

$$\mathbb{C}[V]^W \simeq \mathbb{C}[X_1, \ldots, X_n].$$

### Definition

A *system of basic invariants* for a complex reflection group $W \subseteq \mathrm{GL}(V)$ is a tuple $(f_1, \ldots, f_n)$ of algebraically independent generators of $\mathbb{C}[V]^W$ such each $f_i$ is homogeneous of degree $d_i$ and $d_1 \leq \cdots \leq d_n$.

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All complex reflection groups admit systems of basic invariants. Choosing one amounts to choosing an explicit isomorphism

$$\mathbb{C}[V]^W \xrightarrow{\sim} \mathbb{C}[X_1, \dots, X_n]$$

and an explicit isomorphism

$$W \backslash V = \operatorname{Spec} \mathbb{C}[V]^W \xrightarrow{\sim} \operatorname{Spec} \mathbb{C}[X_1, \dots, X_n] = \mathbb{C}^n.$$

**Theorem**

*Let $W \subseteq \mathrm{GL}(V)$ be an irreducible complex reflection group. The following assertions are equivalent:*

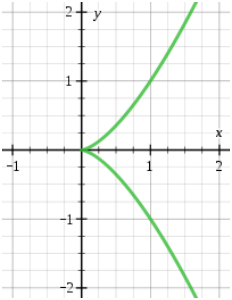
(1) *$W$ is well-generated*

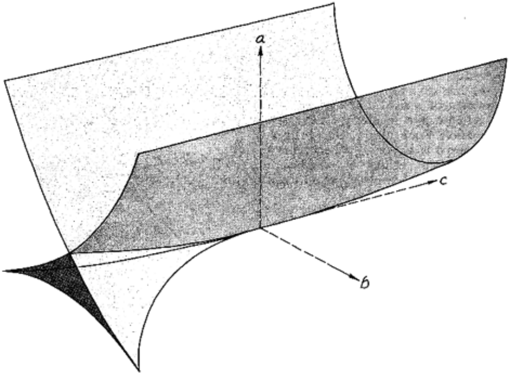
(2) *there exists a system of basic invariants such that the equation of the discriminant $\mathcal{H} \subseteq W\backslash V$ is of the form:*

$$X_n^n + \alpha_2(X_1, \ldots, X_{n-1})X_n^{n-2} + \cdots + \alpha_n(X_1, \ldots, X_{n-1}) = 0$$

Example: type $A_2$ reflection group ($\mathfrak{S}_3$). The discriminant equation can be written:

$$X_2^2 - X_1^3 = 0$$

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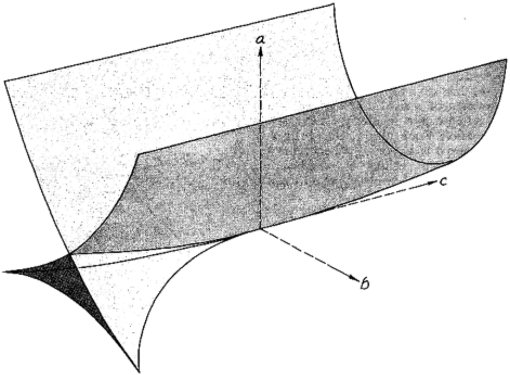
**Definition (audience: please interrupt me, this definition isn't valid)**

The *braid group* $B(W)$ of $W$ is the fundamental group of $W\backslash V - \mathcal{H} = W\backslash V^{\text{reg}}$.

When $W$ is real, then $B(W)$ is isomorphic to the associated Artin group $A(W)$.
The unramified cover $V^{\text{reg}} \twoheadrightarrow W\backslash V^{\text{reg}}$ yields an exact sequence:

$$1 \longrightarrow \pi_1(V^{\text{reg}}) \longrightarrow B(W) = \pi_1(W\backslash V^{\text{reg}}) \longrightarrow W \longrightarrow 1.$$




How do you choose a basepoint?

**Definition (fat basepoint trick)**

Let $X$ be a topological space. Let $\mathcal{U}$ be a contractible subspace of $X$. Let $\pi_1(X)$ be the fundamental groupoid of $X$.
The *fundamental group* of $X$ with respect to the “fat basepoint” $\mathcal{U}$ is defined as the transitive limit

$$\pi_1(X,\mathcal{U}) := \varinjlim_{u,v\in\mathcal{U}} \mathrm{Hom}_{\pi_1(X)}(u,v)$$

for the transitive system of isomorphisms given by homotopy classes of paths within $\mathcal{U}$.

If you don't like transitive limits, just remember this:

- any path starting in $u \in \mathcal{U}$ and ending in $v \in \mathcal{U}$ represents an element of $\pi_1(X, \mathcal{U})$
- if your intuition requires you to really see a loop, draw a path within $\mathcal{U}$ connecting $u$ and $v$
- the product of an element represented by a path with endpoints $u, v \in \mathcal{U}$ with an element represented by a path with endpoints $u', v' \in \mathcal{U}$ is well-defined
- if your intuition requires you to see this product as concatenation, draw a path within $\mathcal{U}$ connecting $v$ and $u'$
- because $\mathcal{U}$ is contractible, all the paths you can draw within $\mathcal{U}$ are homotopic

$$\Delta_f = X_n^n + \alpha_2(X_1, \ldots, X_{n-1})X_n^{n-2} + \cdots + \alpha_n(X_1, \ldots, X_{n-1})$$

Let $Y = \operatorname{Spec} \mathbb{C}[X_1, \ldots, X_{n-1}]$ and let us identify

$$W \backslash V \simeq Y \times \mathbb{C}$$

We can rewrite $\Delta_f$ as:

$$\Delta_f = X_n^n + \alpha_2(Y)X_n^{n-2} + \cdots + \alpha_n(Y).$$

This formula can be viewed as a map from $Y$ to the space $E_n$ of monic degree $n$ one-variable polynomials whose degree $n-1$ coefficient is 0.

The space $E_n$ is itself the orbit space of a type $A$ reflection group: for $\mathfrak{S}_n$ acts irreducibly on the hyperplane $H$ of equation $\sum_i X_i = 0$ in $\mathbb{C}$, and $E_n$

$$\begin{array}{rcl} H & \longrightarrow & E_n \\ (x_1, \ldots, x_n) & \longmapsto & (X - x_1) \ldots (X - x_n) \\ & & = X^n + \sigma_2 X^{n-2} - \sigma_3 X^{n-3} + (-1)^n \sigma_n \end{array}$$

In other words: $E_n$ is the space of centered configurations of $n$ points in a plane.

### Definition (Lyashko-Looijenga)

The *Lyashko-Looijenga* morphism associated with an irreducible well-generated complex reflection group $W$ is the morphism

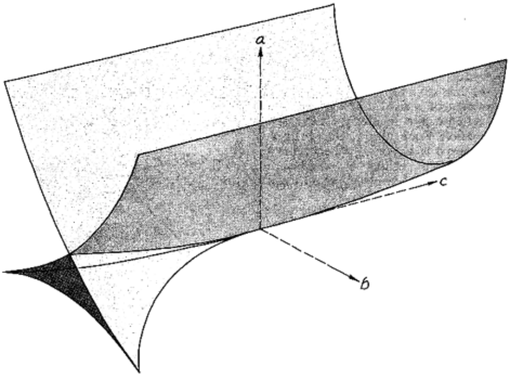
$$\mathrm{LL} : Y \longrightarrow E_n$$

associated with the discriminant equation

$$\Delta_f = X_n^n + \alpha_2(Y) X_n^{n-2} + \cdots + \alpha_n(Y).$$

This depends on the choice of a system of basic invariants.
It is a non-Galois algebraic covering of degree

$$\frac{n! d_n^n}{|W|}.$$



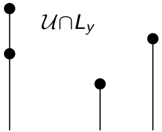

In $W\backslash V \simeq Y \times \mathbb{C}$, consider a "vertical" line $L_y$ obtained fixing the first $n-1$ coordinates.
The intersection $L_y \cap \mathcal{H}$ is the multiset $\{x_1, \ldots, x_n\}$ of roots of $\Delta_f$ at $y$. Set-theoretically,

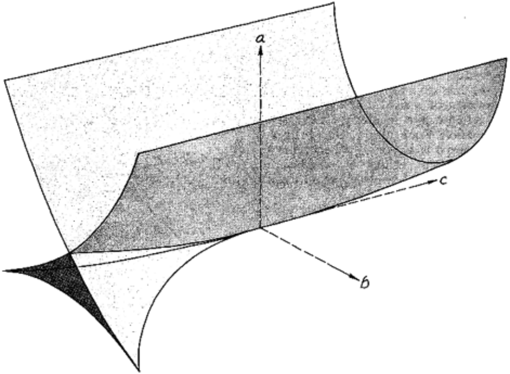
$$\mathrm{LL}(y) = \{x_1, \ldots, x_n\}$$

## Definition

The *standard fat basepoint* for an irreducible well-generated complex reflection group $W$ is the subspace $\mathcal{U} \subseteq V \backslash W - \mathcal{H}$ defined by:

$$\mathcal{U} := \{(y, x) \in Y \times \mathbb{C} | x \text{ is not below any point in } \mathrm{LL}(y)\}.$$

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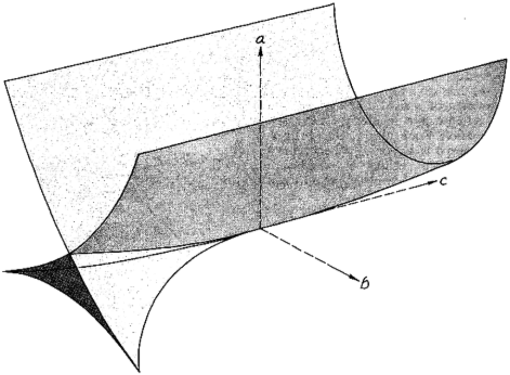



**Lemma**

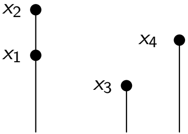
*The standard fat basepoint $\mathcal{U}$ is contractible.*

**Definition**

The braid group of $W$ is

$$B(W) := \pi_1(W\backslash V - \mathcal{H}, \mathcal{U}).$$

Let $y \in Y$. The configuration $\mathrm{LL}(y)$ contains $k$ distinct points (with $1 \leq k \leq n$) that we can order lexicographically $(x_1, \ldots, x_k)$:



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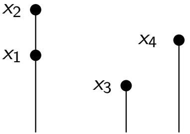
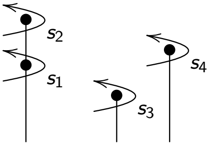



The *reduced label* of $y$ is the sequence $\mathrm{rlbl}(y) = (s_1, \ldots, s_k) \in W^*$ obtained by mapping via $B(W) \to W$ the elements of $B(W)$ corresponding to the paths:



## Definition

The reduced label of 0 consists of a single element, which we denote by $c$. It is the *Coxeter element* of $W$. We set:

$$\mathrm{D}_k(c) := \{(s_1, \ldots, s_k) \in W^k | c = s_1 \ldots s_k \text{ and } l_R(c) = \sum_i l_R(s_i)\}.$$

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## Lemma

*For all $y \in Y$.*

(i) $\mathrm{rlbl}(y) \in \mathrm{D}_\bullet(c)$

(ii) *for all $i$, the reflection length $l_R(s_i)$ coincides with the multiplicity of $x_i$ in $LL(y)$.*

## Definition

Let $x \in E_n$, let $(s_1, \dots, s_k) \in \mathrm{D}_\bullet(c)$. We say that $x$ and $(s_1, \dots, s_k)$ are *compatible* if $x$ contains $k$ distinct points and, for all $i$, the multiplicity of the $i$-th point in $x$ (for the lexicographic ordering) coincides with $l_R(s_i)$.

We denote by

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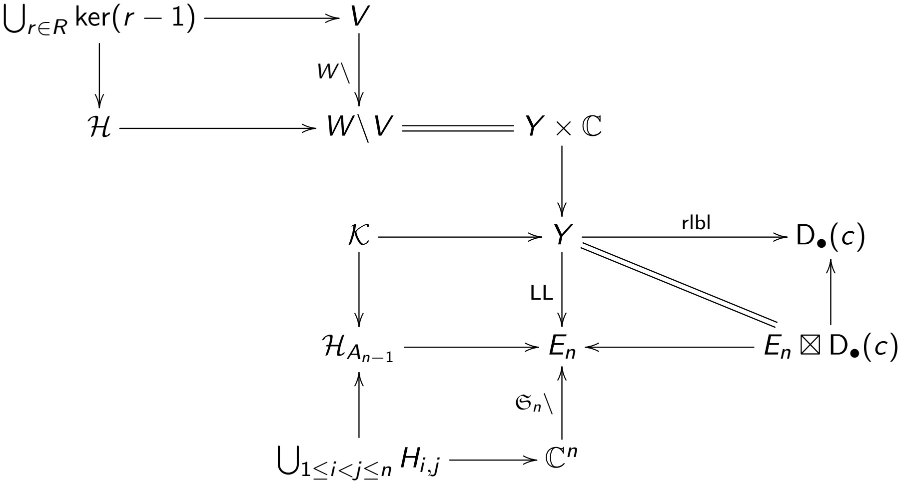
## Theorem (That's the nicest theorem of the talk!)

*The map* $\mathrm{LL} \times \mathrm{rlbl}$ *induces a bijection*

$$\mathrm{LL} \times \mathrm{rlbl} : Y \xrightarrow{\sim} E_n \boxtimes \mathrm{D}_\bullet(c)$$


$\bigcup_{r\in R}\ker(r-1)$
$V$
$W\backslash$
$\mathcal{H}$
$W\backslash V$
$Y\times\mathbb{C}$
$\mathcal{K}$
$Y$
rlbl
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LL
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$\mathfrak{S}_n\backslash$
$\bigcup_{1\leq i<j\leq n}H_{i,j}$
$\mathbb{C}^n$

All elements in sequences in $\mathrm{D}_\bullet(c)$ lie in $NCP(W)$, the lattice of noncrossing partitions of type $W$:
Let $(s_1, s_2) \in \mathrm{D}_2(c)$. The condition $l_R(s_1) + l_R(s_2) = l_R(c)$ implies that $s_1$ lies in $NCP(W)$.

There is a 1-to-1 correspondence

$$(s_1, s_2, \ldots, s_k) \longmapsto 1 \leq s_1 \leq s_1 s_2 \leq \cdots \leq s_1 s_2 \ldots s_k = c$$

between $\mathrm{D}_\bullet(c)$ and the set of chains in $NCP(W)$.

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- face operators

$$(s_1, \dots, s_i, s_{i+1}, \dots, s_k) \mapsto (s_1, \dots, s_i s_{i+1}, \dots, s_k)$$

- degeneracy operators:

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**Hurwitz action**: for each $k$, the braid group $B_k$ acts on $\mathrm{D}_k(c)$ by

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**...and an extra cyclic operator**

$$(s_1, s_2, \ldots, s_k) \mapsto (s_2, \ldots, s_k, s_1^c)$$

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The **simplicial set structure** on $\mathrm{D}_\bullet(c)$

- is closely related to the homotopy-theory of $W\backslash V - \mathcal{H}$ (there exists a natural **geometric realization** functor from the category of simplicial sets to the category of topological spaces)
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The **Hurwitz structure** on $\mathrm{D}_\bullet(c)$ captures all the information, stratum-by-stratum, about the *monodromy theory* of the Lyashko-Looijenga morphism.
The combination of a **simplicial set structure** and a compatible **Hurwitz structure** is what is needed to fully understand a ramified covering. I have found adequate axioms and theory of this generic situation.

Together, the **simplicial set structure** on $\mathrm{D}_\bullet(c)$ and the compatible *cyclic operator* are (a minor generalization) of a **cyclic set structure**, in the sense of Connes.

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$$\mathbb{C}^*\text{-action on } W\backslash V \text{ and } W\backslash V - \mathcal{H} \longleftrightarrow \text{cyclic structure on } \mathrm{D}_\bullet(c)$$

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$$\text{study of } (W\backslash V - \mathcal{H})^{\mu_d} \longleftrightarrow \text{cyclic sieving phenomenon on } \mathrm{D}_\bullet(c)$$

THANKS!