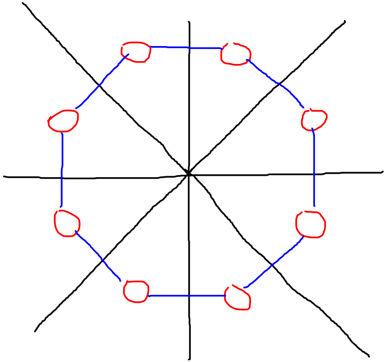
Non-crossing partition lattices and Milnor fibres

Tom Brady Mike Falk Ben Quigley Colum Watt

Bielefeld, 2014

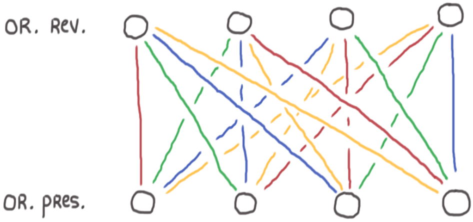
Dihedral Groups I

Cayley graph with the simple reflections as a generating set.



Dihedral Groups II

Cayley graph with **all** reflections as generating set?

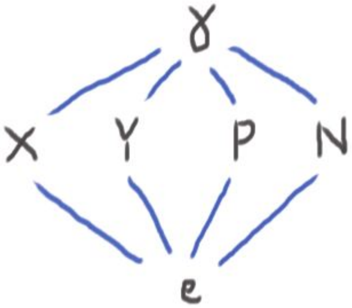


Result is $K_{n,n}$ graph.

This has homotopy type of Milnor fibre of complexified arrangement.

Non-crossing partitions

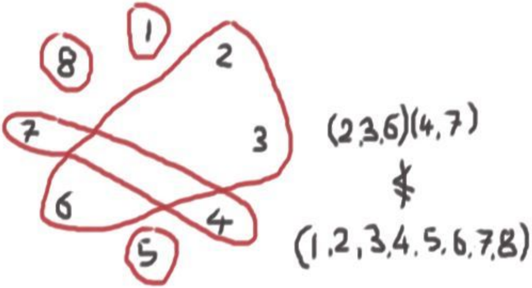
W finite real reflection group of rank n
$|w|$ = total reflection length
partial order: $w_1 \leq w_2 \Leftrightarrow |w_2| = |w_1| + |w_1^{-1} w_2|$.
$(abcd = bda^{bd} c^d)$
γ is a fixed Coxeter element.
NCP= elements in $[e, \gamma]$ (lattice under $\leq$)



Dihedral NCP lattice

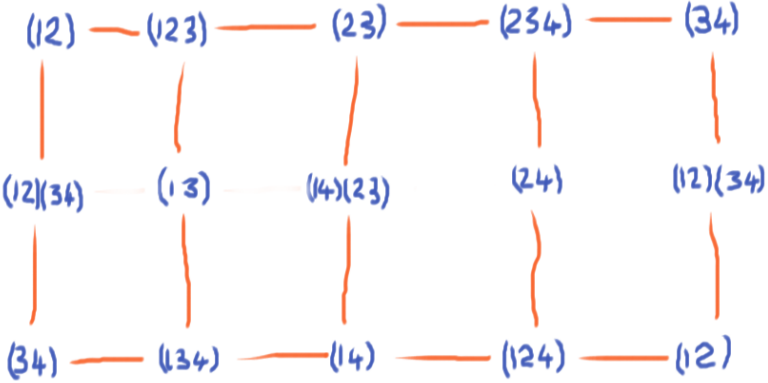
Classical Non-crossing Partitions

Example: $W = \Sigma_{n+1}$
reflections are transpositions
$\gamma = (1, 2, \ldots, n, n+1)$, an $(n+1)$-cycle
$w \leq \gamma$ if and only if
cyclic order in blocks is that induced by γ and
blocks of w are non-crossing.



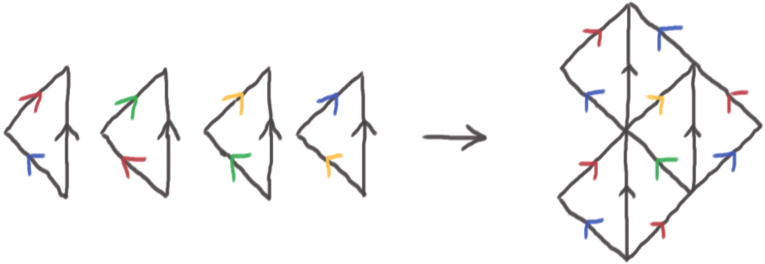
Crossing partition

Proper part of Σ_4 NCP lattice



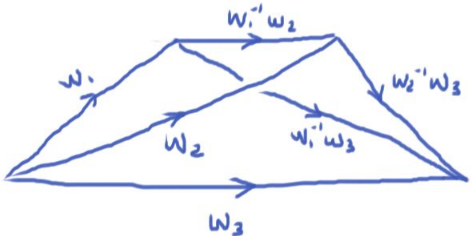
Braid group of W

$B(W)$ is the group with
generating set $\{[w] \mid w \in NCP, w \neq e\}$ and
relations $[w_1][w_1^{-1}w_2] = [w_2]$ whenever $w_1 < w_2$.
Example: $W = D_4$, dihedral group of square.



The classifying space K

Can extend presentation 2-complex of $B(W)$ by filling in higher dimensional cells for longer chains.
$w_1 < w_2$ and $w_2 < w_3$ implies $w_1 < w_3$ but also
$w_1^{-1}w_2 < w_1^{-1}w_3$.



3-cell

Resulting complex, K, is a $K(B(W), 1)$!

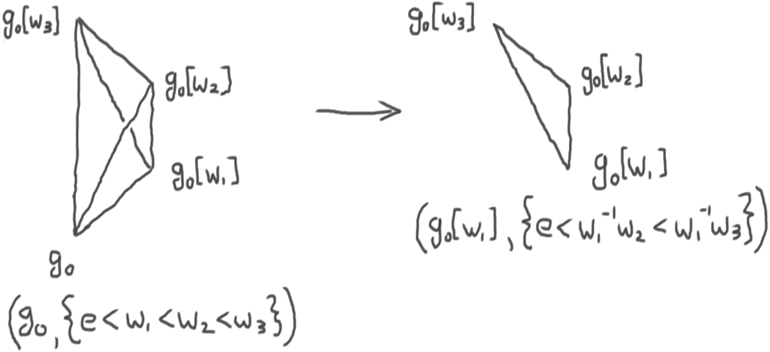
Universal cover of K

Let X be universal cover of K.
This is a simplicial complex with vertex set $B(W)$, the braid group.
It has a k-cell on $\{g_0, g_1, \ldots, g_k\}$ if
$g_i = g_0[w_i]$ for $1 \leq i \leq k$ and $e < w_1 < w_2 < \cdots < w_k$ in NCP.
Write this cell as $(g_0, \{e < w_1 < w_2 < \cdots < w_k\})$.
Of course $B(W)$ acts on X by left multiplication:
$g \cdot (g_0, \{e < w_1 < \cdots < w_k\}) = (gg_0, \{e < w_1 < \cdots < w_k\})$.

Here $(g_0, \{e < w_1 < w_2 < \cdots < w_k\})$ has facets
$(g_0, \{e < w_1 < w_2 < \cdots < \widehat{w}_i < \cdots < w_k\})$ for $1 \leq i \leq k$
and a 'top facet'

$$(g_0[w_1], \{e < w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_k\})$$

Cells of X


g0[w3]
g0[w2]
g0[w1]
g0
(g0, {e < w1 < w2 < w3})
g0[w3]
g0[w2]
g0[w1]
(g0[w1], {e < w1^-1 w2 < w1^-1 w3})

Intermediate covers of K

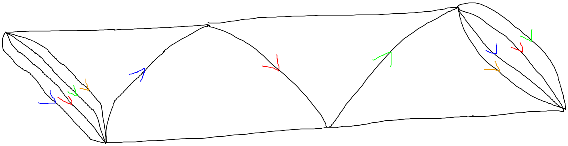
If $H \lhd B(W)$ we can construct an intermediate cover $H\backslash X$
vertex set: $H\backslash B(W)$
cells : $(Hg, \{e < w_1 < w_2 < \cdots < w_k\})$
$H\backslash B(W)$ action
If $H = \ker(\phi)$ for $\phi : B(W) \twoheadrightarrow G$ identify Hg with $\phi(g)$, giving a G action on $H\backslash X$.

$(\phi(g), \{e < w_1 < w_2 < \cdots < w_k\})$ has 'top facet'

$$(\phi(g)\phi([w_1]), \{e < w_1^{-1}w_2 < w_1^{-1}w_3 < \cdots < w_1^{-1}w_k\})$$

Milnor fibre for discriminant

Example: (Milnor fibre of discriminant.)
$\phi : B(W) \twoheadrightarrow \mathbb{Z} : [w] \mapsto |w|$.
$\ker(\phi)\backslash X$ cells $(n, \{e < w_1 < w_2 < \cdots < w_k\}), n \in \mathbb{Z}$.
$(n, \{e < w_1 < w_2 < \cdots < w_k\})$ has top facet
$(n + |w_1|, \{e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\})$



Retraction of cover

Definition: We define N to be the finite subcomplex of $\ker(\phi)\backslash X$ consisting of the cells of the form

$$(m, \{e < w_1 < w_2 < \cdots < w_k\}), 0 \leq m < n - |w_k|$$

Note: We observe that N is the union of cells of the form

$$(0, \{e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1}\}),\ |w_{n-1}| = n - 1$$

and their faces. In particular, N is $(n-1)$-dimensional.

Proposition: The subcomplex N is a strong deformation retract of $\ker(\phi)\backslash X$.

Lemma: If $|w_k| < n$ then, in $H\backslash X$, the cell $(Hg, \{e < w_1 < w_2 < \cdots < w_k\})$ is a facet of precisely two cells of form $(Hg', \{e < u_1 < u_2 < \cdots < u_k < \gamma\})$.
They are $(Hg, \{e < w_1 < w_2 < \cdots < w_k < \gamma\})$ and

$$(Hg[\gamma w_k^{-1}]^{-1}, \{e < \gamma w_k^{-1} < \gamma w_k^{-1} w_1 < \cdots < \gamma w_k^{-1} w_{k-1} < \gamma\})$$

Full Milnor fibre

Example: (Milnor fibre.) $\theta : B(W) \to \mathbb{Z} \times W : [w] \mapsto (|w|, w)$.
cells of $\ker(\theta)\backslash X$: $((n, w), \{e < w_1 < w_2 < \cdots < w_k\})$,
for $n \in \mathbb{Z}, w \in W$ with $\text{parity}(w) = \text{parity}(n)$.

$((n, w), \{e < w_1 < w_2 < \cdots < w_k\})$ has top facet
$((n + |w_1|, ww_1), \{e < w_1^{-1}w_2 < \cdots < w_1^{-1}w_k\})$

Definition: We define M to be the finite subcomplex of $\ker(\theta)\backslash X$ consisting of the cells of the form
$((m, w), \{e < w_1 < w_2 < \cdots < w_k\})$,
where $0 \leq m < n - |w_k|$ and $\text{parity}(w) = \text{parity}(m)$.

Note: We observe that M is the union of cells of the form

$$((0, w), \{e \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_{n-1}\}), \ |w_{n-1}| = n - 1, w \in W_+$$

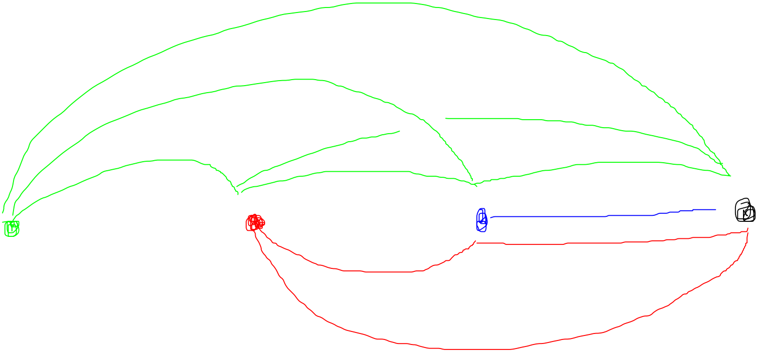
and their faces.

Proposition: The subcomplex M is a strong deformation retract of $\ker(\theta)\backslash X$.

Structure of Milnor fibre for discriminant

Define the following subcomplexes of N:

$$N_i = \{\tau \mid \tau = (j, \sigma) \text{ where } j \geq i\}.$$



Structure of Milnor fibre for discriminant II

Each cell in $N_i \setminus N_{i+1}$ has the form

$$(i, \{e < w_1 < \cdots < w_k\}), \text{ with } i + |w_k| < n.$$

N_i has the structure of the mapping cone of a map

$$\phi_i : |NCP_{[1,n-i-1]}| \to N_{i+1}$$

where the cone point corresponds to the vertex $(i, \{e\})$ of N_i.

The corresponding filtration $\{N_0 \supset N_1 \supset \cdots \supset N_{n-1}\}$ of N can be used to compute the homology of N. This is possible since each quotient space N_i/N_{i+1} has the homotopy type of the suspension of $|NCP_{[1,n-i-1]}|$ while $|NCP_{[1,n-i-1]}|$ in turn has the homotopy type of a wedge of spheres.