Groupoids with weak orders

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Definition of weak order on Coxeter groups

- (W, S): Coxeter system e.g. W a finite Weyl group.
- $\Phi \supseteq \Phi^+$: (real, reduced) root system and positive roots.
- $T := \{ wsw^{-1} \mid w \in W, s \in S \} = \{ s_\alpha \mid \alpha \in \Phi \}$ reflections.
- $W \times \{\pm 1\} \circlearrowright \Phi$, $\quad \Phi_+ \xrightarrow{\cong} \Phi/\{\pm 1\} \cong T \circlearrowleft W$.
- $l(w) := \min\{ n \in \mathbb{N} \mid w \in S^n \}$. ($l$ is the standard length function.)

- For $w \in W$, $\Phi_w := \Phi^+ \cap w(-\Phi^+)$ (inversions of w).
- Weak order: $x \leq y \iff l(y) = l(x) + l(x^{-1}y) \iff \Phi_x \subseteq \Phi_y$.

- $|\Phi_w| = l(w)$.
- Maximal chains from 1 to w in $(W, \leq)$ correspond naturally to reduced expressions of w.

The Boolean ring $\mathcal{P}(T)$ and cocycle $N\colon W \to \mathcal{P}(T)$

- Boolean ring R: $r^2 = r$ for all $r \in R$.
- R is partially ordered: $r \leq s \iff rs = r$.
- For any set X, $\mathcal{P}(X) := \{\, A \mid A \subseteq X \,\}$ is a Boolean ring: $A + B := (A \cup B) \setminus (A \cap B)$, $AB := A \cap B$, $A \leq B \iff A \subseteq B$.
- Stone's theorem: any Boolean ring R is canonically isomorphic to a subring of $\mathcal{P}(X)$ where $X = \mathrm{Spec}(R)$.

- Define $\widehat{\Phi}\colon W \to \mathcal{P}(\Phi)$ by $\widehat{\Phi}(w) := \Phi_w \,\dot{\cup}\, -\Phi_w = \Phi^+ + w(\Phi^+)$.
- $\widehat{\Phi} \in B^1(W, \mathcal{P}(\Phi)) \subseteq Z^1(W, \mathcal{P}(\Phi))$ is a 1-cocycle.

- Define $N\colon W \to \mathcal{P}(T)$ by $N(w) := \{\, s_\alpha \mid \alpha \in \widehat{\Phi}(w) \,\}$.
- $N \in Z^1(W, \mathcal{P}(T))$, $N(w) = \{\, t \in T \mid l(tw) < l(w) \,\}$.

Properties of weak order and the reflection cocycle

Theorem

(a) $N\colon W \to \mathcal{P}(T)$ *is a* 1*-cocycle i.e.* $N(xy) = N(x) + x \cdot N(y)$ *for all* $x, y \in W$*, where* $x \cdot N(y) := xN(y)x^{-1}$*.*

(b) $x \leq y \iff N(x) \subseteq N(y)$*.*

(c) $(W, \leq)$ *is a complete meet semilattice (Björner); it is an even an order ideal (downset) in some complete ortholattice.*

(d) *Intervals in* $(W, \leq)$ *are finite;* S *is the set of atoms of* W*.*

(e) *For* $s \in S$ *and* $w \in W$*, either* $N(s) \subseteq N(w)$ *or* $N(s) \cap N(w) = \emptyset$*.*

Each of the following determines the others up to isomorphism:

$$(W, S) \leftrightsquigarrow (W, I) \leftrightsquigarrow (W, \Phi, \Phi_+) \leftrightsquigarrow (W, \mathcal{P}(T), N) \leftrightsquigarrow (W, \leq).$$

Definition

A groupoid-preorder (GP) is a pair $(G, \leq)$ where

(i) G is a groupoid. ${}_aG_b := \mathrm{Hom}_G(b, a)$, ${}_aG := \dot{\bigcup}_b\, {}_aG_b$ (left star).

(ii) $\leq$ is a preorder on $\mathrm{mor}(G)$ such that morphisms in distinct left stars are incomparable.

The restriction ${}_a\!\leq$ of $\leq$ to ${}_aG$ is called the weak preorder of G at a.

Definition

A protorootoid is a triple $\mathcal{R} = (G, \Lambda, N)$ where

(i) G is a groupoid and $\Lambda\colon G \to \mathbf{BoolRing}$ is a functor.

(iii) $N \in Z^1(G, \Lambda)$ is a 1-cocycle i.e for $a \in \mathrm{ob}(G)$ and $g \in {}_aG$, there is given $N_g = N(g) \in \Lambda(a)$ satisfying $N_{gh} = N_g + (\Lambda(g))(N_h)$.

The underlying GP of $\mathcal{R}$ is $(G, \leq)$ where for $g, h \in {}_aG$, one has $g \leq h$ if $N_g \leq N_h$ in $\Lambda(a)$.

Protorootoidal groupoid-preorders (PGPs)

A GP $(G, \leq)$ is protorootoidal (a PGP) if it is the underlying GP of a protorootoid.

PGP's may be alternatively characterized as GPs coming from a signed groupoid set (G, Φ, Φ^+) via $g \leq h$ if $\Phi_g \subseteq \Phi_h$. We have:

$$(G, \leq) \rightsquigarrow (G, \Lambda, N) \rightsquigarrow (G, \Phi, \Phi^+) \rightsquigarrow (G, \Lambda', N') \rightsquigarrow (G, \leq')$$

(natural constructions). If $(G, \leq)$ is a PGP, then $(G, \leq') = (G, \leq)$.

Theorem

Let $R = (G, \leq)$ be a GP. Then R is a PGP iff for all $a \in \mathrm{ob}(G)$ and $g_l \in {}_aG_{\delta_l}$ ($l = 1, \ldots, n$) one has $g_i^{-1}g_j \,{}_{\delta_i}\!\leq g_i^{-1}g_k$ if ($f(i) = f(j)$ or $f(j) = f(k)$) for all functions $f \colon \{1, \ldots, n\} \to \{0, 1\}$ such that ($f(p) = f(q)$ or $f(q) = f(r)$) whenever $g_p^{-1}g_q \,{}_{\delta_p}\!\leq g_p^{-1}g_r$ ($1 \leq p, q \leq r$).

Example

Writing $g^* := g^{-1}$ for $g \in \text{mor}(G)$, a PGP $(G, \leq)$ satisfies:

- If $f \leq ff^*$ then $f^* \leq f^*g$.
- If $f \leq fg$ and $f^* \leq g$ then $f \leq ff^*$.
- If $f \leq fg$ then $g^* \leq g^*f^*$.
- If $f \leq fg$ and $fg \leq fgh$ then $f \leq fgh$ and $g \leq gh$.
- If $f \leq fg$, $g \leq gh$ and $fgh \leq fghk$ then $g \leq ghk$.
- If $f \leq fg$, $f \leq fh$ and $g^*k \leq g^*h$ then $f \leq fk$.

- For any GP $(G, \leq)$ define an "orthogonality relation" ${}_a\perp$ (or $\perp$) on ${}_aG$ by $g\ {}_a\!\perp h$ if $g^* \leq g^*h$.
- For a *PGP*, $g\ {}_a\!\perp h \iff h\ {}_a\!\perp g \quad (\iff N_gN_h = 0 \text{ in } \Lambda(a))$.

Rootoidal groupoid-preorders (RGPs)

Definition

A PGP $(G, \leq)$ is rootoidal (a RGP) if (i)-(ii) hold for all $a \in \text{ob}(G)$:

(i) $({}_aG, {}_a\leq)$ is a complete meet semilattice.

(ii) JOP (join orthogonality property): Let $(g_i)_{i \in I}$ be a family in ${}_aG$ with an upper bound and set $g := \bigvee_i g_i \in {}_aG$. If $h \in {}_aG$ with $g_i \perp h$ for all i, then $g \perp h$.

A protorootoid is a rootoid if its underlying GP is a RGP.

Theorem

A PGP $(G, \leq)$ is a RGP iff for all $a \in \text{ob}(G)$, $({}_aG, {}_a\leq)$ is embeddable as an order ideal of a complete ortholattice.

(A complete ortholattice is a complete lattice P with an order-reversing map $x \mapsto x^{\complement}$ satisfying $(x^{\complement})^{\complement} = x$, $x^{\complement} \vee x = 1_P$ and $x \wedge x^{\complement} = 0_P$ for all $x \in P$, where 1_P and 0_P are the top and bottom elements of P.)

Principal RGPs and their braid presentations

Definition

A PRGP (principal RGP) is a RGP such that (i)–(ii) below hold:

(i) For all $a \in \mathrm{ob}(G)$, each interval $[g, h]$ in ${}_aG$ is finite.
Notation: ${}_aS := \{\text{atoms of } {}_aG\}$ and $S := \dot\cup_{a\in\mathrm{ob}(G)}\, {}_aS$.

(ii) For all $a \in \mathrm{ob}(G)$, $s \in {}_aS$ and $g \in {}_aG$, one has either $s \;{}_a\!\leq g$ or $s \;{}_a\!\perp g$.

Theorem

Let $(G, \leq)$ be a PRGP and S be as above. Then

(a) $\langle S \rangle = G$, $S = S^*$ *(the atomic generators). Set $l := l_S$ to be the corresponding length function.*

(b) *Braid presentation:* $G \cong \langle\, S \mid \text{trivial relations, braid relations} \,\rangle$.

- Trivial relations: $r = s^{-1}$ for each $r, s \in S$ with $r = s^*$.
- Braid relations: Let $a \in \mathrm{ob}(G)$ and $r \neq s \in {}_aS$ such that $z := r \vee s$ exists. Set $m := l(z)$. There are unique reduced S-expressions $z = r_1 \cdots r_m$ with $r_1 = r$, and $z = s_1 \cdots s_m$ with $s_1 = s$. The "braid relation" corresponding to (a, r, s) is $r_1 \cdots r_m = s_1 \cdots s_m$.

Theorem

(a) *If $r_1 \cdots r_n$ and $s_1 \cdots s_n$ are reduced S-expressions of $g \in \mathrm{mor}(G)$, they are braid equivalent.*

(b) *If the S-expression $r_1 \cdots r_n$ is not reduced, it is braid equivalent to an S-expression $\cdots rr^* \cdots$.*

- There are minimal ("real reduced") (G, Φ, Φ_+) and (G, Λ, N) so $(G, S) \leftrightsquigarrow (G, l) \leftrightsquigarrow (G, \Phi, \Phi_+) \leftrightsquigarrow (G, \Lambda, N) \leftrightsquigarrow (G, \leq)$.

Definitions of the categories **RGP** and **PRGP**

Definition

The category **RGP** has RGPs as objects. Morphisms $\pi\colon (G,\leq)\to(H,\preceq)$ are functors $\pi\colon G\to H$ such that for all $a\in \mathrm{ob}(G)$

(i) ${}_a\pi\colon {}_aG\to {}_{\pi(a)}H$ preserves meets of non-empty subsets of ${}_aG$ and joins of subsets of ${}_aG$ with an upper bound.

(ii) AOP (adjunction orthogonality property) If $h\in {}_{\pi(a)}H$ and ${}_a\pi^{\perp}(h) := \min(\{\, g\in {}_aG \mid h \,{}_{\pi(a)}{\preceq}\, {}_a\pi(g)\,\})$ is defined, then

$$ {}_a\pi^{\perp}(h)\,{}_a{\perp}\, g \iff h\,{}_{\pi(a)}{\perp}\,{}_a\pi(g), \qquad \text{for all for } g\in {}_aG. $$

Call ${}_a\pi^{\perp}\colon \mathrm{dom}({}_a\pi^{\perp})\to {}_aG$ the partial left adjoint of ${}_a\pi$: for $g\in {}_aG$, $h\in {}_{\pi(a)}H$, $h\,{}_{\pi(a)}{\preceq}\,{}_a\pi(g) \iff h\in \mathrm{dom}({}_a\pi^{\perp})$ and $({}_a\pi^{\perp})(h)\,{}_a{\leq}\, g$.

PRGP is the full subcategory of **RGP** with PRGPs as objects.

Completeness properties of **RGP** and **PRGP**

Theorem

(a) **RGP** *is complete (has all small categorical limits).*

(b) **PRGP** *has all limits from categories with finitely many objects.*

Comments on proof

- **GP** has GPs as objects, morphisms $\pi\colon (G, \leq) \to (H, \preceq)$ are functors so all ${}_a\pi$ are order preserving. **PGP** is defined similarly.
- A limit in **GP** has the limit groupoid with limit weak preorders.
- **RGP** $\to$ **PGP** $\to$ **GP** create limits; **PRGP** $\to$ **RGP** creates limits from a category J if ob(J) is finite.
- In (b), atomic generators for the limit come from partial adjoints; AOP implies the limit is principal.
- AOP is proved using formulae for partial left adjoints as joins.

Definition of squares and cubes in a RPG

Definition

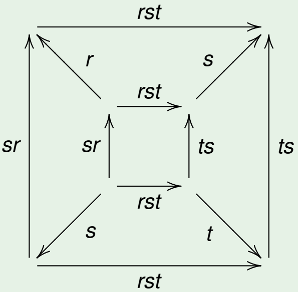
- A square of a RGP $(G, \leq)$ is a square commutative diagram in G such that the two morphisms into each vertex are orthogonal:

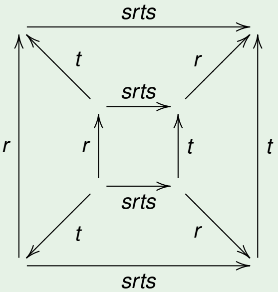
$$\begin{array}{ccc} \cdot & \xleftarrow{\;y\;} & \cdot \\ x\uparrow & & \uparrow w \\ \cdot & \xleftarrow{\;v\;} & \cdot \end{array} \qquad \text{where } yw = xv, x \perp y, y^* \perp w, w^* \perp v^*, v \perp x^*.$$

- Equivalently, $yw = xv$, $y \perp x$ and $(\Lambda(y))(N_w) = N_x$.
- Equivalently, $yw = xv$ and $y(\Phi_w) = \Phi_x$.
- An n-cube of $(G, \leq)$ is an n-cubical diagram in G in which each 2-face is a square of $(G, \leq)$.

Example

$W = S_4$, $S = \{r = (1,2), s = (2,3), t = (3,4)\}$.





- The maximal n for which non-trivial n-cubes exist in $(W, \leq)$ is the maximum rank of a finite parabolic subgroup of (W, S) (non-trivial means there are no identity morphisms on edges).
- (Ferdinands) Every n-cube in a finite Coxeter group is a face of one with long diagonal (the composite from source to sink) w_0.

The RGP of n-cubes

Theorem

Fix a RGP $R = (G, \leq)$. For $n \in \mathbb{N}$, there is a morphism $\pi^{(n)} \colon R^{(n)} \to R$ in **RGP** *such that $R^{(n)} = (G^{(n)}, \leq^{(n)})$, $\mathrm{ob}(G^{(n)}) = \{n\text{-cubes of } R\}$, $\mathrm{mor}(G^{(n)}) = \{(n+1)\text{-cubes of } R\}$,*

$$\mathrm{Hom}(x\!\uparrow, \uparrow\! y) = \{(n+1)\text{-cubes } y \square x\}, \qquad z \square y \circ y \square x = z \square x$$

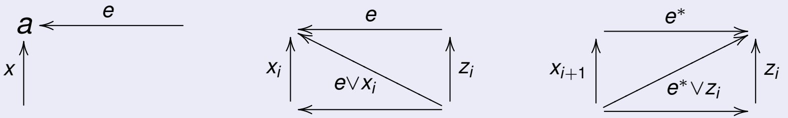
$$\left(y \overset{g}{\square} x \;\; {}_{y}\!\leq^{(n)} \;\; y \overset{h}{\square} z \right) \iff g \,{}_{b}\!\leq h, \quad \pi^{(n)}\left(y \overset{g}{\square} x \right) = \left(b \xleftarrow{g} a \right)$$

where x, y, z are n-cubes with sinks a, b, c indicated by their arrowheads and g, h are morphisms in G between these sinks.

If R is a PRGP, then so is $R^{(n)}$ and $\pi^{(n)}$ is a morphism in **PRGP**.

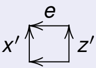
Zig-zag construction

Let $x, e \in {}_aG$ with $x \perp e$. Define (if possible) $x_i, z_i \in \mathrm{mor}(G)$ by $x_0 := x$,

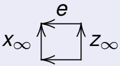


for $i \in \mathbb{N}$, $x_\infty := \bigvee_{i\in\mathbb{N}} x_i$, $z_\infty := \bigvee_{i\in\mathbb{N}} z_i$.

Theorem

There is a square $(*)$: 

with $x\, {}_a\!\leq x'$ iff x_∞ and z_∞ are defined.

In that case, 

is a square $(*)$ with $x\, {}_a\!\leq x_\infty$ and any square $(*)$ with $x\, {}_a\!\leq x'$ satisfies $x_\infty\, {}_a\!\leq x'$ and $z_\infty\, {}_b\!\leq z'$ where $e^* \in {}_bG$.

Subgroupoid of functor groupoid defined with squares

- Let $R = (G, \leq)$ be a RGP, H be a groupoid with G, H connected.
- The functor category G^H is a groupoid. Objects: functors $H \to G$. For functors $\alpha, \beta \colon H \to G$, a morphism $\nu \colon \alpha \to \beta$ is a natural transformation ν i.e. $\nu = (\nu_a : \alpha(a) \to \beta(a))_{a \in \mathrm{ob}(G)}$ where

$$\begin{array}{ccc} \alpha(a) & \xrightarrow{\alpha(h)} & \alpha(b) \\ {\scriptstyle \nu_a}\downarrow & & \downarrow{\scriptstyle \nu_b} \\ \beta(a) & \xrightarrow[\beta(h)]{} & \beta(b) \end{array}$$

  commutes for all $h \colon a \to b$ in $\mathrm{mor}(H)$.
- $G^H_\square$ is the subgroupoid of G^H with all objects but only morphisms $\nu \colon \alpha \to \beta$ for which the above diagrams are squares of R.

Definition of functor RGPs

- Fix $b \in \mathrm{ob}(H)$ and $F \in \mathrm{ob}(G^H)$. Evaluation at b gives a functor $G^H \to G$ which restricts to a functor $\epsilon\colon K \to G$ where K is the component of $G^H_\square$ containing F. Set $a := F(b) = \epsilon(F) \in \mathrm{ob}(G)$.
- Give K the structure of GP $(K, \preceq)$ by pullback along ϵ: for $k \in \mathrm{ob}(K)$ and $\alpha, \beta \in {}_kK$, let $\alpha \,{}_k{\preceq}\, \beta$ if $\epsilon(\alpha) \,{}_{\epsilon(k)}{\leq}\, \epsilon(\beta)$.

Theorem

- *$T := (K, \preceq)$ is a RGP, called a functor RGP of R.*
- *T depends only on the component K of $G^H_\square$ (not on b).*
- *$\epsilon\colon T \to R$ is a morphism in* **RGP**. *It is independent of choice of b up to natural isomorphism.*
- *If R is a PRGP, then T is a PRGP and ϵ is a morphism in* **PRGP**.

Local embeddings of based connected groupoids

- The category E (of local embeddings of based connected groupoids) has connected groupoids with a specified basepoint as objects and basepoint preserving groupoid homomorphisms which are injective on stars as morphisms. Have $(G,a),(K,F)\in \mathrm{ob}(E)$.
- The category $E/(G,a)$ has morphisms $f'\colon (G',a')\to (G,a)$ of E as objects. A morphism $f'\to f''$ is a commutative diagram in E

$$\begin{array}{ccc} (G',a') & \xrightarrow{f'} & (G,a) \\ {\scriptstyle g}\downarrow & \nearrow_{f''} & \\ (G'',a'') & & \end{array}$$

- The set of isomorphism classes $[f']$ of objects f' of $E/(G,a)$ forms a poset $\Lambda=\Lambda_{G,a}$ under $[f']\le[f'']$ above.

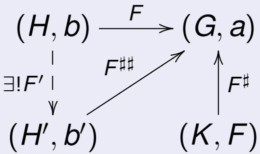
Duality

Theorem

$R = (G, \leq)$ a RGP, H a groupoid with G and H connected, $b \in \mathrm{ob}(G)$, $F\colon H \to G$. Let $a := F(b)$, $\epsilon\colon K \to G$ be as before.

(a) *$F^\sharp := \epsilon\colon (K, F) \to (G, a)$ is an object of $E/(G, a)$, hence so is $F^{\sharp\sharp}$.*

(b) *There is a unique base-point preserving groupoid homomorphism F' making the following diagram commute:*



(c) *The maps $([F] \mapsto [F^\sharp], [F] \mapsto [F^\sharp])$]) restrict to a Galois connection on Λ. The stable classes $([F] = [F^{\sharp\sharp}])$ form a complete lattice.*

Definition of the category **Prtd** of protorootoids

Many results can be formulated equivalently in terms of either GPs, protorootoids or signed groupoid sets (G, Φ, Φ_+); however, signed groupoid sets aren't as convenient for category-theoretic arguments.

Definition

The category **Prtd** of protorooids has protorootoids as objects. A morphism $(G, \Lambda, N) \to (H, \Gamma, M)$ in **Prtd** is a pair (α, ν) where

- $\alpha \colon G \to H$ is a functor.
- $\nu \colon \Lambda \to \Gamma\alpha$ is a natural transformation

$$\begin{array}{ccc} G & \xrightarrow{\Lambda} & \mathbf{BoolRing} \\ {\scriptstyle \alpha}\downarrow & & \Vert \\ H & \xrightarrow{\Gamma} & \mathbf{BoolRing} \end{array}$$

- $M(\alpha(g)) = \nu_a(N_g) \in \Gamma(\alpha(a))$ if $a \in \mathrm{ob}(G)$ and $g \in {}_aH$.

Composition: $(\beta, \mu)(\alpha, \nu) := (\beta\alpha, (\mu\alpha)\nu)$

Abridgement of a protorootoid

Abridgement is a basic construction which produces a “minimal” version of a protorootoid, somewhat analogous to a “real, reduced” root system attached to a more general root system.

- Let $\mathcal{R} = (G, \Lambda, N)$ be a protorootoid.
- Let $\Lambda' \colon G \to \mathbf{BoolRing}$ be the subrepresentation of Λ such that $\Lambda'(a)$ is the subring of $\Lambda(a)$ generated by $\{\, N(g) \mid g \in {}_aG \,\}$.
- Let $N' \in Z^1(G, \Lambda')$ be the restriction of N: $N'(g) := N(g)$, $g \in {}_aG$.
- $\mathfrak{A}(\mathcal{R}) := (G, \Lambda', N')$ is a protorootoid, called the abridgement of $\mathcal{R}$, with the same underlying GP as $\mathcal{R}$.
- There is a morphism $(\mathrm{Id}_G, \iota) \colon \mathfrak{A}(\mathcal{R}) \to \mathcal{R}$ in **Prtd** where $\iota \colon \Lambda' \to \Lambda$ is the natural transformation with component at $a \in \mathrm{ob}(G)$ given by the inclusion $\Lambda'(a) \to \Lambda(a)$.
- Abridgement extends naturally to a functor $\mathfrak{A} \colon \mathbf{Prtd} \to \mathbf{Prtd}$.

Definition of the category **Rtd** of rootoids

- The "underlying GP" construction $(G, \Lambda, N) \rightsquigarrow (G, \leq)$ extends to a functor $F\colon \mathbf{Prtd} \to \mathbf{GP}$.
- The category **Rtd** (resp., **PRtd**) of rootoids (resp., preprincipal rootoids) is the subcategory of **Prtd** containing those objects (morphisms) of **Prtd** such that $F(X)$ is an object (resp., morphism) of **RGP** (resp., **PRGP**).
- The diagram $$\begin{array}{ccc} \mathbf{Rtd} & \to & \mathbf{RGP} \\ \downarrow & & \downarrow \\ \mathbf{Prtd} & \to & \mathbf{GP} \end{array}$$ realizes **Rtd** as a fiber product category $\mathbf{Rtd} = \mathbf{Prtd} \times_{\mathbf{GP}} \mathbf{RGP}$. Similarly $\mathbf{PRtd} = \mathbf{Prtd} \times_{\mathbf{GP}} \mathbf{PRGP}$.

Many general arguments involving rootoids can be decomposed into a (usually tedious but essentially trivial) category-theoretic part involving **Prtd** and a (often non-trivial) order-theoretic part involving **RGP**.

The construction $(G, \leq) \rightsquigarrow (G, \Lambda, N)$.

An example of a good categorical property of **Prtd** in relation to **GP**:

Theorem

The "underlying GP" functor $F\colon \mathbf{Prtd} \to \mathbf{GP}$ *has a left adjoint* $\mathbf{GP} \to \mathbf{Prtd}$ *given on objects by* $(G, \leq) \rightsquigarrow (G, \Lambda, N)$ *where:*

- *For* $a \in \mathrm{ob}(G)$, $\Lambda(a)$ *is the quotient of the free Boolean ring on generators* (g, h) *where* $h \in {}_bG$, $b \in \mathrm{ob}(G)$, $g \in {}_aGb$, *by relations* $(g, h)(g, l) = (g, l)$ *if* $h \,{}_b\!\leq l$, $(g, hk) = (g, h) + (gh, k)$.
- *Denote the image of* (g, h) *in the quotient* $\Lambda(a)$ *by* $[g, h]$.
- $\Lambda(j)([g, k]) := [jg, k]$, $N(g) := (1_a, g)$.
- *So* $[g, h] = (\Lambda(g))(N(h))$.

The underlying GP $(G, \leq')$ *of* (G, Λ, N) *is the coarsest PGP on G such that* $\leq'$ *is finer than* $\leq$ *(given preorders* $\preceq$, $\preceq'$ *on a set* X, $\preceq'$ *is finer than* $\preceq''$, *or* $\preceq''$ *is coarser than* $\preceq'$, *if* $x \preceq'' x' \implies x \preceq' x$*).*

Signed groupoid-sets (G, Φ, Φ^+).

The category **SSet** of definitely signed sets has as objects sets A with a given free action by the group $\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$ and a given set of $\{\pm 1\}$-orbit representatives A^+ (called the positive elements). Morphisms are $\{\pm 1\}$-equivariant functions (not necessarily preserving positive elements) with usual composition.

A signed groupoid-set is a triple (G, Φ, Φ^+) where G is a groupoid, $\Phi\colon G \to$ **SSet** is a functor and for $a \in \mathrm{ob}(G)$, $\Phi^+(a) := (\Phi(a))^+$.

To compare these with protorootoids, introduce the following notation:

- Let $\mathcal{P}\colon$ **Set** $\to$ **BoolRing** be the contravariant power set functor.
- For any groupoid G, let $j_G : G \to G$ be the (contravariant) inversion functor (fixing objects and sending $g \mapsto g^*$ on morphisms).

The construction $(G, \Phi, \Phi_+) \rightsquigarrow (G, \Lambda, N)$

- Let (G, Φ, Φ_+) be a signed groupoid-set.
- Define $\Gamma\colon G \to \textbf{Set}$ by $\Gamma(a) := \Phi(a)/\{\pm 1\}$ for $a \in \mathrm{ob}(G)$.
- Set $\Lambda := \mathcal{P}\Gamma j_G\colon G \to \textbf{BoolRing}$.
- $N(g) := \left(\Phi^+(a) + \Phi(g)(\Phi^+(b))\right)/\{\pm 1\} \in \Lambda(a)$, for $g \in {}_aG_b$.

Theorem

(a) *(G, Λ, N) is a protorootoid.*

(b) *In (G, Γ, N), G is a groupoid, $\Gamma\colon G \to$ **Set** and $N \in Z^1(G, \mathcal{P}\Gamma j_G)$.*

(c) $(G, \Phi, \Phi_+) \leftrightsquigarrow (G, \Gamma, N)$ *(either determines the other up to $\cong$).*

This shows signed groupoid sets correspond naturally (as objects) to special protorootoids which we call set protorootoids. Part (c) underlies classification of principal $\{\pm 1\}$-bundles in the category of G-sets.

The construction $(G, \Lambda, N) \rightsquigarrow (G, \Phi, \Phi_+)$.

Protorootoids give rise to signed groupoid sets via Stone's theorem:

- Let (G, Λ, N) be a protorootoid, so $\Lambda\colon G \to \mathbf{BoolRing}$.
- $\Psi := \operatorname{Spec} \Lambda j_G\colon G \to \mathbf{Set}$, $\Gamma := \mathcal{P}\Psi j_G\colon G \to \mathbf{BoolRing}$.
- By Stone's theorem, there is a natural transformation $\iota\colon \Lambda \to \Gamma$ with injective components defined by $\iota_a(r) := \{\, \mathfrak{p} \in \operatorname{Spec}(\Lambda(a)) \mid r \notin \mathfrak{p} \,\}$ for $r \in \Lambda(a)$, $a \in \operatorname{ob}(G)$.
- $M := \iota N \in Z^1(G, \Gamma)$ where $M(g) := \iota_a(N(g))$ for $g \in {}_aG$.
- Have $\Psi\colon G \to \mathbf{Set}$ and $M \in Z^1(G, \mathcal{P}\Psi j_G)$.

Theorem

(G, Γ, M) is a set prootorootid and $(\mathrm{Id}_G, \iota)\colon (G, \Lambda, N) \to (G, \Gamma, M)$ is a morphism in **Prtd**. *Define the construction $(G, \Lambda, N) \rightsquigarrow (G, \Phi, \Phi_+)$ by letting (G, Φ, Φ_+) be the signed groupoid set from (G, Γ, M).*

The construction $(G, S) \rightsquigarrow (G, \Phi, \Phi^+)$.

- Let G be a groupoid, $S = S^* \subseteq \mathrm{mor}(G)$, $\langle S \rangle = G$.
- $l = l_S \colon G \to \mathbb{N}$: length function. Assume that $l(sg) = l(g) \pm 1$ if $s \in S$, $g \in \mathrm{mor}(G)$ (say (G, S) is even).
- Set ${}_aS = S \cap {}_aG$. For $s \in {}_aS$, let $\alpha_s := \{\, g \in {}_aG \mid l(s^*g) > l(g) \,\}$.
- For $b \in \mathrm{ob}(G)$, $\Phi(b) := \{\, g\alpha_s \mid g \in {}_bG_a, s \in {}_aS, a \in \mathrm{ob}(G) \,\}$ where $g\alpha_s := \{\, gx \mid x \in \alpha_s \,\}$.
- $(\Phi(g))(\alpha) := \{\, gx \mid x \in \alpha \,\} \in \Phi(b)$ for $\alpha \in \Phi(a)$ and $g \in {}_bG_a$.
- Define action of $\{\pm 1\}$ on $\Phi(a)$ by $(-1)X := {}_aG \setminus X$ if $X \in \Phi(a)$.
- Set $\Phi^+(a) := \{\, X \in \Phi(a) \mid 1_a \in X \,\}$.
- Then (G, Φ, Φ^+) is a signed groupoid set.

This extends to the case (G, S) not necessarily even but S contains no identity morphism, beginning by replacing the "simple root" α_s by the pair (α_s, α'_s) where $\alpha'_s := \{\, g \in {}_aG \mid l(s^*g) \geq l(g) \,\}$.

Characterizations of PRGPs amongst RGPs

PRGPs may be characterized among RGPs by familiar properties of a corresponding "minimal" protorootoid or signed groupoid set.

Theorem

Let $R = (G, \leq)$ be an interval finite RGP with atomic generators S. Let $l := l_S \colon \operatorname{mor}(G) \to \mathbb{N}$. Then the following are equivalent:

(a) *R is a PRGP.*

(b) *R is the underlying GP of a signed groupoid set (G, Φ, Φ^+) such that $l(g) = |\Phi_g|$ if $g \in {}_aG_b$ where $\Phi_g := \Phi^+(a) \cap \Phi(g)(-\Phi^+(b))$.*

(c) *R is the underlying GP of a rootoid (G, Λ, N) such that for all $g \in {}_aG$, $[0, N(g)]_{\Lambda(a)} := \{\, z \in \Lambda(a) \mid z \leq N(g) \,\} \cong \mathcal{P}(\{1, \ldots, l(g)\})$.*

In that case, the constructions $(G, S) \rightsquigarrow (G, \Phi, \Phi^+) \rightsquigarrow (G, \Lambda, N)$ provide structures as in (b), (c). Further, the abridgements of all rootoids with $(G, \leq)$ as underlying GP are isomorphic.

Coverings of groupoids

- A groupoid G is connected if ${}_aG_b \neq \emptyset$ for all $a, b \in \mathrm{ob}(G) \neq \emptyset$.
- Any groupoid is the disjoint union (categorical coproduct) of its connected components.
- A groupoid G is simply connected (SC) if $|{}_aG_b| \leq 1$ for all $a, b \in \mathrm{ob}(G)$.
- A groupoid morphism $\pi\colon H \to G$ is called a covering if for each $a \in \mathrm{ob}(H)$, the restriction ${}_a\pi\colon {}_aH \to {}_{\pi(a)}G$ is bijective.
- Any groupoid G has a universal covering $\pi\colon \widehat{G} \to G$, namely a covering π such that $\widehat{G}$ is SC and π induces a bijection between components of $\widehat{G}$ and those of G.

Terminology for groupoids is applied to GPs when it applies to the underlying groupoid e.g. a SC GP $(G, \leq)$ is a GP with G SC.

Coverings of GPs

Let G, H be connected groupoids. A local embedding $\pi\colon G \to H$ is a groupoid homomorphism (functor) such that the induced maps ${}_a\pi\colon {}_aG \to {}_{\pi(a)}G$ on stars are injective. Equivalently, $\pi = ip$ where p is a covering and i the inclusion of a subgroupoid.

- If $R = (G, \leq)$ is a GP and $\pi\colon H \to G$ is a covering, there is a unique preorder $\preceq$ on H making $R' = (H, \preceq)$ a GP such that ${}_a\pi\colon {}_aH \to {}_{\pi(a)}G$ is an order isomorphism for all $a \in \mathrm{ob}(H)$.
- If R is a PGP (resp., RGP, PRGP) then so is R' (and conversely if π is surjective on objects) and π is a morphism in **GP** (resp., **RGP**, **PRGP**). Call R' a covering GP of R, $\pi\colon R' \to R$ a covering.
- Define the universal covering of R as $\pi\colon R' \to R$ taking $\pi\colon H \to G$ a universal covering of groupoids.

GPs as quotients of SC GPs

- Given a SC PGP $R' = (\widehat{G}, \preceq)$, and a group A of automorphisms of R' acting freely on $\mathrm{ob}(\widehat{G})$, there is a covering quotient PGP R'/A and a covering $R' \to R'/A$.
- A connected PGP $R = (G, \leq)$ can be described as a covering quotient of the universal cover $R' = (\widehat{G}, \preceq)$ of R by the group of covering transformations of the universal covering $\pi\colon R' \to R$.

This reduces study of significant parts of the theory of PGPs, RGPs, PRGPs to the case of connected SC PGPs, RGPs, PRGPs.

The connected SC PGPs are (up to $\cong$) the GPs underlying the protorootoids arising as in the following very simple class of examples.

Connected, SC GPs

Example

For a Boolean ring B (which may be taken to be $B = \mathcal{P}(X)$ for a set X, without loss of generality) and a I-indexed family $(b_i)_{i\in I}$, where $I \neq \emptyset$, of elements of B , define a protorootoid $\mathcal{R} = \mathcal{R}(B, (b_i)_{i\in I})$ by $\mathcal{R} := (G, \Lambda, N)$ where:

- G is the connected SC groupoid with $\mathrm{ob}(G) = I$ and ${}_iG_j = \{(i,j)\}$.
- $\Lambda\colon G \to$ **BoolRing** is the constant functor with value B.
- $N(i,j) := b_i + b_j \in \Lambda(i)$ for all $i, j \in I$

The underlying SC GP of $\mathcal{R}$ has weak preorders $({}_iG, {}_i{\leq})$ given by $(i,j)\,{}_i{\leq}\,(i,k)$ if $b_i + b_j \leq b_i + b_k$, for $i, j, k \in I$.

The axioms for PGPs amongst GPs simply characterize the relations amongst the various weak preorders of all such SC GPs.

A left adjoint to an order preserving map $f\colon X \to Y$ between posets X and Y is an order preserving map $f^{\perp}\colon Y \to X$ such that $f^{\perp}y \leq x \iff y \leq fx$. If X, Y are complete lattices and f preserves meets, $f^{\perp}(y) := \bigwedge_{x \in X, f(x) \geq y} x$ is a left adjoint (which preserves joins).

- Let $F\colon J \to \mathbf{Ord}$ be a functor, where **Ord** is the category of posets and J is a small category.
- $P_F := \prod_{j \in \mathrm{ob}(J)} F(j) = \{\, (a_j)_{j \in \mathrm{ob}(J)} \mid a_j \in F(j) \text{ for all } j \in \mathrm{ob}(J) \,\}$.
- $L_F := \lim F = \{\, (a_j)_j \in P \mid (F(e))(a_j) = a_k \text{ if } e\colon j \to k \text{ in } \mathrm{mor}(J) \,\}$.
- $L_F^{\downarrow} := \{\, (a_j)_j \in P \mid (F(e))(a_j) \leq a_k \text{ if } e\colon j \to k \text{ in } \mathrm{mor}(J) \,\}$.
- $L \xrightarrow{V} L^{\downarrow} \xrightarrow{U} P$ and $\iota = UV\colon L \to P$ are inclusions.
- P, L, $L^{\downarrow}$ have the componentwise partial orders.
- If $F(j)$ is a complete lattice for all $j \in \mathrm{ob}(J)$ and $F(e)$ preserves meets whenever $e \in \mathrm{mor}(J)$, then P, L, $L^{\downarrow}$ are complete lattices and V, U, ι preserve meets, hence have left adjoints.

Theorem

Assume that for $j \in \mathrm{ob}(J)$, $F(j)$ has all joins indexed by the left star ${}_jJ$ and for all $e\colon j \to k$ in $\mathrm{mor}(J)$, $F(e)$ preserves such joins. Then U has a left adjoint $U^{\perp}\colon P \to L^{\downarrow}$ given by $U^{\perp}(a) := b$ where for $a = (a_j)_j \in P$, $b := (b_j)_j \in L^{\downarrow}$ with $b_j := \bigvee_{(e:k\to j)\in \mathrm{mor}(J)} (F(e))(a_k)$.

Generalizations and variants of the theorem

- There are dual results about $L^{\uparrow}$ (which is defined dually to $L^{\downarrow}$).
- The theorem extends to $F\colon J \to \mathbf{Cat}$, taking $L^{\downarrow}$ as the category of sections of the split opfibered category $\pi\colon \int F \to J$ attached to F by the/a Grothendieck construction (replace joins by direct sums).
- For a diagram $F\colon J \to \mathbf{Ord}$, J a directed graph, P_F, L_F, $L_F^{\downarrow}$, U_F, V_F, ι_F defined as before coincide with corresponding objects and maps attached to the natural functor $\widehat{F}\colon \widehat{J} \to \mathbf{Ord}$ extending F where $\widehat{J}$ is the free category on J ($P_F = P_{\widehat{F}}$ since $\mathrm{ob}(\widehat{J}) = \mathrm{ob}(J)$).

Application to adjoints for limits

- Suppose $F\colon J \to \mathbf{Ord}$ is a diagram and for each $e\colon j \to k$ in J there is both $F(e)\colon F(j) \to F(k)$ and $F(e)^{\dagger}\colon F(k) \to F(j)$ in **Ord** such that for $a \in F(j)$ and $b \in F(k)$, one has

$$(F(e))(a) = b \iff ((F(e))(a) \leq b \text{ and } (F(e))^{\dagger}(b) \leq a). \qquad (\diamond)$$

- "Double" F to a diagram $H\colon K \to \mathbf{Ord}$ where $\mathrm{ob}(K) = \mathrm{ob}(J)$, $\mathrm{arr}(K) := \mathrm{arr}(J) \dot{\cup} \{\, e^{\sharp}\colon k \to j | (e\colon j \to k) \in \mathrm{arr}(J) \,\}$ with $H(a) := F(a)$ for $a \in \mathrm{ob}(J)$ and $H(e) := F(e)$ and $H(e^{\sharp}) := (F(e))^{\dagger}$ for $e \in \mathrm{mor}(G)$.
- Then $(\iota_F\colon L_F \to P_F) = (U_H\colon L_H^{\downarrow} \to P_H)$, trivially by $(\diamond)$.
- If all $F(j)$ are complete and each $F(e)$ and $(F(e))^{\dagger}$ preserves joins, $\iota_F^{\perp} = U_H^{\perp} = U_{\widehat{H}}^{\perp}\colon P_F \to L_F$ is given by the theorem.

Definition

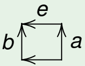
A RGP $(G, \leq)$ is complete if for each $a \in \mathrm{ob}(G)$, $({}_aG, {}_a\leq)$ has a maximum element ${}_a\omega$ (or equivalently, is a complete lattice). Then ${}_a\omega$ is called the longest element of ${}_aG$.

Example

(1) Let $F\colon J \to \mathbf{Ord}$ be a diagram so $F(j)$ is a complete lattice if $j \in \mathrm{ob}(J)$, $F(e)$ preserves both meets and joins for all $e \in \mathrm{mor}(J)$. Take $(F(e))^{\dagger} := (F(e))^{\perp}$ (a left adjoint, so it preserves joins).

Condition $(\diamond)$ holds by definition of adjoints.

(2) Let $(G, \leq)$ be a complete rootoid. Take J a subgraph of G so $\mathrm{ob}(J) \subseteq \mathrm{ob}(G)$, $\mathrm{arr}(J) \subseteq \mathrm{mor}(G)$. For $j \in \mathrm{ob}(J)$, let $F(j) := ({}_jG, {}_j\leq)$. For $(e\colon j \to k) \in \mathrm{arr}(J)$, let $F(e)\colon {}_jG \to {}_kG$ be $x \mapsto e(e^* \vee x))$. Take $(F(e))^{\dagger} := (y \mapsto e^*(e \vee y))$. Then

$(\diamond)$ holds; in fact, each side holds iff there is a square  .

Comments on the applications

- Technical variants of the formulae for $\iota_F^{\perp}$ from (1) (resp., (2)) of the preceding example are key ingredients in checking the AOP in the proof of completeness of **RGP** (resp., in checking the AOP in establishing existence of functor RGPs).
- These variants involve complete meet semilattices instead of complete lattices and are complicated by involvement of partially defined maps.
- For example, the zig-zag construction for a RGP comes from (2) in case J is a directed graph $j \xrightarrow{e} k$. It may be regarded as a partially defined map, which is everywhere defined in the case of a complete RGP.

If $(G, \leq)$ is a complete RGP, the map $g \mapsto g^{\complement} : {}_aG \to {}_aG$ given by $g^{\complement} := g\, {}_b\omega$ for $g \in {}_aG_b$ makes ${}_aG$ a complete ortholattice.

Complete PRPGs generalize finite Coxeter groups in several natural respects (e.g. in properties of longest elements), and complete RGPs are their (possibly non-discrete) analogues. The theory of complete RGPs is simpler, more natural and richer than that of RGPs in general.

Definition

(a) A completion of a RGP $(G, \leq)$ is a complete RGP $(\widehat{G}, \preceq)$ such that G is a subgroupoid of $\widehat{G}$ and the inclusion $\iota\colon G \to \widehat{G}$ is a morphism in **RGP**.

(b) An ideal completion of RGP $(G, \leq)$ is a completion $(\widehat{G}, \preceq)$ of $(G, \leq)$ such that for all $a \in \mathrm{ob}(G)$, ${}_aG$ is an order ideal of ${}_a\widehat{G}$.

Example

- (W, S) a Coxeter system, with standard root system $\Phi \supseteq \Phi^+$.
- The PRGP $(W, \leq)$ is complete iff W is finite.
- Say that $\Gamma \subseteq \Phi$ is closed if $\alpha, \beta \in \Gamma$, $\gamma \in \Phi$ and $\gamma = c\alpha + d\beta$ with $c, d \in \mathbb{R}_{\geq 0}$ implies $\gamma \in \Gamma$.
- Let $\mathcal{B}' := \{\, \Gamma \subseteq \Phi^+ \mid \Gamma, \Phi^+ \setminus \Gamma \text{ both closed} \,\} \subseteq \mathcal{P}(\Phi^+)$.
- Let $\mathcal{B} := \{\, \Gamma \in \mathcal{B}' \mid \Gamma \text{ is finite} \,\} = \{\, \Phi_w \mid w \in W \,\}$.
- There is a SC GP $R := (G, \preceq)$ with $\mathrm{ob}(G) = \mathcal{B}$, ${}_\Gamma G_\Delta := \{(\Gamma, \Delta)\}$ for $\Gamma, \Delta \in \mathcal{B}$ and $(\Gamma, \Delta) \,{}_\Gamma\!\preceq (\Gamma, \Theta)$ if $\Gamma + \Delta \subseteq \Gamma + \Theta$.
- R is a PRGP isomorphic to the universal cover of $(W, \leq)$.

Conjecture

Define the SC GP $R' = (G', \preceq')$ as R is defined above but with $\mathcal{B}$ replaced by $\mathcal{B}'$. Then R' is a complete RGP (equivalently, the weak orders of R' are all lattices, or R' is a SC ideal completion of R).

The preceding conjecture implies $(\mathcal{B}', \subseteq)$ is a complete ortholattice in which weak order of W embeds as an order ideal. Study of this conjectural completion of weak order on W provided the main initial motivations for the definition of RGPs.

Some fundamental open questions

(a) Which RGPs $(G, \leq)$ have (ideal) completions? All RGPs? All SC RGPs? All SC PRGPs? Only very special ones?

(b) Can properties of RGPs $(G, \leq)$ be extended to the case G is a suitable subcategory of a groupoid (or more generally still)?

(c) Can the requirement the weak orders come from a cocycle be relaxed (using only some of the axioms for PGPs amongst GPs)?

Questions (b)–(c) may be relevant to study of Garside structures, where weak-order-like lattice structures appear on submonoids of certain groups (e.g. braid monoids of braid groups). In any case, certain complete PRGPs naturally give rise to Garside structures.