Hurwitz action on presentations of exceptional complex reflection groups

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(joint work with Gunter Malle, 2010)

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The well generated groups are $G(e,1,r)$, $G(e,e,r)$ and the exceptional groups except $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}$ and G_{31}.

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The braid group is generated by *braid reflections* which are elements of $B(W)$ "above" reflections.

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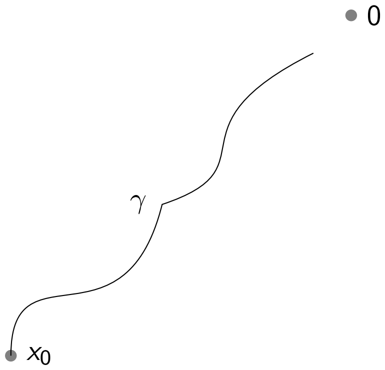

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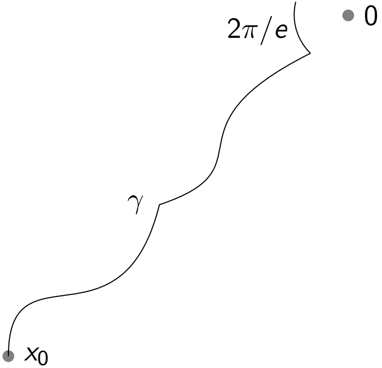

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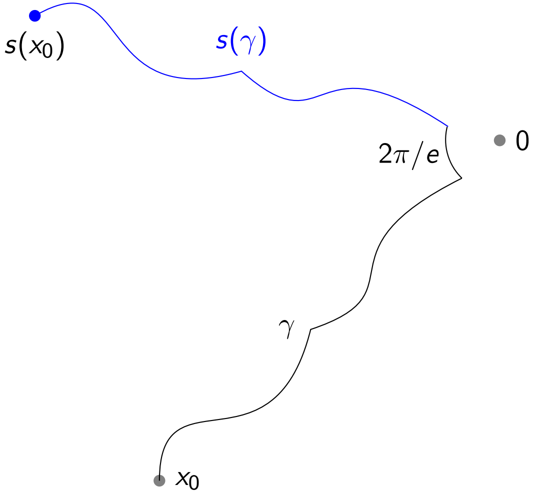

Presentations of $B(W)$

When W is a finite Coxeter group, by the work of Brieskorn (1971) the group $B(W)$ is generated by r braid reflections with presentation

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Further, adding the relations $\mathbf{s}^e = 1$ where e is the order of the image in W of the braid reflection $\mathbf{s}$ gives a presentation of W.

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Here are examples:

group	A_2	G_4	G_8	G_{16}
diagram	(2)—(2)	(3)—(3)	(4)—(4)	(5)—(5)
	s t	s t	s t	s t
reflection degrees	$2, 3$	$4, 6$	$8, 12$	$20, 30$

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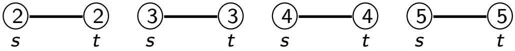
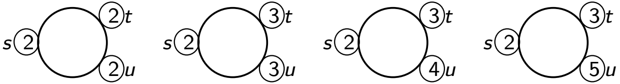


and some examples of not well generated groups

group	$G(4, 2, 2)$	G_7	G_{11}	G_{19}
diagram				
reflection degrees	$4, 4$	$12, 12$	$24, 24$	$60, 60$



Here the circle means the braid relations: $stu = tus = ust$.

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If the 2-reflection group is a Coxeter group, we say that W is a Shephard group. The dimension ≥ 3 groups which are not Shephard groups are G_{24}, G_{27}, G_{29}, G_{31}, G_{33} and G_{34}. They are all 2-reflection groups. Only G_{31} is not well generated.

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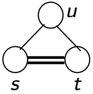
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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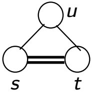
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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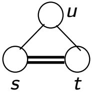
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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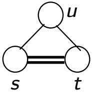
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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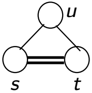
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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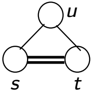
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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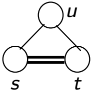
Broué, Malle and Rouquier guessed wrong for G_{24}: 

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With David Bessis, we created in 2004 `VKcurve`, a `GAP3` package which can compute the Π_1 of the complement of any curve in $\mathbb{C}^2$, using the Zariski- Van Kampen method.

Presentations of $B(G_{24})$

In Bessis-M. (2004) we found 3 "simple" presentations of $B(G_{24})$:

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These presentations are obtained by simplifying heuristically those given by the Zariski-Van Kampen method, which have many generators and relations.

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(Bessis 2006) Let c be a Coxeter element. Then there is a "good" lift $\mathbf{c}$ *of c in $B(W)$, an element* $\mathbf{c}$ *which is the product of $r = \dim V$ braid reflexions ("tunnels") which generate $B(W)$, and such that* $\mathbf{c}^h$ *generates the center of the pure braid group $\Pi^1(V^{reg})$.*

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Following Brady and Watt (2002), we define a partial order on $\mathrm{GL}(V)$ by

$$A \preccurlyeq B \Leftrightarrow \dim \mathrm{Image}(A - \mathrm{Id}) + \dim \mathrm{Image}(A^{-1}B - \mathrm{Id}) = \dim \mathrm{Image}(B - \mathrm{Id}).$$

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(Brady and Watt 2002) Let M be maximal for $\preccurlyeq$ and unitary; the set of elements $A \preccurlyeq M$ in the unitary group form a lattice.

Non-crossing partitions

(Bessis 2006) Let c be a Coxeter element. The set of elements $w \in W$ such that $w \preccurlyeq c$ (seen as unitary transformations) form a lattice, called the lattice of non-crossing partitions of type W

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Thus is $c = s_1 \ldots s_r$ is a decomposition into reflections, all the prefixes of this decomposition are non-crossing partitions.

Hurwitz orbit

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(Bessis 2007) The Hurwitz orbit on the decomposition $\mathbf{c} = \mathbf{s}_1 \dots \mathbf{s}_r$ is finite, of cardinality $r!h^r/|W|$. The projection to W is an isomorphism to the Hurwitz orbit on the decompositions of c into r reflections, where the Hurwitz action is transitive.

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The dual braid monoid gives a more efficient way to compute presentations of (at least 5 of the 6) difficult exceptional groups.

Hurwitz action on presentations

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- For instance, for G_{24}, the reflection degrees are $4, 6, 14$ and the Hurwitz orbit has $3!14^3/(4 \cdot 6 \cdot 14) = 49$ elements. The presentation P_1 (resp. P_2, P_3) appears 21, (resp. $21, 7$) times in the Hurwitz orbit.

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- For G_{27}, the reflection degrees are $6, 12, 30$ and the Hurwitz orbit has $3!30^3/(6 \cdot 12 \cdot 30) = 75$ elements, given rise to 5 different presentations appearing each 15 times.

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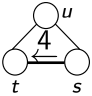
- For type A_3 the orbit is of size 16 giving 12 times the usual presentation and 4 times the presentation 

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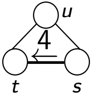
- For type A_3 the orbit is of size 16 giving 12 times the usual presentation and 4 times the presentation 

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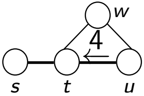
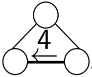
- For type A_4 the orbit is of size 125 giving 60 times the usual presentation, 60 times the presentation 

and 5 times a presentation where the diagram is a tetrahedron, each face being 

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$\prod_{d_i}(q^{d_i}-1)/(q-1) = x^{21}+3x^{20}+6x^{19}+10x^{18}+14x^{17}+18x^{16}+21x^{15}+23x^{14}+24x^{13}+24x^{12}+24x^{11}+24x^{10}+24x^{9}+24x^{8}+23x^{7}+21x^{6}+18x^{5}+14x^{4}+10x^{3}+6x^{2}+3x+1.$

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For the presentations P_1, P_2, P_3 we get respectively for length series:

$$q^{15}+3q^{14}+6q^{13}+12q^{12}+27q^{11}+46q^{10}+55q^{9}+54q^{8}+44q^{7}+31q^{6}+22q^{5}+15q^{4}+10q^{3}+6q^{2}+3q+1$$

$$q^{13}+4q^{12}+16q^{11}+39q^{10}+56q^{9}+58q^{8}+52q^{7}+42q^{6}+29q^{5}+18q^{4}+11q^{3}+6q^{2}+3q+1$$

$$q^{13}+5q^{12}+12q^{11}+24q^{10}+45q^{9}+54q^{8}+59q^{7}+57q^{6}+36q^{5}+21q^{4}+12q^{3}+6q^{2}+3q+1$$

There are reasons to think that the presentation giving the highest degree polunomial (“closest” to the Poincaré polynomial) is “best”.

Hecke algebras

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When W is the Weyl group of the reductive group $\mathbf{G}$ over $\mathbb{F}_q$, we have $H=\operatorname{End}_{\mathbf{G}(\mathbb{F}_q)}\operatorname{Ind}_{\mathbf{B}(\mathbb{F}_q)}^{\mathbf{G}(\mathbb{F}_q)}\operatorname{Id}$, and t is a multiple of the trace of this representation.

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For $\mathbf{b} \in B(W)$ let $T_{\mathbf{b}}$ be the image in $\mathcal{H}$. The conjecture implies that for any section $W \hookrightarrow \mathbf{W} \subset B(W)$ the set $\{T_{\mathbf{w}}\}_{\mathbf{w} \in \mathbf{W}}$ is a $\mathbb{Q}(q)$-basis. We may conjecture further that there exists such a section which gives an a $\mathbb{Z}[q^{\pm 1}]$-basis.

Conjectures on Hecke algebras

It is conjectured that $\mathcal{H}$ is symmetric. In (Broué-Malle-Michel 1999) it is proven that there is at most one symmetrizing trace that specializes to the canonical trace on the group algebra of W for $q = 1$ and satisfies another "natural" condition.

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For finite Coxeter groups, such a section consists of lifts of minimal length representatives; Bremke and Malle (1997) have shown that this works also for $G(d, 1, r)$.

“Quality” of presentations

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- For G_{27} the situation is worse: even for the "best" presentation, there exists one element for which the lift of no minimal length representtive has zero trace. But in each case (including the other presentations where the number of failures may rise to 41 out of the 2160 elements) there are slightly longer words for which $t(T_{\mathbf{w}}) = 0$.