The stabilization functor via the singular Yoneda dg category

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The plan

- The stabilization functor
- The Yoneda dg category
- The singular Yoneda dg category and a comparison theorem

This talk is mainly based on a series of joint work with Zhengfang Wang (汪正方) at Stuttgart.

Section I

- The stabilization functor
- The Yoneda dg category
- The singular Yoneda dg category and a comparison theorem

Recollement

Recollement, as a categorical gluing, arises naturally in

- the study of perverse sheaves
- partial tilting complexes
- qusi-hereditary algebras and highest weight categories
- dg (= differential graded) quotients

Recollement: the definition

Recall from [BBD 1982] that a *recollement* is a diagram:

$$\begin{tikzcd} \mathcal{T}' \arrow[rr, "i"] & & \mathcal{T} \arrow[ll, bend right, "i_\lambda"'] \arrow[ll, bend left, "i_\rho"] \arrow[rr, "j"] & & \mathcal{T}'' \arrow[ll, bend right, "j_\lambda"'] \arrow[ll, bend left, "j_\rho"] \end{tikzcd}$$

such that

- (i_λ, i, i_ρ) and (j_λ, j, j_ρ) are adjoint triples;
- i, j_λ and j_ρ are fully faithful;
- $ji \simeq 0$; by the adjunctions, $i_\lambda j_\lambda \simeq 0$ and $i_\rho j_\rho \simeq 0$.
- For each $X \in \mathcal{T}$, there are *functorial* exact triangles

$$ii_\rho(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} j_\rho j(X) \xrightarrow{\delta_X} \Sigma ii_\rho(X)$$

and

$$j_\lambda j(X) \xrightarrow{\phi_X} X \xrightarrow{\psi_X} ii_\lambda(X) \xrightarrow{\sigma_X} \Sigma j_\lambda j(X).$$

Norm and conorm

- There exists the *norm morphism* $N\colon j_\lambda \to j_\rho$ such that $Nj = \eta \circ \phi$;
- There exists the *conorm morphism* $C\colon i_\rho \to i_\lambda$ such that $iC = \psi \circ \varepsilon$;

The intertwining isomorphism

Proposition (BBD 1982, C.-Le 2022)

There exists a unique isomorphism

$$\xi\colon i_\lambda j_\rho \longrightarrow \Sigma i_\rho j_\lambda$$

making the following two exact diagram commute.

$$\begin{tikzcd}
\Sigma^{-1} i i_\lambda j_\rho \arrow[r, "-j_\lambda \eta' \circ \Sigma^{-1}\sigma j_\rho"] \arrow[d, dotted, "\Sigma^{-1} i\xi"'] & j_\lambda \arrow[r, "N"] \arrow[d, equal] & j_\rho \arrow[r, "\psi j_\rho"] \arrow[d, equal] & i i_\lambda j_\rho \arrow[d, dotted, "i\xi"] \\
i i_\rho j_\lambda \arrow[r, "\varepsilon j_\lambda"] & j_\lambda \arrow[r, "N"] & j_\rho \arrow[r, "\delta j_\lambda \circ j_\rho \phi'"] & \Sigma i i_\rho j_\lambda
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^{-1} i_\lambda j_\rho j \arrow[r, "-\psi' i_\rho \circ \Sigma^{-1} i_\lambda \delta"] \arrow[d, "-\Sigma^{-1}\xi j"'] & i_\rho \arrow[r, "C"] \arrow[d, equal] & i_\lambda \arrow[r, "i_\lambda \eta"] \arrow[d, equal] & i_\lambda j_\rho j \arrow[d, "-\xi j"] \\
i_\rho j_\lambda j \arrow[r, "i_\rho \phi"] & i_\rho \arrow[r, "C"] & i_\lambda \arrow[r, "i_\rho \sigma \circ \varepsilon' i_\lambda"] & \Sigma i_\rho j_\lambda j
\end{tikzcd}$$

The gluing functor

Definition

The *gluing functor* of the following recollement is $i_\lambda j_\rho \colon \mathcal{T}'' \to \mathcal{T}'$.

$$\begin{tikzcd} \mathcal{T}' \arrow[r, "i" description] & \mathcal{T} \arrow[l, bend right, "i_\lambda" description] \arrow[l, bend left, "i_\rho" description] \arrow[r, "j" description] & \mathcal{T}'' \arrow[l, bend right, "j_\lambda" description] \arrow[l, bend left, "j_\rho" description] \end{tikzcd}$$

- In the dg or ∞-categorical setting, the middle category can be recovered from the gluing functor; see [Kuznetsov-Lunts 2015] and [Lurie 2017, Dyckerhoff-Jasso-Walde 2021].
- In the triangulated setting, we recover $\mathcal{T}$ from the *comma category* of the gluing functor, up to an epivalence; see [C.-Le 2022]; compare [Geiss-Keller 2002] and [Keller 2020].

Notation

- Λ a left noetherian ring
- $\Lambda\text{-proj} \subseteq \Lambda\text{-mod} \subseteq \Lambda\text{-Mod}$
- $\mathbf{K}^b(\Lambda\text{-proj}) \subseteq \mathbf{D}^b(\Lambda\text{-mod}) \subseteq \mathbf{D}(\Lambda\text{-Mod})$

The singularity category

Definition (Buchweitz 1986, Orlov 2003)

The *singularity category* of Λ is the Verdier quotient of triangulated categories

$$\mathbf{D}_{\rm sg}(\Lambda) = \frac{\mathbf{D}^b(\Lambda\text{-}\mathrm{mod})}{\mathbf{K}^b(\Lambda\text{-proj})}.$$

- If $\mathrm{gl.dim}\,\Lambda < \infty$, for example, a *regular* commutative ring by Auslander-Buchsbaum-Serre's theorem, then $\mathbf{D}_{\rm sg}(\Lambda) = 0$.
- Buchweitz uses the *stabilized derived category*. Nowadays, we follow [Orlov 2003].

Compactly generated completions

- It is well known that $\mathbf{D}(\Lambda\text{-Mod})$ is a *compactly generated completion* of $\mathbf{K}^b(\Lambda\text{-proj})$, that is, $\mathbf{D}(\Lambda\text{-Mod})$ is compactly generated and $\mathbf{D}(\Lambda\text{-Mod})^c \simeq \mathbf{K}^b(\Lambda\text{-proj})$.
- In general, the inclusion $\mathbf{K}^b(\Lambda\text{-proj}) \subseteq \mathbf{D}^b(\Lambda\text{-mod})$ is proper.

Theorem (Krause 2005)

The homotopy category $\mathbf{K}(\Lambda\text{-}\mathrm{Inj})$ of unbounded complexes of injective modules is a compactly generated completion of $\mathbf{D}^b(\Lambda\text{-}\mathrm{mod})$.

Krause's completion

- The dg-injective resolution functor $\mathbf{i}$ restricts to the embedding $\mathbf{i}\colon \mathbf{D}^b(\Lambda\text{-mod}) \to \mathbf{K}(\Lambda\text{-Inj})$.
- Krause's completion works more generally, namely replacing Λ-Mod by any locally noetherian Grothendieck category.
- In the study of the Grothendieck duality, A. Neeman in a 2008 paper mentioned

  "... *offers a very promising new angle on this old problem.*"

  Here, the "old problem" might mean the Grothendieck duality (in a modern form).

Krause's recollement

Theorem (Krause 2005)

There exists a recollement

$$\begin{tikzcd} \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-}\mathrm{Inj}) \arrow[rr, "\mathrm{inc}"] & & \mathbf{K}(\Lambda\text{-}\mathrm{Inj}) \arrow[ll, bend right, "\bar{\mathbf{a}}"'] \arrow[ll, bend left, "\mathbf{a}'"] \arrow[rr, "\mathrm{can}"] & & \mathbf{D}(\Lambda\text{-}\mathrm{Mod}) \arrow[ll, bend right, "\bar{\mathbf{p}}"'] \arrow[ll, bend left, "\mathbf{i}"] \end{tikzcd}$$

Here, $\mathbf{i}$ *is the dg-injective resolution functor,* $\mathbf{a}'$ *is determined by*

$$\mathbf{a}'I \longrightarrow I \longrightarrow \mathbf{i}I \longrightarrow \Sigma\mathbf{a}'I$$

In general, $\bar{\mathbf{p}}$ *and* $\bar{\mathbf{a}}$ *are mysterious. They are related by*

$$\bar{\mathbf{p}}I \longrightarrow I \longrightarrow \bar{\mathbf{a}}I \longrightarrow \Sigma\bar{\mathbf{p}}I$$

The mysterious upper half

- The localizing subcategory $\mathrm{Loc}\langle \mathbf{i}\Lambda \rangle$ is equivalent to $\mathbf{D}(\Lambda\text{-Mod})$ via the canonical functor $\mathrm{can}\colon \mathbf{K}(\Lambda\text{-Inj}) \to \mathbf{D}(\Lambda\text{-Mod})$.
- $\bar{\mathbf{p}}$ is the composition of its quasi-inverse with the inclusion $\mathrm{Loc}\langle \mathbf{i}\Lambda \rangle \hookrightarrow \mathbf{K}(\Lambda\text{-Inj})$.
- The norm morphism $N\colon \bar{\mathbf{p}} \to \mathbf{i}$ restricts to an isomorphism on $\mathbf{K}^b(\Lambda\text{-proj})$.
- The counit $\bar{\mathbf{p}}\mathrm{can}(I) \to I$ of $(\bar{\mathbf{p}}, \mathrm{can})$ is NOT explicit. So, the functor $\bar{\mathbf{a}}$ is (more) mysterious.

The completion of the singularity category

Applying Ravenel-Thomason-Trobaugh-Yao's localization theorem [Neeman 1992] to the mysterious upper half,

$$\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \xleftarrow{\bar{\mathbf{a}}} \mathbf{K}(\Lambda\text{-Inj}) \xleftarrow{\bar{\mathbf{p}}} \mathbf{D}(\Lambda\text{-Mod})$$

we have

Theorem (Krause 2005)

The homotopy category $\mathbf{K}_{\mathrm{ac}}(\Lambda\text{-}\mathrm{Inj})$ of unbounded acyclic complexes of injective modules is a compactly generated completion of $\mathbf{D}_{\mathrm{sg}}(\Lambda)$ (up to direct summands).

The stabilization functor

Recall Krause's recollement

$$\begin{tikzcd}[column sep=huge] \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}) \arrow[r, "\mathrm{inc}"] & \mathbf{K}(\Lambda\text{-Inj}) \arrow[l, bend right, "\bar{\mathbf{a}}"'] \arrow[l, bend left, "\mathbf{a}'"] \arrow[r, "\mathrm{can}"] & \mathbf{D}(\Lambda\text{-Mod}) \arrow[l, bend right, "\bar{\mathbf{p}}"'] \arrow[l, bend left, "\mathbf{i}"] \end{tikzcd}$$

Definition (Krause 2005)

The *stabilization functor* of Λ is

$$\mathbb{S} = \bar{\mathbf{a}}\mathbf{i} \colon \mathbf{D}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj}).$$

Therefore, $\mathbb{S}$ is the gluing functor of the recollement above.

Moreover, we have $\mathbb{S} \simeq \Sigma\mathbf{a}'\bar{\mathbf{p}}$.

The stabilization functor, continued

- The functor $\mathbb{S}$ induces the embedding $\mathbf{D}_{\rm sg}(\Lambda) \hookrightarrow \mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$. To be more precise,

$$\mathbf{D}^b(\Lambda\text{-mod}) \xrightarrow{\mathbf{i}} \mathbf{K}(\Lambda\text{-Inj}) \xrightarrow{\bar{\mathbf{a}}} \mathbf{K}_{\rm ac}(\Lambda\text{-Inj})$$

  vanishes on $\mathbf{K}^b(\Lambda\text{-proj})$, and induces the required embedding.

- If Λ is Gorenstein, $\mathbb{S}$ gives functorial *Gorenstein injective resolutions* for Λ-modules. More specifically, if Λ is quasi-Frobenius, then $\mathbb{S}M$ is given by

$$\mathbf{p}M \longrightarrow \mathbf{i}M \longrightarrow \mathbb{S}M \longrightarrow \Sigma\mathbf{p}M.$$

  It is the *complete resolution* of M; [Tate 1952].

The dg singularity category

- The dg quotient [Keller 1999, Drinfeld 2004] enhances the Verdier quotient.
- For example, the dg derived category $\mathbf{D}^b_{\rm dg}(\Lambda\text{-mod})$ is a canonical dg enhancement for $\mathbf{D}^b(\Lambda\text{-mod})$.

Definition (Keller 2018, Blanc-Robalo-Töen-Vezzosi 2018, Brown-Dyckerhoff 2020 ...)

The *dg singularity category* of Λ is the dg quotient

$$\mathbf{S}_{\rm dg}(\Lambda) = \mathbf{D}^b_{\rm dg}(\Lambda\text{-mod})/C^b_{\rm dg}(\Lambda\text{-proj}).$$

$\mathbf{S}_{\rm dg}(\Lambda)$ *enhances* $\mathbf{D}_{\rm sg}(\Lambda)$, that is, it is pretriangulated such that $H^0(\mathbf{S}_{\rm dg}(\Lambda)) = \mathbf{D}_{\rm sg}(\Lambda)$.

The dg approach to Krause's recollement

Set $\mathcal{D} = \mathbf{D}^b_{\mathrm{dg}}(\Lambda\text{-mod})$, $\mathcal{P} = C^b_{\mathrm{dg}}(\Lambda\text{-proj})$ and $\mathcal{S} = \mathbf{S}_{\mathrm{dg}}(\Lambda)$.

By a general result of [Keller 1999, Drinfeld 2004], we have

Theorem (Krause 2005, implicitly)

Krause's recollement is "isomorphic" to the canonical recollement associated to the dg quotient:

$$\begin{tikzcd}[column sep=huge]
\mathbf{D}(\mathcal{S}) \arrow[r, "\mathrm{can}"] & \mathbf{D}(\mathcal{D}) \arrow[l, bend right, "-\otimes^{\mathbb{L}}_{\mathcal{D}}\mathcal{S}"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathcal{S},-)"] \arrow[r, "\mathrm{res}"] & \mathbf{D}(\mathcal{P}) \arrow[l, bend right, "-\otimes^{\mathbb{L}}_{\mathcal{P}}\mathcal{D}"'] \arrow[l, bend left, "\mathbb{R}\mathrm{Hom}_{\mathcal{P}}(\mathcal{D},-)"]
\end{tikzcd}$$

Consequently, the stabilization functor $\mathbb{S}$ is "isomorphic" to $\mathbb{R}\mathrm{Hom}_{\mathcal{P}}(\mathcal{D}, -) \otimes^{\mathbb{L}}_{\mathcal{D}} \mathcal{S}$.

The goal

- Goal: to describe $\mathbb{S}$ explicitly.
- Assume that Λ is over a field k.
- Tool: the Yoneda dg category and its strict localization, namely, the singular Yoneda dg category.

Section II

- The stabilization functor
- The Yoneda dg category
- The singular Yoneda dg category and a comparison theorem

The bar resolution

- Set $\overline{\Lambda} = \Lambda/k$, and $\Sigma(\overline{\Lambda}) = s\overline{\Lambda}$. Its typical element is written $s\bar{a}$, which is of degree -1.
- The normalized *bar resolution*

$$\mathbb{B} = \Lambda \otimes T(s\overline{\Lambda}) \otimes \Lambda = \prod_{n\geq 0} \Lambda \otimes s\overline{\Lambda}^{\otimes n} \otimes \Lambda$$

has differential given

$$d(a \otimes s\bar{a}_{1,n} \otimes b) = aa_1 \otimes s\bar{a}_{2,n} \otimes b + (-1)^n a \otimes s\bar{a}_{1,n-1} \otimes a_n b$$
$$+ \sum_{i=1}^{n-1} (-1)^i a \otimes s\bar{a}_{1,i-1} \otimes s\overline{a_i a_{i+1}} \otimes s\bar{a}_{i+2,n} \otimes b.$$

Here, $s\bar{a}_{i,j} = s\bar{a}_i \otimes \cdots \otimes s\bar{a}_j$.

The *fundamental* bar resolution

- "*One of the fundamental constructions in homological algebra is the bar resolution, which has been successfully applied to ...*", taken from [Buchsbaum-Rota 1994].
- "*... is fundamental in governing the homological properties of the algebra*", taken from [Butler-King 1999].

The bar resolution, continued

- $\epsilon\colon \mathbb{B} \to \Lambda \otimes \Lambda \to \Lambda$ is a quasi-isomorphism.
- It provides a *functorial* dg-projective resolution: for any complex X of Λ-modules, $\mathbb{B} \otimes_\Lambda X$ is a dg-projective resolution of X.
- $\mathbb{B}$ is a *coalgebra* in the category of complexes of bimodules: we have

$$\Delta\colon \mathbb{B} \longrightarrow \mathbb{B} \otimes_\Lambda \mathbb{B}$$

given by

$$\Delta(a \otimes s\bar{a}_{1,n} \otimes b) = \sum_{i=0}^{n} (a \otimes s\bar{a}_{1,i} \otimes 1) \otimes_\Lambda (1 \otimes s\bar{a}_{i+1,n} \otimes b).$$

The Yoneda dg category

We define the *Yoneda dg category* $\mathcal{Y}$; compare [Keller 1994, Anno-Logvinenko 2021].

- The objects are just complexes of Λ-modules.
- For any complexes X and Y, we have

$$\begin{aligned}\mathcal{Y}(X, Y) &= \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y)\\ &= \prod_{n\geq 0} \mathrm{Hom}(s\overline{\Lambda}^{\otimes n} \otimes X, Y).\end{aligned}$$

This is NOT a product of complexes. Elements in $\mathrm{Hom}(s\overline{\Lambda}^{\otimes n} \otimes X, Y)$ will be said to have *filtration-degree* n.

The Yoneda dg category, continued

- The composition of $f \in \mathcal{Y}(X, Y)$ and $g \in \mathcal{Y}(Y, Z)$ is defined as

$$g \odot f \colon \mathbb{B} \otimes_\Lambda X \xrightarrow{\Delta} \mathbb{B} \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \xrightarrow{f} \mathbb{B} \otimes_\Lambda Y \xrightarrow{g} Z.$$

- If f is represented by $f \in \mathrm{Hom}(s\overline{\Lambda}^{\otimes n} \otimes X, Y)$ and g is represented by $g \in \mathrm{Hom}(s\overline{\Lambda}^{\otimes m} \otimes Y, Z)$, then $g \odot f$ is represented by the following element in $\mathrm{Hom}(s\overline{\Lambda}^{\otimes n+m} \otimes X, Z)$:

$$s\bar{a}_{m+n} \otimes x \longmapsto (-1)^{m|f|} g(s\bar{a}_{1,m} \otimes f(s\bar{a}_{m+1,m+n} \otimes x)).$$

- $\mathcal{Y}$ might be viewed as the coKleisli category of the comonad $\mathbb{B} \otimes_\Lambda -$ on the dg category $C_{\mathrm{dg}}(\Lambda\text{-Mod})$ of complexes.

The Yoneda dg category as a dg enhancement

Proposition (Keller 1994 implicitly, C.-Wang)

$\mathcal{Y}$ is a dg enhancement of $\mathbf{D}(\Lambda\text{-}\mathrm{Mod})$, that is, it is pretriangulated with $H^0(\mathcal{Y}) = \mathbf{D}(\Lambda\text{-}\mathrm{Mod})$.

- Reason: $\mathcal{Y}(X, Y) = \mathrm{Hom}_\Lambda(\mathbb{B} \otimes_\Lambda X, Y)$ computes $\mathrm{Hom}_{\mathbf{D}(\Lambda\text{-}\mathrm{Mod})}(X, \Sigma^i(Y))$.
- To justify the terminology: the dg endomorphism algebra $\mathcal{Y}(M, M)$ computes the *Yoneda Ext-algebra* $\mathrm{Ext}^*_\Lambda(M, M)$.

The advantages of $\mathcal{Y}$

- Noncommutative differential forms as a dg endofunctor $\Omega_{\rm nc}$;
- A closed natural transformation $\theta\colon \mathrm{Id}_{\mathcal{Y}} \to \Omega_{\rm nc}$;
- A explicit homotopy inverse ι_X for the dg-projective resolution $\epsilon \otimes_\Lambda \mathrm{Id}_X \colon \mathbb{B} \otimes_\Lambda X \to X$.
- These data are compatible.

Noncommutative differential forms

- For a complex X, we define the *GRADED noncommutative differential* 1-*forms* with values in X to be

$$\Omega_{\mathrm{nc}}(X) = s\overline{\Lambda} \otimes X.$$

As a complex, $|s\bar{a} \otimes x| = |x| - 1$, $d(s\bar{a} \otimes x) = -s\bar{a} \otimes d_X(x)$. The (somewhat nontrivial) Λ-action is given by

$$b \blacktriangleright (s\bar{a} \otimes x) = s\overline{ba} \otimes x - s\bar{b} \otimes ax.$$

- Observe that $\Omega_{\mathrm{nc}}(\Lambda)$ is the *graded bimodule of right noncommutative differential* 1-*forms* in [Wang 2016]; compare [Cuntz-Quillen 1995].

Noncommutative differential forms, continued

- We have $\Omega_{\rm nc}(X) \simeq \Omega_{\rm nc}(\Lambda) \otimes_\Lambda X$.
- For a Λ-module M, we have an exact sequence

$$0 \longrightarrow \overline{\Lambda} \otimes M \longrightarrow \Lambda \otimes M \longrightarrow M \longrightarrow 0,$$

where the left arrow $\bar{a} \otimes x \mapsto a \otimes x - 1_\Lambda \otimes ax$.

Therefore, we have

$$\Omega_{\rm nc}(M) \simeq \Sigma\Omega(M).$$

Here, $\Omega(M)$ denotes the first syzygy of M. This isomorphism justifies the notation "$\Omega_{\rm nc}$"!

Noncommutative differential forms as a dg endofunctor

- $\Omega_{\rm nc}\colon \mathcal{Y} \to \mathcal{Y}$ is a dg endofunctor, whose action on morphisms is given by

$$(f\colon X \to Y) \longmapsto (\mathrm{Id}_{s\overline{\Lambda}} \otimes f\colon \Omega_{\rm nc}(X) \to \Omega_{\rm nc}(Y)).$$

- There is a closed natural transformation of degree zero:

$$\theta\colon \mathrm{Id}_{\mathcal{Y}} \longrightarrow \Omega_{\rm nc}$$

such that $\theta_Y\colon Y \to \Omega_{\rm nc}(Y)$ is of filtration-degree *one*, represented by $\mathrm{Id}_{\Omega_{\rm nc}(Y)}\colon s\overline{\Lambda} \otimes Y \to \Omega_{\rm nc}(Y)$.

- $\theta\Omega_{\rm nc} = \Omega_{\rm nc}\theta$.

An explicit homotopy inverse

- We define a closed morphism of degree zero in $\mathcal{Y}$:

$$\iota_X \colon X \longrightarrow \mathbb{B} \otimes_\Lambda X$$

such that its entry $(\iota_X)_p$ of filtration-degree p is given by

$$(s\bar{\Lambda})^{\otimes p} \otimes X \longrightarrow \mathbb{B}^{-p} \otimes_\Lambda X \subseteq \mathbb{B} \otimes_\Lambda X,\ s\bar{a}_{1,p} \otimes x \longmapsto (1 \otimes s\bar{a}_{1,p} \otimes 1) \otimes_\Lambda x.$$

- $(\varepsilon \otimes_\Lambda \mathrm{Id}_X) \odot \iota_X = \mathrm{Id}_X$. It follows that ι_X is a *homotopy inverse* of the dg-projective resolution $\epsilon \otimes_\Lambda \mathrm{Id}_X \colon \mathbb{B} \otimes_\Lambda X \to X$.

A commutative diagram in $\mathcal{Y}$

Proposition

The following diagram strictly commutes.

$$\begin{tikzcd}
X \arrow[rr, "\theta_X"] \arrow[d, "\iota_X"'] & & \Omega_{\rm nc}(X) \arrow[d, "\iota_{\Omega_{\rm nc}(X)}"] \\
\mathbb{B} \otimes_\Lambda X \arrow[r, "\rm pr"] & \mathbb{B}_{\geq 1} \otimes_\Lambda X \arrow[r, "\sim"] & \mathbb{B} \otimes_\Lambda \Omega_{\rm nc}(X)
\end{tikzcd}$$

In comparison,

$$\begin{tikzcd}
X \arrow[r, "\theta_X"] & \Omega_{\rm nc}(X) \\
\mathbb{B} \otimes_\Lambda X \arrow[u, "\varepsilon \otimes_\Lambda \mathrm{Id}_X"] \arrow[r] & \mathbb{B} \otimes_\Lambda \Omega_{\rm nc}(X) \arrow[u, "\varepsilon \otimes_\Lambda \mathrm{Id}_{\Omega_{\rm nc}(X)}"']
\end{tikzcd}$$

does NOT commute because of different filtration-degree.

The dg-injective resolutions via $\mathcal{Y}$

- For any complex X, $\mathcal{Y}(\Lambda, X) = \prod_{n\geq 0} \mathrm{Hom}((s\overline{\Lambda})^{\otimes n} \otimes \Lambda, X)$ is a dg-injective complex of Λ-*modules*.
- The natural embedding $X \to \mathcal{Y}(\Lambda, X)$ is a quasi-isomorphism; consequently, we have $\mathbf{i} \simeq \mathcal{Y}(\Lambda, -)$.
- $\mathcal{Y}(\Lambda, \Lambda)$ is a complex of Λ-bimodules, and $\Lambda \to \mathcal{Y}(\Lambda, \Lambda)$ is a quasi-isomorphism between complexes of bimodules.

An explicit form of Krause's recollement

We rewrite Krause's recollement.

$$\begin{tikzcd}[column sep=huge] \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-}\mathsf{Inj}) \arrow[r, "\mathrm{inc}"] & \mathbf{K}(\Lambda\text{-}\mathsf{Inj}) \arrow[l, bend right, "\bar{\mathbf{a}}"'] \arrow[l, bend left, "\mathbf{a}'=\mathrm{Hom}_\Lambda(\mathrm{Cone}(\varepsilon),-)"] \arrow[r, "\mathrm{can}"] & \mathbf{D}(\Lambda\text{-}\mathsf{Mod}) \arrow[l, bend right, "\bar{\mathbf{p}}=\mathcal{Y}(\Lambda,\Lambda)\otimes_\Lambda \mathbb{B}\otimes_\Lambda -"'] \arrow[l, bend left, "\mathbf{i}=\mathcal{Y}(\Lambda,-)"] \end{tikzcd}$$

Here, $\bar{\mathbf{p}} \simeq \mathcal{Y}(\Lambda,\Lambda) \otimes_\Lambda \mathbf{p}-$, where $\mathbf{p}$ is the dg-projective resolution functor, and identified with $\mathbb{B} \otimes_\Lambda -$.

However, we can NOT describe the counit $\bar{\mathbf{p}}\mathrm{can}I \to I$ explicitly.

The stabilization functor as a cone

For each complex X, we have a chain map

$$\kappa_X \colon \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \longrightarrow \mathcal{Y}(\Lambda, X)$$

$$f \otimes_\Lambda (a_0 \otimes s\bar{a}_{1,q} \otimes 1) \otimes_\Lambda x \longmapsto (s\bar{b}_{1,p} \otimes b \mapsto \delta_{q,0} f(s\bar{b}_{1,p} \otimes b) a_0 x).$$

Proposition

We have a functorial exact triangle

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \xrightarrow{\kappa_X} \mathcal{Y}(\Lambda, X) \longrightarrow \mathbb{S}X \longrightarrow \Sigma \mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X$$

Section III

- The stabilization functor
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A reminder

- The Yoneda dg category $\mathcal{Y}$, composition denoted by $\odot$;
- A dg endofunctor $\Omega_{\rm nc}\colon \mathcal{Y} \to \mathcal{Y}$;
- A closed natural transformation $\theta\colon \mathrm{Id}_{\mathcal{Y}} \to \Omega_{\rm nc}$;
- Restricting to bounded complexes of finitely generated modules, we have $\mathcal{Y}^f \simeq \mathbf{D}^b_{\rm dg}(\Lambda\text{-mod})$.

The singular Yoneda dg category

The *singular Yoneda dg category* $\mathcal{SY}$ of Λ is defined as follows.

- It has the same objects as $\mathcal{Y}$.
- The Hom complex $\mathcal{SY}(X, Y)$ is the colimit of

$$\mathcal{Y}(X, Y) \to \mathcal{Y}(X, \Omega_{\rm nc}(Y)) \to \mathcal{Y}(X, \Omega^2_{\rm nc}(Y)) \to \cdots,$$

  where the structure maps send $f\colon X \to \Omega^p_{\rm nc}(Y)$ to $\theta_{\Omega^p_{\rm nc}(Y)} \odot f$. The canonical image of such an f in $\mathcal{SY}(X, Y)$ is denoted by $[f; p]$.
- The composition of $[f; p]\colon X \to Y$ and $[g; q]\colon Y \to Z$ is

$$[g; q] \odot_{\rm sg} [f; p] = [\Omega^p_{\rm nc}(g) \odot f; p + q].$$

The singular Yoneda dg category vs the dg singularity category

Recall $\mathbf{S}_{\rm dg}(\Lambda) = \mathbf{D}^b_{\rm dg}(\Lambda\text{-mod})/C^b_{\rm dg}(\Lambda\text{-proj})$.

Theorem (C.-Wang)

We have $\mathcal{SY}^f \simeq \mathbf{S}_{\rm dg}(\Lambda)$. Consequently, the singular Yoneda dg category $\mathcal{SY}$ contains a dg enhancement of $\mathbf{D}_{\rm sg}(\Lambda)$.

Therefore, $\mathcal{SY}(M, M)$ computes the *singular Yoneda Ext algebra* or *Tate cohomology algebra* $\widehat{\mathrm{Ext}}^*_\Lambda(M, M)$ of M.

The Hom complexes in $\mathcal{SY}$

- An observation: $\mathcal{Y}(\Lambda, X) = \prod_{n\geq 0} \mathrm{Hom}((s\overline{\Lambda})^{\otimes n} \otimes \Lambda, X)$ is a complex of INJECTIVE Λ-modules.
- $\mathcal{SY}(\Lambda, X) = \mathrm{colim}\ \mathcal{Y}(\Lambda, \Omega^p_{\mathrm{nc}}(X))$ is also a complex of injective modules, ACYCLIC!

Proposition

There is a well-defined triangle functor

$$\mathcal{SY}(\Lambda, -) \colon \mathbf{D}(\Lambda\text{-}\mathrm{Mod}) \longrightarrow \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-}\mathrm{Inj}).$$

The Hom functor $\mathcal{SY}(\Lambda, -)$ as a cone

Proposition

There is a functorial exact triangle

$$\operatorname{colim} \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X) \xrightarrow{\vartheta_X} \mathcal{Y}(\Lambda, X) \longrightarrow \mathcal{SY}(\Lambda, X) \longrightarrow$$

where ϑ_X *is induced by* $\mathbb{B}_{\leq p} \otimes_\Lambda X \to X$.

The proof uses the commutative diagram in $\mathcal{Y}$ and its generalization:

$$\begin{tikzcd} X \arrow[rr, "\theta_X"] \arrow[d, "\iota_X"'] & & \Omega_{\mathrm{nc}}(X) \arrow[d, "\iota_{\Omega_{\mathrm{nc}}(X)}"] \\ \mathbb{B} \otimes_\Lambda X \arrow[r, "\mathrm{pr}"] & \mathbb{B}_{\geq 1} \otimes_\Lambda X \arrow[r, "\sim"] & \mathbb{B} \otimes_\Lambda \Omega_{\mathrm{nc}}(X) \end{tikzcd}$$

The comparison theorem

Recall both $\mathbb{S}$, $\mathcal{SY}(\Lambda, -)\colon \mathbf{D}(\Lambda\text{-Mod}) \to \mathbf{K}_{\mathrm{ac}}(\Lambda\text{-Inj})$, as certain explicit cones.

Theorem (C.-Wang)

There is a natural transformation

$$c\colon \mathbb{S} \longrightarrow \mathcal{SY}(\Lambda, -)$$

such that its restriction to $\mathbf{D}^+(\Lambda\text{-Mod})$ *is an isomorphism. If* Λ *is Gorenstein, then* c *is an isomorphism.*

Remark: it is not clear whether c is an isomorphism for non-Gorenstein Λ.

The key ingredient of the proof

We compare the two cones:

$$
\begin{tikzcd}
\mathcal{Y}(\Lambda,\Lambda)\otimes_\Lambda \mathbb{B}\otimes_\Lambda X \arrow[r] \arrow[d, dotted] & \mathcal{Y}(\Lambda,X) \arrow[r] \arrow[d, equal] & \mathbb{S}X \arrow[r] \arrow[d, dotted, "c_X"] & {} \\
\mathrm{colim}\ \mathcal{Y}(\Lambda,\mathbb{B}_{\leq p}\otimes_\Lambda X) \arrow[r] & \mathcal{Y}(\Lambda,X) \arrow[r, "\mathrm{can}"] & \mathcal{S}\mathcal{Y}(\Lambda,X) \arrow[r] & {}
\end{tikzcd}
$$

We have to analyze the leftmost dotted arrow.

A colimit of cochain maps

The above dotted arrow

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B} \otimes_\Lambda X \longrightarrow \mathrm{colim}\, \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X)$$

is the colimit of the following

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B}_{\leq p} \otimes_\Lambda X \longrightarrow \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X).$$

Proposition

Assume that X is bounded-below, or Λ is Gorenstein. Then the above map

$$\mathcal{Y}(\Lambda, \Lambda) \otimes_\Lambda \mathbb{B}_{\leq p} \otimes_\Lambda X \longrightarrow \mathcal{Y}(\Lambda, \mathbb{B}_{\leq p} \otimes_\Lambda X)$$

is a homotopy equivalence.

Main references

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Thank you very much for your attention!

谢谢大家!

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