Two results, both developments of a 2015 article by Krause

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Overview

1. k-linear categories and dg categories
2. Existence and uniqueness of enhancements of triangulated categories, and enhanceability of functors
3. Key new ideas
4. Grothendieck's K_0
5. Higher K–theory
6. The counterexample

Let k be a commutative ring; we fix it throughout.

Let $k\text{–}\mathrm{Mod}$ be the category of k-modules and k-linear maps.

Let $\mathbf{C}(k)$ be the category of cochain complexes of k-modules. The objects are cochain complexes A^* of k-modules, and the morphisms are the cochain maps. In pictures

Objects:

$$\cdots \longrightarrow A^{-2} \xrightarrow{\partial^{-2}} A^{-1} \xrightarrow{\partial^{-1}} A^{0} \xrightarrow{\partial^{0}} A^{1} \xrightarrow{\partial^{1}} A^{2} \longrightarrow \cdots$$

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Morphisms:

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 & & \downarrow{\scriptstyle f^{-2}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^{0}} & & \downarrow{\scriptstyle f^{1}} & & \downarrow{\scriptstyle f^{2}} & & \\
\cdots & \longrightarrow & B^{-2} & \xrightarrow{\partial^{-2}} & B^{-1} & \xrightarrow{\partial^{-1}} & B^{0} & \xrightarrow{\partial^{0}} & B^{1} & \xrightarrow{\partial^{1}} & B^{2} & \longrightarrow & \cdots
\end{array}$$

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In $\mathbf{C}(k)$: if $A^*, B^* \in \mathrm{Ob}(\mathbf{C}(k))$ then $A^* \otimes B^*$ and $\mathcal{H}\mathrm{om}(A^*, B^*)$ are objects of $\mathbf{C}(k)$ given by the formulas

$$\left[A^* \otimes B^*\right]^n \quad = \quad \bigoplus_{i \in \mathbb{Z}} \left[A^{-i} \otimes_k B^{n+i}\right]$$

$$\left[\mathcal{H}\mathrm{om}(A^*, B^*)\right]^n \quad = \quad \prod_{i \in \mathbb{Z}} \left[\mathrm{Hom}_k(A^i, B^{n+i})\right]$$

The unit $\mathbb{1} \in \mathrm{Ob}(\mathbf{C}(k))$ is the complex with k in degree zero

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

k-linear and dg (differential graded) categories

Definition (k linear and dg categories)

A k-linear category is a category enriched over $k\text{–}\mathrm{Mod}$, and a dg category is a category enriched over $\mathbf{C}(k)$.

G. Maxwell Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, Cambridge-New York, 1982.

G. Maxwell Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, Reprint of the 1982 original [Cambridge Univ. Press].

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1. Give a set of objects $\mathrm{Ob}(\mathcal{C})$.
2. For every pair of objects $C_1, C_2 \in \mathrm{Ob}(\mathcal{C})$ we must give a cochain complex $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$.
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3. There must be a composition law. Thus for every triple of objects $C_1, C_2, C_3 \in \mathrm{Ob}(\mathcal{C})$ we must be given a cochain map

$$\mathrm{Hom}_{\mathcal{C}}(C_2, C_3) \otimes \mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C_1, C_3)$$

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4. For every object $C \in \mathrm{Ob}(\mathcal{C})$ we must give a map $\mathbb{1} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, C)$.

dg functors

Let $\mathcal{C}, \mathcal{D}$ be dg categories. A dg functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is the following:

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$$\mathrm{Hom}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(f(C), f(C'))$$
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1. A function $\mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D})$.
2. For every pair of objects $C, C' \in \mathrm{Ob}(\mathcal{C})$, we must be given a cochain map
$$\mathrm{Hom}_{\mathcal{C}}(C, C') \longrightarrow \mathrm{Hom}_{\mathcal{D}}(f(C), f(C'))$$
3. Composition and identities are respected.

Let $\mathcal{C}$ be a dg category. The k-linear category $H^0(\mathcal{C})$ has

1. $$\text{Ob}(H^0(\mathcal{C})) \quad = \quad \text{Ob}(\mathcal{C})$$

2. $$\mathrm{Hom}_{H^0(\mathcal{C})}(C_1, C_2) \quad = \quad H^0\left[\mathrm{Hom}_{\mathcal{C}}(C_1, C_2)\right]$$

If $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a dg functor between dg categories, then $H^0(f) : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$ is the obvious k-linear functor between k-linear categories.

Pretriangulated dg categories

Every dg category $\mathcal{C}$ has a Yoneda functor

$$\mathcal{Y}\colon \mathcal{C} \longrightarrow \mathrm{Hom}_{\mathsf{dg}}(\mathcal{C}^{\mathrm{op}}, \mathbf{C}(k))$$

On objects $$\mathcal{Y}(C) = \mathrm{Hom}_{\mathcal{C}}(-, C)$$

The category $\mathcal{C}$ is pretriangulated if the essential image of the functor $\mathcal{Y}$ is closed in $\mathrm{Hom}_{\mathsf{dg}}(\mathcal{C}^{\mathrm{op}}, \mathbf{C}(k))$ under mapping cones.

The information relevant to us

If $\mathcal{C}$ is pretriangulated then $H^0(\mathcal{C})$ is a k-linear triangulated category. If $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a dg functor of pretriangulated dg categories, then $H^0(f) : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$ is a k-linear exact functor of k-linear triangulated categories.

Quasi-equivalences and unique enhancements

Definition

Let $\mathcal{C}, \mathcal{D}$ be dg categories. A dg functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ is a quasi-equivalence if $H^0(f) : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$ is an equivalence of categories.

Definition

Let $\mathcal{T}$ be a k-linear triangulated category. We say that $\mathcal{T}$ has a unique enhancement if any two pretriangulated dg categories $\mathcal{D}$ and $\mathcal{D}'$, with $H^0(\mathcal{D}) \cong \mathcal{T} \cong H^0(\mathcal{D}')$ as triangulated categories, admit a zigzag of dg functors, all quasi-equivalences

$$\begin{array}{ccccccccccc} & & \mathcal{C}_1 & & & & \mathcal{C}_2 & & \cdots & & \mathcal{C}_n & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & & \swarrow & & \searrow & \\ \mathcal{D} & & & & \mathcal{C}'_1 & & & & \mathcal{C}'_2 & \cdots & \mathcal{C}'_{n-1} & & \mathcal{D}' \end{array}$$

Enhancement of functors

If $\mathcal{D}, \mathcal{D}'$ are pretriangulated dg categories and $F : H^0(\mathcal{D}) \longrightarrow H^0(\mathcal{D}')$ is a k-linear exact functor of k-linear triangulated categories, we say that F is enhanceable to $\mathcal{D}, \mathcal{D}'$ if there is a zigzag of dg functors

$$
\begin{array}{ccccccccccc}
 & \mathcal{C}_1 & & \mathcal{C}_2 & & \cdots & & \mathcal{C}_n & \\
\swarrow & & \searrow \swarrow & & \searrow \swarrow & & \searrow \swarrow & & \searrow \\
\mathcal{D} & & \mathcal{C}'_1 & & \mathcal{C}'_2 & \cdots & \mathcal{C}'_{n-1} & & \mathcal{D}'
\end{array}
$$

whose image under H^0 composes to F.

Questions

Does every k-linear triangulated category have a dg enhancement?

Can a k-linear triangulated category have two or more nonequivalent enhancements?

Does every k-linear triangulated functor $F : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$ have a dg enhancement to $\mathcal{C}, \mathcal{D}$?

Note

If $H^0(\mathcal{C})$ and/or $H^0(\mathcal{D})$ have more than one enhancement, saying that a functor $F : H^0(\mathcal{C}) \longrightarrow H^0(\mathcal{D})$ is enhanceable depends on the choice of enhancements.

Alexey I. Bondal and Mikhail M. Kapranov, *Enhanced triangulated categories*, Mat. Sb. **181** (1990), no. 5, 669–683.

This paper is the origin of dg enhancements—it sets up the theory.

Amnon Neeman, *Stable homotopy as a triangulated functor*, Inventiones Mathematicae **109** (1992), 17–40.

Gave the first example of a non-enhanceable exact functor of triangulated categories. More precisely: it produces a non-enhanceble exact functor $F : \mathbf{D}^b(\mathbb{Z}[\frac{1}{2}]) \longrightarrow \mathcal{T}^b[\frac{1}{2}]$.

The category $\mathcal{T}^b[\frac{1}{2}]$ is topological; it is the category of finite spectra with 2 inverted. It doesn't have a dg enhancement—but it is topologically enhanceable.

Example (Triangulated category with two nonequivalent enhancements)

Let $k = \mathbb{Z}/p$, the field with p elements. For any integer $n > 0$ consider the (tensor) triangulated category $\mathcal{T}$, defined by

1. $A[2p^n - 2] = A$.
2. Every object $A \in \mathcal{T}$ is a direct sum of shifts of $\mathbb{1}$, the identity of the tensor product.
3. $$\mathrm{Hom}(\mathbb{1}, \mathbb{1}[r]) = \begin{cases} k & \text{if } (2p^n - 2)|r \\ 0 & \text{otherwise} \end{cases}$$
4. The exact triangles are all isomorphic to direct sums of rotations of $0 \longrightarrow \mathbb{1} \xrightarrow{1} \mathbb{1} \longrightarrow 0$.

Obvious dg enhancement

The category of graded vector spaces over the graded field $k[x, x^{-1}]$, where degree$(x) = 2p^n - 2$.

Second enhancement, topological

The category of modules over the ring spectrum $K(n)$, the nth Morava K-theory at the prime p.

I first learned about this, as a conjectured counterexample, from Jeff Smith in the mid 1990s. For a proof that the enhancements are different see Section 2.1 of

Stefan Schwede, *The stable homotopy category has a unique model at the prime 2*, Adv. Math. **164** (2001), no. 1, 24–40.

Uniqueness of enhancements of triangulated categories, second counterexample

Marco Schlichting, *A note on K-theory and triangulated categories*, Invent. Math. **150** (2002), no. 1, 111–116.

Provided the second example of a triangulated category with two nonequivalent enhancements.

Let k be a perfect field of characteristic $p > 0$. Then it's easy to see that there is an equivalence of k-linear triangulated categories

$$\mathbf{D}_{\mathrm{sg}}(k[\varepsilon]/\varepsilon^2) \quad \cong \quad \mathbf{D}_{\mathrm{sg}}(W_2(k)) ,$$

where $W_2(k)$ is the length-2 Witt ring of k.

Thus the triangulated category comes with two natural dg enhancements, one k-linear and one $W_2(k)$-linear.

If $k = \mathbb{Z}/p$ Schlichting proves that these aren't equivalent.

Enhanceability for functors $F : \mathbf{D}^b(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^b(\mathrm{coh}(Y))$, with X, Y smooth and projective over a field k and with the standard enhancements for $\mathbf{D}^b(\mathrm{coh}(X))$ and $\mathbf{D}^b(\mathrm{coh}(Y))$

Dmitri O. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381, Algebraic geometry, 7.

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- Dmitri O. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381, Algebraic geometry, 7.
- Yujiro Kawamata, *Equivalences of derived categories of sheaves on smooth stacks*, Amer. J. Math. **126** (2004), no. 5, 1057–1083.
- Alberto Canonaco and Paolo Stellari, *Twisted Fourier-Mukai functors*, Adv. Math. **212** (2007), no. 2, 484–503.
- Alice Rizzardo, *On the existence of Fourier-Mukai functors*, Math. Z. **287** (2017), no. 1-2, 155–179.

Alice Rizzardo, Michel Van den Bergh, and Amnon Neeman, *An example of a non-Fourier-Mukai functor between derived categories of coherent sheaves*, Invent. Math. **216** (2019), no. 3, 927–1004.

Vadim Vologodsky, *Triangulated endofunctors of the derived category of coherent sheaves which do not admit DG liftings*, Arnold Math. J. **5** (2019), no. 1, 139–143.

The article with brave conjectures

Alexey I. Bondal, Michael Larsen, and Valery A. Lunts, *Grothendieck ring of pretriangulated categories*, Int. Math. Res. Not. (2004), no. 29, 1461–1495.

Conjectured that (a) reasonable categories, such as $\mathbf{D}^b(\mathrm{coh}(X))$, have unique enhancements, and (b) exact functors between them are all enhanceable.

Existence of enhancement, counterexamples

Fernando Muro, Stefan Schwede, and Neil Strickland, *Triangulated categories without models*, Invent. Math. **170** (2007), no. 2, 231–241.

Shows that there exist non-enhanceable categories. More precisely: the category of free $\mathbb{Z}/4$-modules.

The triangulated structure is given as follows.

1. $A[1] = A$.
2. The exact triangles are isomorphs of direct sums of rotations of two basic triangles:

$$0 \longrightarrow \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/4 \longrightarrow 0$$

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

Existence of enhancement, counterexamples

Alice Rizzardo and Michel Van den Bergh, *A k-linear triangulated category without a model*, Ann. of Math. (2) **191** (2020), no. 2, 393–437.

Shows that there exist non-enhanceable categories linear over a field k.

Uniqueness of enhancements, early positive results

Stefan Schwede, *The stable homotopy category has a unique model at the prime 2*, Adv. Math. **164** (2001), no. 1, 24–40.

Stefan Schwede, *The stable homotopy category is rigid*, Ann. of Math. (2) **166** (2007), no. 3, 837–863.

Uniqueness of enhancements, positive results

Valery A. Lunts and Dmitri O. Orlov, *Uniqueness of enhancement for triangulated categories*, J. Amer. Math. Soc. **23** (2010), no. 3, 853–908.

Alberto Canonaco and Paolo Stellari, *Uniqueness of dg enhancements for the derived category of a Grothendieck category*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 11, 2607–2641.

Benjamin Antieau, *On the uniqueness of infinity-categorical enhancements of triangulated categories*, `arXiv:1812.01526`.

The categories $\mathbf{D}^?(\mathcal{A})$

Theorem (Canonaco, N- and Stellari)

Let $\mathcal{A}$ be an abelian category. For $? \in \{b, +, -, \emptyset\}$ we have that the category $\mathbf{D}^?(\mathcal{A})$ has a unique enhancement.

1. If $\mathcal{A}$ is a Grothendieck category with a small set of compact generators, the uniqueness of the enhancement of $\mathbf{D}(\mathcal{A})$ is due to Lunts and Orlov.
2. If $\mathcal{A}$ is an arbitrary Grothendieck abelian category, the fact that $\mathbf{D}(\mathcal{A})$ has a unique enhancement is due to Canonaco and Stellari.
3. The special case, showing $\mathbf{D}^b(\mathrm{coh}(X))$ has a unique enhancement, was shown by Lunts and Orlov if X is quasi-projective and by Canonaco and Stellari if X has the resolution property.
4. Antieau proved that $\mathbf{D}^b(\mathcal{A})$, $\mathbf{D}^-(\mathcal{A})$ and $\mathbf{D}^+(\mathcal{A})$ have unique enhancements, for any abelian category $\mathcal{A}$.

Henning Krause, *Deriving Auslander's formula*, Doc. Math. **20** (2015), 669–688.

Maurice Auslander, *Representation theory of Artin algebras*, Queen Mary College, London (1971).

Henning Krause, *Deriving Auslander's formula*, Doc. Math. **20** (2015), 669–688.

Maurice Auslander, *Representation theory of Artin algebras*, Queen Mary College, London (1971).

G. Maxwell Kelly, *Chain maps inducing zero homology maps*, Proc. Cambridge Philos. Soc. **61** (1965), 847–854.

Henning Krause, *Deriving Auslander's formula*, Doc. Math. **20** (2015), 669–688.

Key new ideas

Let $\mathcal{A}$ be an abelian category, and consider the Verdier quotient functor $\pi : \mathbf{K}^?(\mathcal{A}) \longrightarrow \mathbf{D}^?(\mathcal{A})$.

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We will study a diagram

$$\begin{array}{ccc}
\mathbf{V}^?(\mathcal{A}) & \longrightarrow & \mathbf{B}^?(\mathcal{A}) \\
\big\downarrow & & \big\downarrow \\
\mathbf{K}^?(\mathcal{A}) & \xrightarrow{\ \pi\ } & \mathbf{D}^?(\mathcal{A})
\end{array}$$

where the vertical maps are inclusions of full subcategories.

The objects, of either $\mathbf{V}^?(\mathcal{A})$ or $\mathbf{B}^?(\mathcal{A})$, are the complexes with zero differentials

$$\cdots \longrightarrow A^{-2} \xrightarrow{\ 0\ } A^{-1} \xrightarrow{\ 0\ } A^{0} \xrightarrow{\ 0\ } A^{1} \xrightarrow{\ 0\ } A^{2} \longrightarrow \cdots$$

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In the category $\mathbf{V}^?(\mathcal{A})$ we have

$$\bigoplus_{i=-\infty}^{\infty} A^i[-i] \quad = \quad \coprod_{i=-\infty}^{\infty} A^i[-i] \quad = \quad \prod_{i=-\infty}^{\infty} A^i[-i] \ .$$

Key lemmas

Lemma (1)

With the notation as above we have that, in the category $\mathbf{D}^?(\mathcal{A})$,

$$\bigoplus_{i=-\infty}^{0} A^i[-i] = \coprod_{i=-\infty}^{0} A^i[-i] \,, \qquad \bigoplus_{i=0}^{\infty} A^i[-i] = \prod_{i=0}^{\infty} A^i[-i] \,.$$

Lemma (2)

With the notation as on the next slide, we have $\langle \mathbf{V}^?(\mathcal{A}) \rangle_3 = \mathbf{K}^?(\mathcal{A})$ *and therefore also* $\langle \mathbf{B}^?(\mathcal{A}) \rangle_3 = \mathbf{D}^?(\mathcal{A})$.

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Reminder of the terminology in the second lemma

Alexey I. Bondal and Michel Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

Reminder

Let $\mathcal{T}$ be a small triangulated category, and let $S \subset \mathcal{T}$ be a set of objects. We define

1. $\langle S \rangle_1$ is the set of all direct summands of finite direct sums of shifts of objects in S.
2. An object y belongs to $\langle S \rangle_{n+1}$ if there exists a triangle

$$x \longrightarrow y \oplus y' \longrightarrow z \longrightarrow x[1]$$

with $x \in \langle S \rangle_n$ and $z \in \langle S \rangle_1$.

Where Auslander's formula comes in

Given an abelian category $\mathcal{A}$, Auslander considered the embedding $\mathcal{A} \longrightarrow \text{mod}\,\mathcal{A}$. That is $\mathcal{A}$ embeds in the category of finitely presented additive functors $F : \mathcal{A} \longrightarrow \mathcal{AB}$.

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Recall: the functor $F : \mathcal{A} \longrightarrow \mathcal{AB}$ is finitely presented if there exists an exact sequence

$$\mathrm{Hom}(-, A) \longrightarrow \mathrm{Hom}(-, B) \longrightarrow F(-) \longrightarrow 0 \ .$$

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And one notes (1) the representable functors are the projective objects in $\operatorname{mod} \mathcal{A}$, and (2) every object in $\operatorname{mod} \mathcal{A}$ has projective dimension ≤ 2. Let K be the kernel of $A \longrightarrow B$ above, and we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}(-, K) \longrightarrow \mathrm{Hom}(-, A) \longrightarrow \mathrm{Hom}(-, B) \longrightarrow F(-) \longrightarrow 0 \ .$$

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Because every object in $\operatorname{mod}\mathcal{A}$ has finite projective dimension we have

$$\mathbf{K}^?(\mathcal{A}) \quad = \quad \mathbf{K}^?(\operatorname{proj}(\operatorname{mod}\mathcal{A})) \quad = \quad \mathbf{D}^?(\operatorname{mod}\mathcal{A})$$

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Because every object in $\operatorname{mod}\mathcal{A}$ has finite projective dimension we have

$$\mathbf{K}^?(\mathcal{A}) \quad = \quad \mathbf{K}^?(\operatorname{proj}(\operatorname{mod}\mathcal{A})) \quad = \quad \mathbf{D}^?(\operatorname{mod}\mathcal{A})$$

And because every object in $\operatorname{mod}\mathcal{A}$ has projective dimension ≤ 2, Kelly's 1965 theorem tells us that the projectives generate the category in three steps: that is

$$\mathbf{K}^?(\mathcal{A}) \quad = \quad \langle \mathbf{V}^?(\mathcal{A}) \rangle_3 \; .$$

Sketch how to pass from the key lemmas to the theorem

Let $\mathcal{C}$ be a dg enhancement of $\mathbf{D}^?(\mathcal{A})$, and let $\widetilde{\mathcal{C}}$ be the canonical enhancement of $\mathbf{K}^?(\mathcal{A})$. We would like to enhance the canonical Verdier quotient map $\pi : \mathbf{K}^?(\mathcal{A}) \longrightarrow \mathbf{D}^?(\mathcal{A})$ to a dg functor $\widetilde{\pi} : \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}$, which would then have to factor as $\widetilde{\mathcal{C}} \longrightarrow \frac{\widetilde{\mathcal{C}}}{\mathsf{Acy}(\widetilde{\mathcal{C}})} \longrightarrow \mathcal{C}$.

Let $\mathcal{V} \subset \widetilde{\mathcal{C}}$ be the inverse image of $\mathbf{V}^?(\mathcal{A}) \subset \mathbf{K}^?(\mathcal{A})$, and let $\mathcal{B} \subset \mathcal{C}$ be the inverse image of $\mathbf{B}^?(\mathcal{A}) \subset \mathbf{D}^?(\mathcal{A})$.

By the second lemma, which says that $\langle \mathbf{V}^?(\mathcal{A}) \rangle_3 = \mathbf{K}^?(\mathcal{A})$ and $\langle \mathbf{B}^?(\mathcal{A}) \rangle_3 = \mathbf{D}^?(\mathcal{A})$, we have quasi-equivalences $\mathcal{P}\mathrm{erf}(\mathcal{V}) \cong \widetilde{\mathcal{C}}$ and $\mathcal{P}\mathrm{erf}(\mathcal{B}) \cong \mathcal{C}$.

It suffices to produce a quasi-functor $\mathcal{V} \longrightarrow \mathcal{B}$, as it would induce a quasi-functor $\mathcal{P}\mathrm{erf}(\mathcal{V}) \longrightarrow \mathcal{P}\mathrm{erf}(\mathcal{B})$, that is $\widetilde{\pi} : \widetilde{\mathcal{C}} \longrightarrow \mathcal{C}$.

The objects of $\mathcal{V}$ and $\mathcal{B}$ are the same, we need to come up with a map $\mathrm{Hom}_{\mathcal{V}}(A,B) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(A,B)$. For the objects of $\mathcal{V}$ we have, in the dg category $\mathcal{V}$, isomorphisms

$$\bigoplus_{i=-\infty}^{\infty} A^i[-i] \quad = \quad \coprod_{i=-\infty}^{\infty} A^i[-i] \quad = \quad \prod_{i=-\infty}^{\infty} A^i[-i] \ .$$

If the same were true for $\mathcal{B}$ then, for $(-) \in \{\mathcal{B}, \mathcal{V}\}$

$$\mathrm{Hom}_{(-)}\left(\bigoplus_{i=-\infty}^{\infty} A^i[-i]\ ,\ \bigoplus_{j=-\infty}^{\infty} B^j[-j]\right)$$

would rewrite as

$$\prod_{i=-\infty}^{\infty}\prod_{j=-\infty}^{\infty} \mathrm{Hom}_{(-)}(A^i[-i], B^j[-j])$$

and in the derived category $\mathbf{D}(k)$ the map

$$\begin{array}{c}\prod_{i=-\infty}^{\infty}\prod_{j=-\infty}^{\infty}\mathrm{Hom}_{\mathcal{V}}\big(A^i[-i],B^j[-j]\big)\\ \Big\downarrow \\ \prod_{i=-\infty}^{\infty}\prod_{j=-\infty}^{\infty}\mathrm{Hom}_{\mathcal{B}}\big(A^i[-i],B^j[-j]\big)\end{array}$$

would rewrite as

$$\begin{array}{c}\prod_{i=-\infty}^{\infty}\prod_{j=-\infty}^{\infty}\mathrm{Hom}_{\mathcal{B}}\big(A^i[-i],B^j[-j]\big)^{\leq j-i}\\ \Big\downarrow \\ \prod_{i=-\infty}^{\infty}\prod_{j=-\infty}^{\infty}\mathrm{Hom}_{\mathcal{B}}\big(A^i[-i],B^j[-j]\big)\end{array}$$

In the category $\mathcal{B}$, write

$$A^* \quad = \quad \bigoplus_{i=-\infty}^{\infty} A^i \quad = \quad \left[\bigoplus_{i=-\infty}^{-n-1} A^i\right] \oplus \bigoplus_{i=-n}^{n} A^i \oplus \left[\bigoplus_{i=n+1}^{\infty} A^i\right]$$

and

$$B^* \quad = \quad \bigoplus_{i=-\infty}^{\infty} B^i \quad = \quad \left[\bigoplus_{i=-\infty}^{-n-1} B^i\right] \oplus \bigoplus_{i=-n}^{n} B^i \oplus \left[\bigoplus_{i=n+1}^{\infty} B^i\right]$$

Then $\mathrm{Hom}_{\mathcal{B}}(A^*, B^*)$ is a finite product, which we truncate as much as we can. And then we take inverse limits as $n \longrightarrow \infty$.

Grothendieck's K_0

Definition

An exact category is an additive category $\mathcal{E}$, with a collection of admissible short exact sequences

$$E' \longrightarrow E \longrightarrow E'',$$

satisfying some axioms.

Example

1. $R\text{–mod}$, the category of finite modules over a noetherian ring R.
2. $R\text{–proj}$, the category of finitely generated projective R-modules.
3. $\mathrm{Vect}(X)$, the category of vector bundles over X.

Example

Given any additive category $\mathcal{E}$, we can turn it into an exact category by declaring the sequences

$$E' \longrightarrow E' \oplus E'' \longrightarrow E''$$

to be the admissible exact sequences. We will write $\mathcal{E}^{\oplus}$ for this exact category.

Remark

If $\mathcal{E} = R\text{–proj}$, then $\mathcal{E} = \mathcal{E}^{\oplus}$. But for $\mathcal{E} = R\text{–mod}$ and $\mathcal{E} = \mathrm{Vect}(X)$, the split exact structure doesn't agree with the ordinary exact structure.

Definition

Let $\mathcal{E}$ be an essentially small exact category. The abelian group $K_0(\mathcal{E})$ is defined by the formula

$$K_0(\mathcal{E}) = \frac{\text{free abelian group generated by objects } E \in \mathcal{E}}{(E - E' - E'') \mid \text{ there exists a an admissible } E' \longrightarrow E \longrightarrow E''}$$

Definition

When X is a reasonable scheme, we define $K_0(X) = K_0\big[\mathrm{Vect}(X)\big]$.

Higher K–theory

Mayer-Vietoris sequence

Given a scheme X and two open sets $U, V \subset X$, there are obvious restriction functors

$$\begin{array}{ccc} \mathrm{Vect}(X) & \longrightarrow & \mathrm{Vect}(U) \\ \downarrow & & \downarrow \\ \mathrm{Vect}(V) & \longrightarrow & \mathrm{Vect}(U \cap V) \end{array}$$

giving maps

$$K_0(X) \longrightarrow K_0(U) \oplus K_0(V) \longrightarrow K_0(U \cap V)$$

This turns out to be exact. We would like to extend to

Mayer-Vietoris sequence, continued

$$
\begin{array}{ccccc}
 & & & & \swarrow \\
K_1(X) & \longrightarrow & K_1(U)\oplus K_1(V) & \longrightarrow & K_1(U\cap V) \\
 & \swarrow & & & \\
K_0(X) & \longrightarrow & K_0(U)\oplus K_0(V) & \longrightarrow & K_0(U\cap V) \\
 & \swarrow & & & \\
K_{-1}(X) & \longrightarrow & K_{-1}(U)\oplus K_{-1}(V) & \longrightarrow & K_{-1}(U\cap V) \\
\swarrow & & & &
\end{array}
$$

This turns out to be possible. It is the culmination of the work of many people.

Vanishing observations

Two conjectures

1. **Weibel's conjecture:** If X is a noetherian scheme of dimension n, then $K_r(X) = 0$ for all $r < -n$.
2. **Schlichting's conjecture:** It is a theorem that, if X is a noetherian, regular and finite dimensional scheme, then $K_r(X) = 0$ for all $r < 0$.

   Schlichting conjectured a major generalization.

Weibel's conjecture is true, it was proved in

Moritz Kerz, Florian Strunk, and Georg Tamme, *Algebraic K-theory and descent for blow-ups*, Invent. Math. **211** (2018), no. 2, 523–577.

Focus on Schlichting's conjecture

Schlichting conjecture isn't only about schemes.

Remember: given any exact category $\mathcal{E}$ there is a recipe to produce a K-theory out of it. And until now we have focused on the case $\mathcal{E} = \mathrm{Vect}(X)$.

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Marco Schlichting, *Negative K-theory of derived categories*, Math. Z. **253** (2006), no. 1, 97–134.

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(Conjecture A)

Marco Schlichting, *Negative K-theory of derived categories*, Math. Z. **253** (2006), no. 1, 97–134.

Schlichting proved:

1. If the abelian category $\mathcal{A}$ is noetherian, then $K_n(\mathcal{A}) = 0$ for $n < 0$.
2. For any abelian category $\mathcal{A}$, we have $K_{-1}(\mathcal{A}) = 0$.

Plausibility argument

Theorem (Quillen). Suppose $\mathcal{B}$ is an abelian category, assume $\mathcal{A} \subset \mathcal{B}$ is a Serre subcategory, and let $\mathcal{C} = \mathcal{B}/\mathcal{A}$.

$$
\begin{array}{ccccc}
 & & & & \swarrow \\
K_1(\mathcal{A}) & \longrightarrow & K_1(\mathcal{B}) & \longrightarrow & K_1(\mathcal{C}) \\
 & & \swarrow & & \\
K_0(\mathcal{A}) & \longrightarrow & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{C}) \\
 & & \swarrow & & \\
0 & & & &
\end{array}
$$

Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Lecture Notes in Math., vol. 341, Springer verlag, 1973, pp. 85–147.

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$$\cdots \longrightarrow K_1(\mathcal{A}) \longrightarrow K_1(\mathcal{B}) \longrightarrow K_1(\mathcal{C}) \longrightarrow K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{B}) \longrightarrow K_0(\mathcal{C}) \longrightarrow K_{-1}(\mathcal{A}) \longrightarrow K_{-1}(\mathcal{B}) \longrightarrow K_{-1}(\mathcal{C}) \longrightarrow \cdots$$

How difficult can it be?

Given $\mathcal{A}$ we want to construct

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The reason is: we can form $F : \mathcal{B} \longrightarrow \mathcal{B}$ by the formula

$$F(B) = \coprod_{i=1}^{\infty} B$$

We notice

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$$F(B) = \coprod_{i=1}^{\infty} B$$

We notice

$$F(B) \cong B \oplus F(B) \qquad \text{hence} \qquad K_n(F) = K_n(\mathrm{id}) + K_n(F)$$

The problem with this:

Given $\mathcal{A}$, we can let $\mathcal{B}$ be the smallest abelian category containing $\mathcal{A}$ and closed under coproducts.

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The image of a map

$$\coprod_{i=1}^{\infty} A_i \longrightarrow A$$

will not usually lie in $\mathcal{A}$.

Let $\mathcal{T}$ be a model category with a bounded t–structure. Antieau, Gepner and Heller proved the following generalization of Schlichting's results:

1.

2.

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1. If the abelian category $\mathcal{T}^{\heartsuit}$ is noetherian, then $K_n(\mathcal{T}) = 0$ for $n < 0$.
2. Unconditionally we have $K_{-1}(\mathcal{T}) = 0$.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

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If $\mathcal{A}$ is an abelian category, Schlichting's results come about by putting $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ with the standard t–structure.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

The generalized Schlichting conjecture (Conjecture B)

For any $\mathcal{T}$ with a bounded t–structure, $K_n(\mathcal{T}) = 0$ for all $n < 0$.

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Yet another conjecture, in case the above are false (Conjecture C)

For any $\mathcal{T}$ with a bounded t–structure, the natural map $K_n(\mathcal{T}^{\heartsuit}) \longrightarrow K_n(\mathcal{T})$ is an isomorphism for $n < 0$.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Plausibility argument

Let $\mathcal{R} \subset \mathcal{S}$ be model categories with $\mathcal{T} = \mathcal{S}/\mathcal{R}$. Then

$$
\begin{array}{ccccc}
\swarrow & & & & \\
K_{-1}(\mathcal{R}) & \longrightarrow & K_{-1}(\mathcal{S}) & \longrightarrow & K_{-1}(\mathcal{T}) \\
& & \swarrow & & \\
K_{-2}(\mathcal{R}) & \longrightarrow & K_{-2}(\mathcal{S}) & \longrightarrow & K_{-2}(\mathcal{T}) \\
& & \swarrow & & \\
K_{-3}(\mathcal{R}) & \longrightarrow & K_{-3}(\mathcal{S}) & \longrightarrow & K_{-3}(\mathcal{T}) \\
& & \swarrow & &
\end{array}
$$

Punchline

Schlichting's conjecture (Conjecture A)

and the generalized Schlichting conjecture (Conjecture B)

are both false.

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The counterexample appeared in

Amnon Neeman, *A counterexample to vanishing conjectures for negative K-theory*, Invent. Math. **225** (2021), no. 2, 427–452.

Back to Auslander's formula

- Maurice Auslander, *Representation theory of Artin algebras*, Queen Mary College, London (1971).
- Henning Krause, *Deriving Auslander's formula*, Doc. Math. **20** (2015), 669–688.

Let $\mathcal{A}$ be an abelian category, and let $\operatorname{mod}\mathcal{A}$ be the category of finitely presented functors $\mathcal{A} \longrightarrow \mathcal{AB}$. We learned

$$\mathbf{K}^b(\mathcal{A}) \quad = \quad \mathbf{K}^b(\operatorname{proj}(\operatorname{mod}\mathcal{A})) \quad = \quad \mathbf{D}^b(\operatorname{mod}\mathcal{A})$$

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Auslander's formula tells us that there is an exact functor $\Lambda : \operatorname{mod}\mathcal{A} \longrightarrow \mathcal{A}$, expressing $\mathcal{A}$ as the Gabriel quotient of $\operatorname{mod}\mathcal{A}$ by some Serre subcategory $\operatorname{eff}\mathcal{A}$.

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Krause's article, which derives Auslander's formula, gives

$$\mathbf{D}^b_{\text{eff}\,\mathcal{A}}(\text{mod}\,\mathcal{A}) \longrightarrow \mathbf{D}^b(\text{mod}\,\mathcal{A}) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

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$$\begin{array}{ccccc}
\mathbf{Ac}^b(\mathcal{A}) & \longrightarrow & \mathbf{K}^b(\mathcal{A}) & \longrightarrow & \mathbf{D}^b(\mathcal{A}) \\
\downarrow \wr & & \downarrow \wr & & \Vert \\
\mathbf{D}^b_{\operatorname{eff}\mathcal{A}}(\operatorname{mod}\mathcal{A}) & \longrightarrow & \mathbf{D}^b(\operatorname{mod}\mathcal{A}) & \longrightarrow & \mathbf{D}^b(\mathcal{A})
\end{array}$$

The categories $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ and $\mathbf{D}^b(\mathcal{E}) = \mathbf{K}^b(\mathcal{E})/\mathbf{Ac}^b(\mathcal{E})$

Let $\mathcal{E}$ be any idempotent-complete exact category. Let $\mathbf{K}^b(\mathcal{E})$ be the category whose objects are bounded cochain complexes in $\mathcal{E}$, meaning

$$\cdots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

with $E^i = 0$ for $|i| \gg 0$.

The full subcategory $\mathbf{Ac}^b(\mathcal{E})$ of acyclics contains those cochain complexes for which there exist admissible short exact sequences

$$0 \longrightarrow K^i \xrightarrow{\alpha^i} E^i \xrightarrow{\beta^i} K^{i+1} \longrightarrow 0$$

such that $\partial^i = \alpha^{i+1} \circ \beta^i$.

And $\mathbf{D}^b(\mathcal{E}) = \mathbf{K}^b(\mathcal{E})/\mathbf{Ac}^b(\mathcal{E})$.

The t–structure on $\mathbf{Ac}^b(\mathcal{E})$

$$\mathbf{Ac}^b(\mathcal{E})^{\leq 0} \quad = \quad \{E^* \in \mathbf{Ac}^b(\mathcal{E}) \mid E^i = 0 \text{ for all } i > 0\}$$

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$$\mathbf{Ac}^b(\mathcal{E})^{\geq 0} \quad = \quad \{E^* \in \mathbf{Ac}^b(\mathcal{E}) \mid E^i = 0 \text{ for all } i < -2\}$$

Proof that this is a t–structure

A morphism from an object $E^* \in \mathbf{Ac}^b(\mathcal{E})^{\leq 0}$ to an object $F^* \in \mathbf{Ac}^b(\mathcal{E})^{\geq 1}$ may be represented by a cochain map

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\partial^{-1}} & E^{0} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^{-1} & \xrightarrow[\widetilde{\partial}^{-1}]{} & F^{0} & \xrightarrow[\widetilde{\partial}^{0}]{} & F^{1} & \xrightarrow[\widetilde{\partial}^{1}]{} & \cdots
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 & & \downarrow & & \downarrow f & \overset{\theta}{\swarrow} & \downarrow g & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^{-1} & \xrightarrow[\widetilde{\partial}^{-1}]{} & F^0 & \xrightarrow[\widetilde{\partial}^{0}]{} & F^1 & \xrightarrow[\widetilde{\partial}^{1}]{} & \cdots
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that is:

$$g = \widetilde{\partial}^{-1} \circ \theta \ .$$

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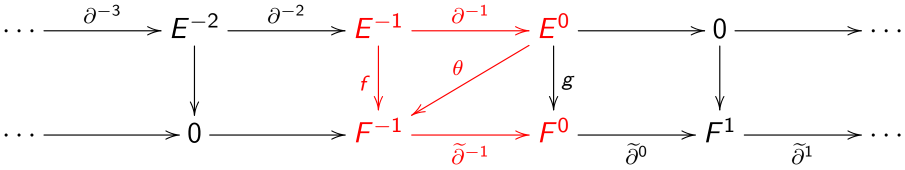
$$
\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\partial^{-1}} & E^{0} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow f & \swarrow \theta & \downarrow g & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & F^{-1} & \xrightarrow[\widetilde{\partial}^{-1}]{} & F^{0} & \xrightarrow[\widetilde{\partial}^{0}]{} & F^{1} & \xrightarrow[\widetilde{\partial}^{1}]{} & \cdots
\end{array}
$$

that is:

$$g = \widetilde{\partial}^{-1} \circ \theta \ .$$

Proof that this is a t–structure

A morphism from an object $E^* \in \mathbf{Ac}^b(\mathcal{E})^{\leq 0}$ to an object $F^* \in \mathbf{Ac}^b(\mathcal{E})^{\geq 1}$ may be represented by a cochain map



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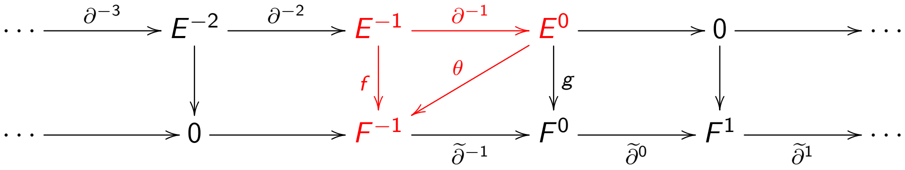

that is:

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Proof that this is a t–structure, continued

Next choose any object $E^* \in \mathbf{Ac}^b(\mathcal{E})$, that is a complex

$$\cdots \xrightarrow{\partial^{i-2}} E^{i-1} \xrightarrow{\partial^{i-1}} E^i \xrightarrow{\partial^i} E^{i+1} \xrightarrow{\partial^{i+1}} \cdots$$

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Write $\partial^{-1} : E^{-1} \longrightarrow E^0$ as a composite $E^{-1} \xrightarrow{\beta^{-1}} K^0 \xrightarrow{\alpha^0} E^0$.

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Write $\partial^{-1} : E^{-1} \longrightarrow E^0$ as a composite $E^{-1} \xrightarrow{\beta^{-1}} K^0 \xrightarrow{\alpha^0} E^0$. Now consider the cochain maps

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\beta^{-1}} & K^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha^0} & & \downarrow & & \\
\cdots & \xrightarrow{\partial^{-3}} & E^{-2} & \xrightarrow{\partial^{-2}} & E^{-1} & \xrightarrow{\partial^{-1}} & E^0 & \xrightarrow{\partial^{0}} & E^1 & \xrightarrow{\partial^{1}} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \beta^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & K^0 & \xrightarrow{\alpha^0} & E^0 & \xrightarrow{\partial^{0}} & E^1 & \xrightarrow{\partial^{1}} & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{-\partial^{-2}} & E^{-1} & \xrightarrow{-\beta^{-1}} & K^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

The heart

The heart of this t–structure, denoted $\mathbf{Ac}^b(\mathcal{E})^\heartsuit$, is by definition the full subcategory

$$\mathbf{Ac}^b(\mathcal{E})^\heartsuit \quad = \quad \mathbf{Ac}^b(\mathcal{E})^{\leq 0} \cap \mathbf{Ac}^b(\mathcal{E})^{\geq 0} \ .$$

The objects are the acyclic cochain complexes

$$0 \longrightarrow E^{-2} \longrightarrow E^{-1} \longrightarrow E^{0} \longrightarrow 0$$

and the morphisms are the homotopy equivalence classes of cochain maps.

Formal consequence of the general theory

The category $\mathbf{Ac}^b(\mathcal{E})^\heartsuit$ is abelian.

Now we have $\mathbf{Ac}^b(\mathcal{E}) \subset \mathbf{K}^b(\mathcal{E})$ with quotient $\mathbf{D}^b(\mathcal{E})$, giving

$$\begin{array}{ccccc}
 & & & & \swarrow \\
K_{-1}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathbf{K}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathbf{D}^b(\mathcal{E})) \\
 & \swarrow & & & \\
K_{-2}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathbf{K}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathbf{D}^b(\mathcal{E})) \\
 & \swarrow & & & \\
K_{-3}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathbf{K}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathbf{D}^b(\mathcal{E})) \\
\swarrow & & & &
\end{array}$$

which rewrites as

$$
\begin{array}{ccccc}
 & & & & \swarrow \\
K_{-1}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-1}(\mathcal{E}) \\
 & \swarrow & & & \\
K_{-2}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-2}(\mathcal{E}) \\
 & \swarrow & & & \\
K_{-3}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-3}(\mathcal{E}) \\
\swarrow & & & &
\end{array}
$$

Thus the vanishing of $K_n(\mathbf{Ac}^b(\mathcal{E}))$ for all $n < 0$ would imply that the map

$$K_n(\mathcal{E}^{\oplus}) \longrightarrow K_n(\mathcal{E})$$

would have to be an isomorphism for all $n < 0$. Hence, for a counterexample to the generalized Schlichting conjecture, all we need to do is find an $\mathcal{E}$ for which this fails.

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would have to be an isomorphism for all $n<0$. Hence, for a counterexample to the generalized Schlichting conjecture, all we need to do is find an $\mathcal{E}$ for which this fails.

If we want to disprove the (ungeneralized) Schlichting conjecture and/or to study the yet another conjecture, then it might be helpful to look at the natural map

$$K_n\Big(\mathbf{Ac}^b(\mathcal{E})^{\heartsuit}\Big) \longrightarrow K_n(\mathbf{Ac}^b(\mathcal{E}))\ .$$

Theorem

Let $\mathcal{E}$ be an idempotent-complete exact category. Then the natural functor

$$D^b\left(\mathbf{Ac}^b(\mathcal{E})^{\heartsuit}\right) \longrightarrow \mathbf{Ac}^b(\mathcal{E})$$

is an equivalence of triangulated categories if and only if $\mathcal{E}$ is hereditary, meaning $\mathrm{Ext}^i(E,E')=0$ for all $i>1$ and $E,E'\in\mathcal{E}$.

Corollary

If $\mathcal{E}$ is hereditary then the map

$$K_n\left(\mathbf{Ac}^b(\mathcal{E})^{\heartsuit}\right) \longrightarrow K_n(\mathbf{Ac}^b(\mathcal{E}))$$

must be an isomorphism for all $n\in\mathbb{Z}$.

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must be an isomorphism for all $n \in \mathbb{Z}$.

Example

If Y is any algebraic curve, then the category $\mathcal{E} = \mathrm{Vect}(Y)$ is hereditary.

After all: there is a spectral sequence

$$H^i(\mathcal{E}xt^j(E,E')) \Longrightarrow Ext^{i+j}(E,E'),$$

For vector bundles we know the vanishing of $\mathcal{E}xt^j(E,E')$ for $j>0$, and for curves we have the vanishing of H^i for $i>1$.

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The corollary on the previous page informs us that, for any algebraic curve Y and with $\mathcal{E} = \mathrm{Vect}(Y)$, the natural map

$$K_n\left(\mathbf{Ac}^b(\mathcal{E})^{\heartsuit}\right) \longrightarrow K_n(\mathbf{Ac}^b(\mathcal{E}))$$

is an isomorphism for all $n \in \mathbb{Z}$.

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Recall the general exact sequence

$$
\begin{array}{ccccc}
 & & & & \swarrow \\
K_{-1}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-1}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-1}(\mathcal{E}) \\
 & & & \swarrow & \\
K_{-2}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-2}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-2}(\mathcal{E}) \\
 & & & \swarrow & \\
K_{-3}(\mathbf{Ac}^b(\mathcal{E})) & \longrightarrow & K_{-3}(\mathcal{E}^{\oplus}) & \longrightarrow & K_{-3}(\mathcal{E}) \\
 & \swarrow & & &
\end{array}
$$

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LOUSY ARGUMENT!

The right approach would have been to prove the more general statement:

Theorem

Let k be a field and let $\mathcal{E}$ be an idempotent-complete, additive, k-linear category. Assume that, for all objects $E, E' \in \mathcal{E}$, $\mathrm{Hom}(E, E')$ is finite-dimensional as a k-vector space.

Then $K_n(\mathcal{E}^{\oplus}) = 0$ for all $n < 0$.

Let $\mathcal{T}$ be a model category with a bounded t–structure. Antieau, Gepner and Heller proved the following generalization of Schlichting's results:

1. If the abelian category $\mathcal{T}^\heartsuit$ is noetherian, then $K_n(\mathcal{T}) = 0$ for $n < 0$.
2. Unconditionally we have $K_{-1}(\mathcal{T}) = 0$.

If $\mathcal{A}$ is an abelian category, Schlichting's results come about by putting $\mathcal{T} = \mathbf{D}^b(\mathcal{A})$ with the standard t–structure.

Benjamin Antieau, David Gepner, and Jeremiah Heller, *K-theoretic obstructions to bounded t-structures*, Invent. Math. **216** (2019), no. 1, 241–300.

Thank you,

Thank you,

and HAPPY BIRTHDAY

TO HENNING!