# “Semiorthogonal decompositions for representations of algebraic groups”, Bielefeld, 1-3 September 2025

**Abstract**

Semiorthogonal decompositions are a well-established tool for the study of derived categories in algebraic geometry and in the representation theory of Artin algebras. The goal of this meeting is to discuss semiorthogonal decompositions that arise in the representation theory of algebraic groups, going over the main proofs of the recent work [17] of Samokhin and van der Kallen, and exploring emerging connections to geometric representation theory.

## Contents

# 1 Introduction

The goal of the meeting is to go through the proofs of the main theorems of [17] and to discuss emerging connections to geometric representation theory, some of which are described in Section 1 below. We first recall the main theorems of *loc.cit.* Let $\mathbb{G} \to \mathbb{Z}$ be a split simply connected semisimple Chevalley group scheme (a smooth split affine group scheme over $\mathrm{Spec}(\mathbb{Z})$ whose geometric fibers are connected simply connected semisimple algebraic groups), $\mathbb{B} \subset \mathbb{G}$ a Borel subgroup scheme, and $\mathbb{G}/\mathbb{B} \to \mathbb{Z}$ be the corresponding Chevalley flag scheme (resp., the corresponding generalized flag scheme $\mathbb{G}/\mathbb{P} \to \mathbb{Z}$ for a standard parabolic subgroup scheme $\mathbb{P} \subset \mathbb{G}$ over $\mathbb{Z}$). Let $\mathbf{B}$ (resp., $\mathbf{G}$) denote the group schemes over $\mathsf{k}$ obtained by base change from $\mathbb{B}$ (resp., from $\mathbb{G}$) along $\mathrm{Spec}(\mathsf{k}) \to \mathrm{Spec}(\mathbb{Z})$, where $\mathsf{k}$ is a field. Let $\mathrm{rep}(\mathbf{G})$ (resp., $\mathrm{rep}(\mathbf{B})$) denote the category of rational modules over the group scheme $\mathbf{G}$ (resp., over the group scheme $\mathbf{B}$) that are finite-dimensional over $\mathsf{k}$. Then

**Theorem 1.1.** *The category $\mathcal{D} = \mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ has a $\mathbf{G}$-linear semiorthogonal decomposition*

$$\mathcal{D} = \langle \mathsf{X}_\mathsf{v} \rangle_{\mathsf{v} \in \mathsf{W}} \tag{1}$$

*with respect to a total order $\prec$ on the Weyl group $W$ that refines the Bruhat order. Each subcategory $\mathsf{X}_\mathsf{v}$ is equivalent to $\mathrm{D}^b(\mathrm{rep}(\mathbf{G}))$.*

There is an integral version of Theorem 1.1 with $\mathbf{B}$ (resp., $\mathbf{G}$) replaced by $\mathbb{B}$ (resp., by $\mathbb{G}$). One needs a little bit of work to introduce the corresponding derived category $\mathrm{D}^b(\mathrm{rep}(\mathbb{B}))$ (resp., $\mathrm{D}^b(\mathrm{rep}(\mathbb{G}))$) in order to state the integral counterpart of Theorem 1.1, see [17, Sections 8.1 and 11].

Next, let $\mathbf{P}$ be a parabolic subgroup containing $\mathbf{B}$. Let $W_\mathbf{P}$ be the parabolic Weyl group corresponding to $\mathbf{P}$, and $W^\mathbf{P}$ be the set of minimal coset representatives of $W/W_\mathbf{P}$. Let $\prec_\mathbf{P}$ denote the restriction to $W^\mathbf{P}$ of the chosen total order $\prec$ on $W$ from Theorem 1.1 above. Then:

**Theorem 1.2.** *The category* $\mathcal{D} = \mathrm{D}^b(\mathrm{rep}(\mathbf{P}))$ *has a* $\mathbf{G}$*-linear semiorthogonal decomposition*

$$\mathcal{D} = \langle \hat{\mathsf{X}}_v \rangle_{v \in W^{\mathbf{P}}} \tag{2}$$

*with respect to the order* $\prec_{\mathbf{P}}$ *on* $W^{\mathbf{P}}$*. Each subcategory* $\hat{\mathsf{X}}_v$ *is equivalent to* $\mathrm{D}^b(\mathrm{rep}(\mathbf{G}))$*.*

Similarly, there is an integral version of Theorem 1.2 with $\mathbf{P}$ (resp., $\mathbf{G}$) replaced by $\mathbb{P}$ (resp., by $\mathbb{G}$)., see [17, Sections 8.1 and 11].

Theorems 1.1 and 1.2 imply the following:

**Theorem 1.3.** *Let* $\prec$ *be the same total order on* $W$ *as in Theorem 1.1, and let* $\mathcal{D} = \mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{B}))$*. Let* $v, w$ *denote elements of* $W$*. Then there are objects* $\mathcal{X}_v \in \mathcal{D}$ *such that*

1. $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X}_v, \mathcal{X}_v[i]) = \begin{cases} \mathsf{k} & \text{if } i = 0 \\ 0 & \text{else.} \end{cases}$.

2. *If* $w \succ v$ *then* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X}_v, \mathcal{X}_w[i]) = 0$ *for all* $i$*.*

3. *The triangulated hull of* $\{\mathcal{X}_v \mid v \in W\}$ *is* $\mathcal{D}$*.*

*In other words, the collection of objects* $(\mathcal{X}_v)_{v \in W}$ *is a full exceptional collection in* $\mathcal{D}$*.*

Once again, there is an integral version of Theorem 1.3, see [17, Sections 12]. Similarly as in the case of Theorem 1.2 there is

**Theorem 1.4.** *Let* $\prec_{\mathbf{P}}$ *be the same total order on* $W^{\mathbf{P}}$ *as in Theorem 1.2, and let* $\mathcal{D} = \mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{P}))$*. Let* $v, w \in W^{\mathbf{P}}$*. Then there are objects* $\hat{\mathcal{X}}_v \in \mathcal{D}$ *such that*

1. $\mathrm{Hom}_{\mathcal{D}}(\hat{\mathcal{X}}_v, \hat{\mathcal{X}}_v[i]) = \begin{cases} \mathsf{k} & \text{if } i = 0, \\ 0 & \text{else.} \end{cases}$

2. *If* $w \succ_{\mathbf{P}} v$ *then* $\mathrm{Hom}_{\mathcal{D}}(\hat{\mathcal{X}}_v, \hat{\mathcal{X}}_w[i]) = 0$ *for all* $i$*.*

3. *The triangulated hull of* $\{\hat{\mathcal{X}}_v \mid v \in W^{\mathbf{P}}\}$ *is* $\mathcal{D}$*.*

*In other words, the collection of objects* $(\hat{\mathcal{X}}_v)_{v \in W^{\mathbf{P}}}$ *is a full exceptional collection in* $\mathcal{D}$*.*

See [17, Sections 13] for the integral counterpart of Theorem 1.4.

As it turns out, Theorem 1.3 (resp., Theorem 1.4) is an almost immediate consequence of Theorem 1.1 (resp., Theorem 1.2). Further, all the essential ingredients for proving Theorem 1.2 are worked out in the course of the proof of Theorem 1.1. Therefore, the main focus of the meeting is going to be on the proof of Theorem 1.1.

### Connections to geometric and modular representation theory

We expect Theorems 1.1 and 1.2 to have further connections to geometric representation theory. Some of these connections are going to be discussed in the research session (see Section 5 below), once details of the proof of Theorem 1.1 have been worked out.

In some more detail, one of the crucial inputs for proving Theorem 1.1 is the Steinberg basis, [18], which can be thought of a basis of $\mathbf{T}$-equivariant $\mathsf{K}$-theory of the flag variety $\mathbf{G}/\mathbf{B}$ as a $\mathsf{K}^0(\mathrm{rep}(\mathbf{T}))$-module. The group $\mathsf{K}^0_{\mathbf{T}}(\mathbf{G}/\mathbf{B})$ is equipped with a natural $\mathsf{K}^0(\mathrm{rep}(\mathbf{T}))$-valued non-degenerate pairing, [12, Proposition 1.6]. The basis of $\mathsf{K}^0(\mathbf{G}/\mathbf{B})$ that is dual to the Steinberg basis appears in the context of Lusztig's asymptotic Hecke algebra, [7]. Theorem 1.1 furnishes another basis of $\mathsf{K}^0(\mathbf{G}/\mathbf{B})$ through the classes of objects $[\mathcal{X}_v], v \in W$ which make it possible to compute the dual Steinberg basis. The problem is to relate the classes $[\mathcal{X}_v] \in \mathsf{K}^0_{\mathbf{T}}(\mathbf{G}/\mathbf{B}), v \in W$ of the present paper to Lusztig's canonical basis, [13].

The category of $\mathbf{B}$-equivariant $\mathcal{D}$-modules on $\mathbf{G}/\mathbf{B}$ (the finite Hecke category) plays a fundamental role in geometric representation theory. Its quasi-coherent counterpart, the category of $\mathbf{B}$-equivariant (quasi)-coherent sheaves on $\mathbf{G}/\mathbf{B}$ is called the coherent Hecke category, [2, Section 4.2]. Theorem 1.1 can be formulated as a statement about semiorthogonal decompositions of the coherent Hecke category as a $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$-linear category. In this regard, the arising question is about relations to coherent Springer theory, [3], and to semiorthogonal decompositions in that context (See Section 2.3 of *loc.cit.*).

Associated to an almost simple simply connected linear algebraic group $\mathbf{G}$ over $\bar{\mathbb{F}}_p$ is the finite reductive group $\mathbf{G}(\mathbb{F}_p)$. A recent paper [4] proves a number of Lusztig's conjectures from [14] about reductions modulo $p$ of complex representations of $\mathbf{G}(\mathbb{F}_p)$. In its own turn, it puts forward a conjecture (Conjecture 7.5 of *loc.cit.*) relating lifts of unipotent principal series representations of $\mathbf{G}(\mathbb{F}_p)$ to virtual representations of $\mathbf{G}$ constructed with the help of objects $\mathcal{X}_v, v \in W$ from Theorems 1.1 and 1.2. This leads to questions about origins of the objects $\mathcal{X}_v, v \in W$ stemming from the Deligne-Lusztig's theory.

## 2 Background

### Summary

The goal of this section is to introduce the necessary background from representation theory and rational cohomology of reductive groups culminating in Corollary 2.21 of [17]. That corollary will serve as one of the key inputs for proving the main $\mathbf{B}$-cohomology vanishing statements in Section 6 of [17].

### Talk 1 (Juan Omar Gomez). Affine (reductive) groups schemes

This is a very basic talk that serves to introduce/to recall the main notions from the theory of linear algebraic groups and their representations that are used throughout the paper; essentially, these are listed in [17, Section 1.5.1]. The speaker will recall representations of group schemes and will mention the relationship between $\mathbb{G}$-modules and $\mathsf{k}[\mathbb{G}]$-comodules for a flat group scheme $\mathbb{G}$ over a base ring $\mathsf{k}$. Recall the definition of a split Chevalley group scheme over $\mathbb{Z}$. Given a closed subgroup scheme $\mathbb{H} \subset \mathbb{G}$, recall the induction functor $\mathrm{ind}_{\mathbb{H}}^{\mathbb{G}}$, its right derived functor $\mathrm{Rind}_{\mathbb{H}}^{\mathbb{G}}$, and the adjoint pair $(\mathrm{res}_{\mathbb{H}}^{\mathbb{G}}, Rind_{\mathbb{H}}^{\mathbb{G}})$. Recall the definition of the functor $\mathcal{L} : \mathrm{Rep}(\mathbb{B}) \to QCoh^{\mathbb{G}}(\mathbb{G}/\mathbb{B})$ following [11, I, Chapter V] ("associated sheaf"). Cohomology of (quasi)-coherent sheaves on $\mathbb{G}/\mathbb{B}$ that are associated to rational representations of $\mathbb{B}$ are computed via $R\mathrm{ind}_{\mathbb{B}}^{\mathbb{G}}$, [11, I, Proposition 5.12] (recall here that our Borel subgroup $\mathbb{B}$ corresponds to *negative* roots).

The speaker will then recall the dominance order on the weight lattice $X(\mathbb{T})$, the definitions of Weyl and dual Weyl modules. Finally, recall Universal coefficient Theorem, [17, Theorem 1.8], following [11, Proposition I 4.18] and its sheafy version, which is [17, Theorem 1.8, (2)]. These will be instrumental for transferring the main statements over a field $\mathsf{k}$ to $\mathbb{Z}$.

## Talk 2 (Antoine Touzé). Rational cohomology of algebraic groups

Let $\mathbf{G}$ (resp., $\mathbf{B}$) denote the split semisimple simply connected algebraic group obtained from a split simply connected Chevalley group scheme $\mathbb{G}$ (resp., from a Borel subgroup scheme $\mathbb{B}$) by base change along $\mathrm{Spec}(\mathsf{k}) \to \mathrm{Spec}(\mathbb{Z})$. The speaker will recall the basics of rational cohomology of algebraic groups following [11, I, Chapter 4]. Relation between $\mathbf{B}$-cohomology and $\mathbf{G}$-cohomology via the spectral sequence involving $R\mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}$ (cf. also [17, Section 8.2]). The beginning of [17, Section 6.1] also fits here ("Cohomological descent from $\mathbf{G}/\mathbf{B}$ to $\mathbf{B}$").

## Talk 3 (Kostiantyn Tolmachov). Highest weight category structure on Rep(G)

The speaker will recall the definition of a highest weight category structure on a finite length abelian category, [6], then will give the definition of standard and costandard objects. Recall the parametrization of simple modules in $\mathrm{Rep}(\mathbf{G})$ by dominant weighs $X_+(\mathbf{T}) \subset X(\mathbf{T})$. State the theorem saying that $\mathrm{Rep}(\mathbf{G})$ has the structure of a highest weight category with respect to the dominance order on $X(\mathbf{T})$, [8]. The speaker will then describe standard modules as Weyl modules and costandard objects as dual Weyl modules, and recall the definition of a good filtration on a $\mathbf{G}$-module. Explain briefly cohomological criteria for detecting modules with good filtrations, [17, Section 2.10].

## Talk 4 (Dmitry Kubrak). Highest weight category structure on Rep(B)

$\mathrm{Rep}(\mathbf{B})$ has two highest weight category structures, [19]. Recall the definition of excellent order on $X(\mathbf{T})$, [Section 1.2 of *loc.cit.*] and [17, Section 2.5]. Recall the definition of Joseph-Demazure modules $P(\lambda), \lambda \in X(\mathbf{T})$ (costandard modules with respect to the highest weight structure given by the excellent order), [20, Section 2.2] and [17, Section 2.2] (note that [20], despite being the main source for **B**-module theory, doesn't use the framework of highest weight categories). Recall the definition of antipodal excellent order on $X(\mathbf{T})$, [19, Section 1.2] and [17, Section 2.5]. Recall the definition of relative Schubert modules $Q(\lambda), \lambda \in X(\mathbf{T})$ (costandard modules with respect to the highest weight structure given by the antipodal excellent order).

Having set up the framework of highest weight category structure on $\mathrm{Rep}(\mathbf{B})$, state the two main **B**-cohomology statements of this section: one is [17, Theorem 2.9] (originally in [19, Theorem 2.20(i)] and [20, Theorem 3.2.6]): $H^p(\mathbf{B}, P(\lambda) \otimes Q(\mu)) = 0$ for $p > 0$ and $\lambda, \mu \in X(\mathbf{T})$. Then state and comment on proof of [17, Theorem 2.20] (originally [20, Corollary 5.1.7]): for $\lambda \in X(\mathbf{T})$ and $\mu \in X(\mathbf{T})_+$, the tensor product $P(\lambda) \otimes \nabla_\mu$ has excellent filtration. Finally, explain how combining these two statements one arrives at [17, Corollary 2.21]: for $\lambda, \mu \in X(\mathbf{T})$ and $\nu \in X(\mathbf{T})_+$ one has $H^p(\mathbf{B}, P(\lambda) \otimes Q(\mu) \otimes \nabla_\nu) = 0$ for $p > 0$. The latter cohomology vanishing will be a crucial ingredient for proving in [17, Section 6] the main **B**-cohomology vanishings of the paper.

# 3 T-equivariant K-theory of G/B and B-cohomology vanishing theorems

## Summary

The goal of this section is to prove the main vanishing theorems which are Theorem 6.6 and Corollary 6.7. The background part of this section consists of introducing the Steinberg basis $e_v, v \in W$ of $\mathsf{K}^0(\mathrm{rep}(\mathbf{B}))$ as a $\mathsf{K}^0(\mathrm{rep}(\mathbf{G}))$-module, the **T**-equivariant $\mathsf{K}$-theory of $\mathbf{G}/\mathbf{B}$ and its two bases, which are the Schubert basis and the opposite Schubert cell basis, and an orthogonality result by Graham-Kumar (attributed to Knutson) from [10] recalled in [17, Section 4.3]. It is at this point when starts the fusion of costandard modules in $\mathrm{rep}(\mathbf{B})$ (for both highest weight category structures) with the Steinberg basis, and the modules $P(-e_v), v \in W$ and $Q(e_v), v \in W$ are introduced in [17, Section 4.3]. The modules $P(-e_v), v \in W$ (resp., $Q(e_v), v \in W$) give rise to **G**-equivariant coherent sheaves on $\mathbf{G}/\mathbf{B}$ denoted $\mathcal{P}_v, v \in W$ (resp., $\mathcal{Q}(e_v), v \in W$).

Graham-Kumar's orthogonality result is an intermediary between the "$P$"-basis $\{\mathcal{P}_v, v \in W\}$ and the "$Q$"-basis $\{\mathcal{Q}(e_v), v \in W\}$ of $\mathsf{K}_{\mathbf{T}}(\mathbf{G}/\mathbf{B})$: the transition matrix between the two bases is computed in two steps, by first computing the transition matrix between the opposite Schubert cell basis and the "$P$"-basis in Section 5.1 (the matrix $(\beta_{vw})_{v,w\in W}$ and then computing the transition matrix between the "$Q$"-basis and the Schubert basis and in Section 5.2 (the matrix $(\alpha_{vw})_{v,w\in W}$. Theorem 4.1 states that both matrices $(\beta_{vw})_{v,w\in W}$ and $(\alpha_{vw})_{v,w\in W}$ are upper-triangular after a suitable reordering of rows and columns and invertible; their diagonal entries can be read off the Steinberg basis (Theorem 4.1, (3)).

Combining all these statements together, orthogonality property of the Schubert and the opposite Schubert cell bases and upper-triangularity of the matrices $(\beta_{vw})_{v,w\in W}$ and $(\alpha_{vw})_{v,w\in W}$ then lead to semiorthogonality at the $\mathsf{K}$-theory level of the "$P$"-basis against the "$Q$"-basis with respect to the Bruhat order. Now, combined with the higher $\mathbf{B}$-cohomology vanishing from Corollary 2.21 and the basic results from the beginning of Section 6 (which have been both covered on the previous day), this leads to genuine semiorthogonality at the categorical level of the "$P$"-basis against the "$Q$"-basis with respect to the Bruhat order, [17, Theorem 6.2, Corollary 6.3, Theorems 6.5 and 6.6].

This section culminates in [17, Corollary 6.7] that ties Theorems 6.2 and 6.6 of [17] together. It will be crucial for showing in the subsequent Section 4 that semiorthogonal decompositions of the $\mathsf{k}$-linear category $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ are in fact $\mathbf{G}$-linear.

### Talk 5 (Alexey Ananyevskiy). Steinberg basis. Schubert basis and dual Schubert cell basis. Pairing. Orthogonality of two bases with respect to the pairing

Give the definition and state main properties of the Steinberg basis, [18], [1]. Introduce the bases in the title, the pairing on $\mathsf{K}_{\mathbf{T}}(\mathbf{G}/\mathbf{B})$, and state [10, Proposition 2.1].

### Talk 6 (Wilberd van der Kallen). Triangularity theorem [17, Theorem 4.1]

The proof of triangularity of the transition matrices is broken up into a series of lemmas; their proofs draw on [20, Section 2] concerning Frobenius splitting and on combinatorics of the Weyl group.

### Talk 7 (Dmitry Kubrak). B-cohomology vanishing statements [17, Theorems 6.2, 6.5, and 6.6]

The goal of this section is to derive higher ind-vanishig statements for $\mathbf{B}$-modules of interest. These vanishing statements will be crucial for the construction of semiorthogonal decompositions of $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ in Section 9 of [17]. One can relate coherent cohomology of $\mathbf{G}$-equivariant vector bundles on $\mathbf{G}/\mathbf{B}$ (the derived functor of $\mathrm{ind}_{\mathbf{B}}^{\mathbf{G}}$) to $\mathbf{B}$-cohomology ("cohomological descent from $\mathbf{G}/\mathbf{B}$ to $\mathbf{B}$"). This will be applied first to showing the higher ind-vanishing for the tensor product $P(\lambda)\otimes Q(\mu)$ for $\lambda,\mu\in X(\mathbf{T})$ in Theorem 6.2. After having recalled [17, Lemma 6.1], the speaker should first explain how [17, Theorem 6.2] follows from [17, Corollary 2.21]. This leads to [17, Corollary 6.3] that reduces the output of pairing of two $\mathsf{K}$-theoretic classes $[\mathcal{L}(P(\lambda))]$ and $[\mathcal{L}(Q(\mu))]$ to a single term which is a genuine $\mathbf{G}$-representation. Finally, for the modules $P(-e_v), v\in W$ and $Q(e_v), v\in W$ Theorem 6.5 of [17] computes the resulting representation to be either 0 or the trivial representation $\mathsf{k}$. That theorem is the cornerstone for [17, Section 9] in which both semiorthogonal decompositions of $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ and the ($\mathsf{k}$-linear) exceptional objects $X_v\in\mathrm{D}^b(\mathrm{rep}(\mathbf{B})), v\in W$ are going to be constructed.

Discuss the integral versions of vanishing statements in this section (Theorems 6.4 and 6.6 of [17]).

### Talk 8 (Kostiantyn Tolmachov). G-linear categories and G-linear semiorthogonal decompositions

This covers Section 8.3 of [17]. The speaker will briefly recall main definitions concerning triangulated categories, admissible subcategories, semiorthogonal decompositions, and exceptional collections [17, Section 7]. The speaker will then introduce the notion of a $\mathbf{G}$-linear triangulated category and that of a $\mathbf{G}$-linear semiorthogonal decomposition of such a category. One can draw an analogy with the classical geometric situation when the base is a (classical) scheme: one can also consult the subsequent [17, Remark 12.2, Section 12]. The speaker will then discuss the main example of a a $\mathbf{G}$-linear triangulated category which is that of $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$, [17, Section 8.2]. The speaker might want to emphasize on the relation connecting $\mathbf{B}$-cohomology to $\mathbf{G}$-cohomology via $\mathrm{Rind}_{\mathbf{B}}^{\mathbf{G}}$, [17, Proposition 8.6]. Propositions 8.9 and 8.11 of [17], being short and easy, can be given with complete proofs.

# 4 Semiorthogonal decompositions of $\mathbf{D}^b(\mathbf{rep(B)})$ as a G-linear category

### Summary

Assembling together all the previous coohomology statements, we can now proceed to constructing semiorthogonal decompositions of the category $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$. The eventual goal is in having $\mathbf{G}$-linear semiorthogonal decompositions of $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$; this will be achieved in two steps. The first step is the generation property for $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ as a $\mathbf{G}$-linear category by the objects of interest. The second step produces a collection of objects $X_p \in \mathrm{D}^b(\mathrm{rep}(\mathbf{B})), p \in W$ which are first shown to be exceptional in $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ as a $\mathsf{k}$-linear category. Finally, by virtue of the vanishing theorems for $\mathbf{B}$-cohomology from [17, Section 6], the objects $X_p \in \mathrm{D}^b(\mathrm{rep}(\mathbf{B})), p \in W$ turn out to be exceptional in $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ as a $\mathbf{G}$-linear category.

### Talk 9 (Alexey Ananyevskiy). Generating the categories rep(B) and $\mathbf{D}^b$(rep(B))

This talk aims at proving Theorem 8.19 asserting that for any element $p \in W$, the triangulated hull in $\mathrm{D}(\mathrm{Rep}(\mathbf{B}))$ of the two categories

$$\mathrm{hull}(\{\nabla_\lambda \otimes Q(e_v)\}_{v \succ p, \lambda \in X(\mathbf{T})_+}), \tag{3}$$
$$\mathrm{hull}(\{\nabla_\lambda \otimes P(-e_v)^*\}_{v \preceq p, \lambda \in X(\mathbf{T})_+}) \tag{4}$$

is $\mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$.

The proof is essentially a categorical upgrade of the $\mathsf{K}$-theoretic statement proven in [1, Theorem 2]. Various notions of generation can also be discussed, noting that direct summands in the triangulated hull of the above two categories will appear automatically, so that the triangulated hull thus obtained will be idempotent-complete.

## Talk 10 (Andreas Krug). Cut at $p \in W$. The construction of objects $X_p, p \in W$

Using the first step in the previous talk concerning generation property, one concludes that the two subcategories from (3) form a **G**-linear semiorthogonal decomposition of $D^b(\mathrm{rep}(\mathbf{B}))$. For that, using Corollary 6.7 (and Corollary 8.3 when working over $\mathbb{Z}$), one first checks that the two **G**-linear triangulated subcategories of $D^b(\mathrm{rep}(\mathbf{B}))$ are semiorthogonal to each other. As those subcategories split generate $D^b(\mathrm{rep}(\mathbf{B}))$, they will give rise to a **G**-linear semiorthogonal decomposition of $D^b(\mathrm{rep}(\mathbf{B}))$. This decomposition and its variants ([17, Section 9.1]) will allow to produce the sought-for objects $X_v \in D^b(\mathrm{rep}(\mathbf{B})), v \in W$ that will give rise to **G**-linear semiorthogonal decompositions of $D^b(\mathrm{rep}(\mathbf{B}))$ the pieces of which will be labelled by elements of the Weyl group.

The speaker will first define the objects $X_p$ and $Y_p$ for a given $p \in W$, following [17, Section 9.2]. The next step is in computing the Hom-groups in $D^b(\mathrm{rep}(\mathbf{B}))$ between $X_p$ and $Y_p$. This is [17, Lemma 9.5], the proof of which can be sketched. The final step is in establishing the isomorphism between $X_p$ and $Y_p$. The speaker will state [17, Theorem 10.1], which is a variant of Theorem 8.20 of [17] from the previous talk, indicating the relation of the former theorem to the latter ("induction along the antipodal excellent order $<_a$"). Then state Corollary 10.4 and sketch the proof of the isomorphism $X_p = Y_p$ (the rest of [17, Section 10]). That isomorphism, combined with [17, Lemma 9.5], implies that the objects $X_p, p \in W$ are exceptional in $D^b(\mathrm{rep}(\mathbf{B}))$ considered as a $\mathsf{k}$-linear category ("**B**-exceptional" for short).

## Talk 11 (Marco Rampazzo). Cohomological properties of the objects $X_p, p \in W$. Full exceptional collections on G/B

### Summary

Theorems 11.1 of [17] (resp., Theorem 11.3 over $\mathbb{Z}$) proves that the **B**-exceptional objects $X_p, p \in W$ constructed in the previous Talk 10 are in fact **G**-exceptional, that is those objects are exceptional in $D^b(\mathrm{rep}(\mathbf{B}))$ considered as a **G**-linear category. In turn, full exceptional collections on **G**/**B** will appear as sheafifications of the objects $X_v \in D^b(\mathrm{rep}(\mathbf{B})), v \in W$. Keeping in mind the construction of $X_v, v \in W$ and using [17, Corollary 6.7], the speaker will sketch the proofs of Theorem 11.1 and of Corollary 11.4 of [17]. These two statements lead to Theorem 11.6, the proof of which can be given a full account.

Theorem 12.1 of [17] now follows formally from what has been achieved by now. Relation to base change for semiorthogonal decompositions can also be discussed, following [17, Remark 12.2].

**Talk 12 (Stefan Dawydiak). Parabolic Steinberg basis and full exceptional collections on $\mathbf{G}/\mathbf{P}$**

**Summary**

This section cover the parabolic case. Its main statements, which are Theorems 1.3 and 1.4, follow essentially the same path that has been set out in Theorems 1.1 and 1.2. Given a standard parabolic subgroup $\mathbf{P} \supset \mathbf{B}$, there is a rational morphism $\pi_{\mathbf{P}} : \mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{P}$; thus, there is a fully faithful pullback functor $\pi_{\mathbf{P}}^* : \mathrm{D}^b(\mathbf{G}/\mathbf{P}) \to \mathrm{D}^b(\mathbf{G}/\mathbf{B})$. It is natural to expect that the objects $\hat{\mathsf{X}}_v, v \in W^{\mathbf{P}}$ that give rise to semiorthogonal decompositions of $\mathrm{D}^b(\mathrm{rep}(\mathbf{P}))$ as a $\mathbf{G}$-linear category (resp., the objects $\hat{\mathcal{X}}_v \in \mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{P}), v \in W^{\mathbf{P}}$ giving full exceptional collections in $\mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{P}))$ are contained among the objects $\mathsf{X}_v, v \in W$ of Theorem 1.1 (resp., among the objects $\mathcal{X}_v, v \in W$ of $\mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{B})$ of Theorem 1.2). One has therefore to recognize those objects among $\mathsf{X}_v, v \in W$ (resp., among $\mathcal{X}_v, v \in W$) that are obtained by the restriction functor $\mathrm{res}_{\mathbf{G}}^{\mathbf{P}} : \mathrm{D}^b(\mathrm{rep}(\mathbf{P})) \to \mathrm{D}^b(\mathrm{rep}(\mathbf{B}))$ (resp., by the pullback $\pi_{\mathbf{P}}^* : \mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{P}) \to \mathrm{D}^b(\mathsf{Coh}(\mathbf{G}/\mathbf{B})$ along the projection $\pi_{\mathbf{P}} : \mathbf{G}/\mathbf{B} \to \mathbf{G}/\mathbf{P}$). The fundamental fact that both functors $\mathrm{res}_{\mathbf{G}}^{\mathbf{P}}$ and $\pi_{\mathbf{P}}^*$ are $t$-exact and fully faithful on the respective derived categories makes it possible to recognize the sought-for exceptional objects on $\mathbf{G}/\mathbf{P}$ by applying the induction functor $\mathrm{Rind}_{\mathbf{B}}^{\mathbf{P}}$ to appropriate objects $\mathsf{X}_v, v \in W$ (resp., the pushforward $R\pi_{\mathbf{P}*}$ to $\mathcal{X}_v, v \in W$).

It turns out the Steinberg weights $e_v, v \in W^{\mathbf{P}}$ for a given parabolic $\mathbf{P}$ behave nicely with respect to the induction functor $\mathrm{Rind}_{\mathbf{B}}^{\mathbf{P}}$ suggesting a natural parabolic analogue of the key $\mathbf{B}$-modules from [17, Section 2] covered in Talk 4. As the objects $X_v, v \in W$ are built out of $P(-e_v)^*, v \in W$ and $Q(e_w), w \in W$, one first detects which $\mathbf{B}$-modules $P(-e_v)^*, v \in W$ are restricted from $\mathbf{P}$. The key statement about the behavior of modules $P(-e_v), v \in W$ under the induction functor $\mathrm{Rind}_{\mathbf{B}}^{\mathbf{P}}$ is Lemma 13.9. A parabolic analogue of [17, Theorem 6.6] is Theorem 13.16.

That allows to further apply the arguments of [17, Sections 8-9] covered in Talks 8 and 9 in the parabolic case obtaining Theorems 1.3 and 1.4.

# 5 Research talks. Emerging connections to geometric representation theory

Below is a tentative list of research talks on the 4th of September

- On the semiorthogonal decompositions for twisted flag varieties (A.Ananyevskiy)
- Coherent categorification of Lusztig's asymptotic affine Hecke algebra (S.Dawydiak)
- Singular cohomology of $BG$ via representation theory (D.Kubrak)

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