Higher torsion classes and $\tau_n$-tilting theory

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GOAL: Generalise results about torsion classes and $\tau$-tilting theory to a “higher setting”.

1. Recall the classical definitions and results.
2. Introduce the “higher” setting = $n$-abelian categories.
3. Characterise $n$-torsion classes within an $n$-abelian category.
4. Relate $n$-torsion classes with $\tau_n$-rigid pairs.
Throughout, let $A$ be a basic finite-dimensional algebra, so $\text{mod} \ A$ is an abelian length category.

**Definition (Dickson)**

A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ in $\text{mod} \ A$ is called a torsion pair if:

1. $\text{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$
2. For all $M \in \text{mod} \ A$, there is a short exact sequence

$$0 \to tM \to M \to fM \to 0$$

where $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$.

In this case, we call $\mathcal{T}$ a torsion class and $\mathcal{F}$ a torsion-free class.
Theorem (Dickson)

A full additive subcategory $\mathcal{T} \subset \text{mod } A$ is a torsion class if and only if it is closed under quotients and extensions.
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Example

With $A$ the path algebra of $1 \to 2 \to 3$, any torsion class containing $\frac{2}{3}$ and 1 must also contain:
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Example

With $A$ the path algebra of $1 \to 2 \to 3$, any torsion class containing $2/3$ and $1$ must also contain:

```
1 2 3
\rightarrow
2 3
\rightarrow
1 2
\rightarrow
3 2
\rightarrow
1

Very easy to describe all torsion classes - there are 14!
Classical story - torsion pairs

Theorem (Dickson)

A full additive subcategory $T \subset \text{mod} A$ is a torsion class if and only if it is closed under quotients and extensions.

Example

With $A$ the path algebra of $1 \rightarrow 2 \rightarrow 3$, any torsion class containing $\frac{2}{3}$ and $1$ must also contain:

Very easy to describe all torsion classes - there are 14!
Classical story - τ-tilting theory

Let τ denote the Auslander–Reiten translation.

**Definition (Adachi–Iyama–Reiten)**

A pair $(M, P)$ with $M \in \text{mod } A$, $P \in \text{proj } A$ is called:

1. **τ-rigid** if $\text{Hom}_A(M, \tau M) = 0$ and $\text{Hom}_A(P, M) = 0$.
2. **support τ-tilting** if it is τ-rigid and $|M| + |P| = |A|$.

In this case, AIR showed that being support τ-tilting is equivalent to being maximal with respect to the τ-rigid property.
Classical story - relationship between torsion classes and $\tau$-tilting theory

Theorem (Adachi–Iyama–Reiten)

In $\text{mod } A$, there is a bijection

\[
\{ \text{functorially finite torsion classes} \} \longleftrightarrow \{ \text{support } \tau\text{-tilting pairs} \}.
\]

1. $\mathcal{T} \subseteq \text{mod } A$ is functorially finite if every $M \in \text{mod } A$ has both a left and right approximation by $\mathcal{T}$.

2. $f : M \to T$ is a left $\mathcal{T}$-approximation if $T \in \mathcal{T}$ and

\[
\begin{array}{c}
M \\
\downarrow \quad \exists \\
T'
\end{array}
\xrightarrow{f}
\begin{array}{c}
T
\end{array}
\]

3. If $A$ is representation finite (or even $\tau$-tilting finite) then the functorial finiteness is automatic.
Classical story - relationship between torsion classes and \( \tau \)-tilting theory

How does the bijection work?

**Definition**

An object \( X \in T \subset \text{mod} \ A \) is called Ext\(^1\)-projective in \( T \) if \( \text{Ext}^1_A(X, T) = 0 \) for all \( T \in T \).

Given a functorially finite torsion class \( T \subset \text{mod} \ A \),

1. set \( T \) to be a basic additive generator of the Ext\(^1\)-projectives in \( T \);
2. set \( P \) to be the maximal basic projective module such that \( \text{Hom}_A(P, T) = 0 \).

Then \( (T, P) \) is a support \( \tau \)-tilting pair.
Classical story - relationship between torsion classes and $\tau$-tilting theory

How does the bijection work?

**Definition**

An object $X \in \mathcal{T} \subset \text{mod } A$ is called $\Ext^1$-projective in $\mathcal{T}$ if $\Ext^1_A(X, T) = 0$ for all $T \in \mathcal{T}$.

Given a functorially finite torsion class $\mathcal{T} \subset \text{mod } A$,

1. set $T$ to be a basic additive generator of the $\Ext^1$-projectives in $\mathcal{T}$;
2. set $P$ to be the maximal basic projective module such that $\text{Hom}_A(P, T) = 0$.

Then $(T, P)$ is a support $\tau$-tilting pair.

Given $(T, P)$ support $\tau$-tilting pair, $\text{Fac}(T)$ is a torsion class.
Goal: study $n$-torsion classes and $\tau_n$-rigid pairs in $n$-abelian categories.
Motivated by generalising the Auslander Correspondence, Iyama found the notion of $n$-cluster-tilting subcategories appearing naturally in representation theory.

**Definition**

A subcategory $\mathcal{M} \subset \text{mod } A$ is called $n$-cluster-tilting if it is functorially finite and

$$\mathcal{M} = \{X \in \text{mod } A \mid \text{Ext}^{1 \leq i \leq n-1}_A(X, \mathcal{M}) = 0\}$$

$$= \{Y \in \text{mod } A \mid \text{Ext}^{1 \leq i \leq n-1}_A(\mathcal{M}, Y) = 0\}.$$

- $n = 1$; $\text{mod } A$ is the only 1-cluster-tilting subcategory.
- $n = 2$; the “usual” notion of cluster-tilting.

Introduced by Jasso, $n$-abelian categories were designed to axiomatise the structure of $n$-cluster-tilting subcategories.
What is $n$-abelian?

Very loosely, the idea is to replace the importance of short exact sequences (and the kernels/cokernels that they consist of) with exact sequences of longer length.

Definition (Jasso)

1. Given a morphism $f : X_0 \rightarrow X_1$, an $n$-cokernel of $f$ is a sequence of morphisms

$$X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n+1}$$

such that the sequence

$$0 \rightarrow (X_{n+1}, Y) \rightarrow (X_n, Y) \rightarrow \cdots \rightarrow (X_1, Y) \rightarrow (X_0, Y)$$

is exact for all $Y$. ($n$-kernel is defined dually)

2. A sequence of morphisms $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n+1}$ is called $n$-exact if
   - $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_{n+1}$ is an $n$-cokernel of $X_0 \rightarrow X_1$;
   - $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n$ is an $n$-kernel of $X_n \rightarrow X_{n+1}$.
What is \( n \)-abelian?

- Then \( \mathcal{M} \subset \text{mod} \ A \) is \( n \)-abelian if
  - every morphism has both an \( n \)-kernel and \( n \)-cokernel;
  - every monomorphism and its \( n \)-cokernel forms an \( n \)-exact sequence (plus dual requirement);
  - plus other technical things.

- Jasso showed \( n \)-cluster tilting subcategories of \( \text{mod} \ A \) are \( n \)-abelian. And the converse is also true (Kvamme, Ebrahimi–Nasr-Isfahani).

- In an \( n \)-cluster-tilting subcategory, \( n \)-exact sequences are exact! And vice versa!
Example

Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$.

- This has a 2-cluster-tilting subcategory shown in yellow.
- There is a 2-exact sequence
  
  $0 \rightarrow 3 \rightarrow \frac{2}{3} \rightarrow \frac{1}{2} \rightarrow 1 \rightarrow 0$.

- The 2-cokernel of $\frac{2}{3} \rightarrow \frac{1}{2}$ is
  
  $\frac{1}{2} \rightarrow 1 \rightarrow 0 \rightarrow 0$. 
Recall that for a torsion pair \((\mathcal{T}, \mathcal{F})\), any \(M \in \text{mod} \ A\) must have a short exact sequence

\[
0 \rightarrow tM \rightarrow M \rightarrow fM \rightarrow 0
\]

where \(tM \in \mathcal{T}\), and \(fM \in \mathcal{F}\).
Recall that for a torsion pair \((\mathcal{T}, \mathcal{F})\), any \(M \in \text{mod } A\) must have a short exact sequence

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where \(tM \in \mathcal{T}\), and \(fM \in \mathcal{F}\).

Notice that, since \(\text{Hom}_A(\mathcal{T}, X) = 0 \iff X \in \mathcal{F}\), this is the same as asking for a short exact sequence

\[
0 \to tM \to M \to fM \to 0
\]

where \(tM \in \mathcal{T}\) and \(0 \to \text{Hom}_A(\mathcal{T}, fM) \to 0\) is exact for all \(T \in \mathcal{T}\).
**n-torsion classes**

**Definition (Jørgensen)**

Let $\mathcal{M} \subset \text{mod } A$ be an $n$-cluster-tilting subcategory. A full subcategory $\mathcal{U} \subset \mathcal{M}$ is called an $n$-torsion class if, for each $M \in \mathcal{M}$, there exists an $n$-exact sequence

$$0 \to U_M \to M \to V_1 \to \cdots \to V_n \to 0$$

where $U_M \in \mathcal{U}$ and

$$0 \to \text{Hom}_A(U, V_1) \to \cdots \to \text{Hom}_A(U, V_n) \to 0$$

is exact for each $U \in \mathcal{U}$. 
Torsion and $n$-torsion

Theorem (Asadollahi–Jørgensen–Schroll–Treffinger)

Let $\mathcal{M} \subset \text{mod} \ A$ be an $n$-cluster-tilting subcategory. Any $n$-torsion class $\mathcal{U} \subset \mathcal{M}$ can be obtained as $\mathcal{T} \cap \mathcal{M}$ for a torsion class $\mathcal{T} \subset \text{mod} \ A$.

Moreover, given $\mathcal{U}$, we can always choose $\mathcal{T}$ in such a way that:

1. $tM \in \mathcal{U}$ for all $M \in \mathcal{M}$;
2. $\text{Ext}^{n-1}_A(tM, fM') = 0$ for all $M, M' \in \mathcal{M}$.
3. For any $M \in \mathcal{M}$, the corresponding $n$-exact sequence is

$$0 \to tM \to M \to V_1 \to \cdots \to V_n \to 0.$$
Recall:

**Theorem (Dickson)**

A full additive subcategory $\mathcal{T} \subset \text{mod} \ A$ is a torsion class if and only if it is closed under extensions and quotients.
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Our result:


Let $\mathcal{M} \subset \text{mod} A$ be an $n$-cluster-tilting subcategory. Then a full additive subcategory $\mathcal{U} \subset \mathcal{M}$ is an $n$-torsion class if and only if it is contravariantly finite, closed under $n$-extensions and $n$-quotients.
Closure under $n$-quotients

Let $\mathcal{M} \subset \text{mod } A$ be an $n$-cluster-tilting subcategory and let $\mathcal{C} \subset \mathcal{M}$. Suppose that

$$X \xrightarrow{f} Y \rightarrow Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_n \rightarrow 0$$

is an exact sequence in $\mathcal{M}$, so $Z_1 \rightarrow Z_2 \rightarrow \cdots \rightarrow Z_n$ is an $n$-cokernel of $f$. 
Closure under $n$-quotients

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By Herschend–Jørgensen, such an $n$-cokernel can always be chosen to be “minimal” - essentially getting rid of null-homotopic summands.
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$\mathcal{C}$ is closed under $n$-cokernels if, for any “minimal” exact sequence as above with $X, Y \in \mathcal{C}$, we must have $Z_1, \ldots, Z_n \in \mathcal{C}$. 
Closure under $n$-quotients

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$\mathcal{C}$ is closed under $n$-quotients if, for any “minimal” exact sequence as above with $Y \in \mathcal{C}$, we must have $Z_1, \ldots, Z_n \in \mathcal{C}$.
Example

Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$. For an $n$-torsion class $\mathcal{U} \subset \mathcal{M}$:

2-cluster-tilting subcategory $\mathcal{M}$ shown in yellow.
Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$. For an $n$-torsion class $\mathcal{U} \subset \mathcal{M}$:

- The 2-extension

$$0 \to 3 \to \frac{2}{3} \to \frac{1}{2} \to 1 \to 0$$

shows $1 \oplus 3 \in \mathcal{U} \implies \mathcal{U} = \mathcal{M}$.

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Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$. For an $n$-torsion class $U \subset \mathcal{M}$:

- The 2-extension

$$0 \to 3 \to \frac{2}{3} \to \frac{1}{2} \to 1 \to 0$$

shows $1 \oplus 3 \in U \implies U = \mathcal{M}$.

- The same sequence and closure under 2-quotients shows $\frac{2}{3} \in U \implies \frac{1}{2}, 1 \in U$.
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- The same sequence and closure under 2-quotients shows $\frac{2}{3} \in \mathcal{U} \implies \frac{1}{2}, 1 \in \mathcal{U}$.

- The 2-cokernel $\frac{1}{2} \rightarrow 1 \rightarrow 0 \rightarrow 0$ of $\frac{2}{3} \rightarrow \frac{1}{2}$ shows $\frac{1}{2} \in \mathcal{U} \implies 1 \in \mathcal{U}$.
Example

Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$. For an $n$-torsion class $U \subset M$:

- The 2-extension

\[
0 \rightarrow 3 \rightarrow \frac{2}{3} \rightarrow \frac{1}{2} \rightarrow 1 \rightarrow 0
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shows $1 \oplus 3 \in U \implies U = M$.

- The same sequence and closure under 2-quotients shows $\frac{2}{3} \in U \implies \frac{1}{2}, 1 \in U$.

- The 2-cokernel $\frac{1}{2} \rightarrow 1 \rightarrow 0 \rightarrow 0$ of $\frac{2}{3} \rightarrow \frac{1}{2}$ shows $\frac{1}{2} \in U \implies 1 \in U$. 

2-cluster-tilting subcategory $\mathcal{M}$ shown in yellow.
Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$. For an $n$-torsion class $U \subset M$:

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shows $1 \oplus 3 \in U \implies U = M$.

- The same sequence and closure under 2-quotients shows $\frac{2}{3} \in U \implies \frac{1}{2}, 1 \in U$.

- The 2-cokernel $\frac{1}{2} \to 1 \to 0 \to 0$ of $\frac{2}{3} \to \frac{1}{2}$ shows $\frac{1}{2} \in U \implies 1 \in U$.

So there are 6 2-torsion classes: $0, 3, 1, 1 \oplus \frac{1}{2}, 1 \oplus \frac{1}{2} \oplus \frac{2}{3}, M$. 

2-cluster-tilting subcategory $M$ shown in yellow.
$\tau_n$-tilting theory

The $n^{th}$- Auslander–Reiten translation is defined as $\tau_n := \tau \circ \Omega^{n-1}$, where $\Omega$ denotes the syzygy functor.

**Definition**

Let $\mathcal{M} \subset \text{mod} A$ be an $n$-cluster-tilting category. A pair $(M, P)$, where $M \in \mathcal{M}$, $P \in \text{proj} A$ is called $\tau_n$-rigid if

$$\text{Hom}_A(M, \tau_n M) = 0 \quad \text{and} \quad \text{Hom}(P, M) = 0.$$

Warning: there are definitions for support $\tau_n$-tilting pairs, and maximal $\tau_n$-rigid pairs but they are subtly different for $n > 1$. 


Definition

An object $X \in \mathcal{C} \subset \text{mod } A$ is called $\text{Ext}^n$-projective in $\mathcal{C}$ if $\text{Ext}_A^n(X, C) = 0$ for all $C \in \mathcal{C}$. 

Proposition (AHJKPT, McMahon)

Let $M$ be an $n$-cluster tilting subcategory of $\text{mod } A$. Consider an object $U$ in an $n$-torsion class $U = T \cap M$ of $M$ (where $T$ is as in AJST Theorem). The following statements are equivalent:

1. $U$ is $\text{Ext}^n$-projective in $M$.
2. $\tau^n U \in F$ (where $F$ is the torsion-free class associated to $T$).
3. $\text{Hom}(U', \tau^n U) = 0$ for all $U'$ in $U$.

In particular, $\text{Ext}^n$-projective objects in an $n$-torsion class are $\tau^n$-rigid.
Definition

An object \( X \in \mathcal{C} \subseteq \text{mod } A \) is called \( \text{Ext}^n \)-projective in \( \mathcal{C} \) if \( \text{Ext}_A^n(X, C) = 0 \) for all \( C \in \mathcal{C} \).

Proposition (AHJKPT, McMahon)

Let \( \mathcal{M} \) be an \( n \)-cluster tilting subcategory of \( \text{mod } A \). Consider an object \( U \) in an \( n \)-torsion class \( \mathcal{U} = \mathcal{T} \cap \mathcal{M} \) of \( \mathcal{M} \) (where \( \mathcal{T} \) is as in AJST Theorem). The following statements are equivalent:

1. \( U \) is \( \text{Ext}^n \)-projective in \( \mathcal{U} \).
2. \( \tau_n U \in \mathcal{F} \) (where \( \mathcal{F} \) is the torsion-free class associated to \( \mathcal{T} \)).
3. \( \text{Hom}(U', \tau_n U) = 0 \) for all \( U' \) in \( \mathcal{U} \).

In particular, \( \text{Ext}^n \)-projective objects in an \( n \)-torsion class are \( \tau_n \)-rigid.
Theorem (AHJKPT)

Let $\mathcal{M}$ be an $n$-cluster tilting subcategory of $\text{mod} \ A$. In $\mathcal{M}$ there is an injective map

$$\{ \text{ff } n\text{-torsion classes} \} \longrightarrow \{ \tau_n\text{-rigid pairs with } |A| \text{ summands} \},$$

which maps a ff $n$-torsion class to the pair $(M, P)$ where

1. $M$ is a basic additive generator of the $\text{Ext}^n$-projectives in $\mathcal{U}$;
2. $P$ is the maximal basic projective module such that $\text{Hom}_A(P, \mathcal{U}) = 0$.

Moreover, a partial inverse is given by $(M, P) \mapsto \text{Fac}(M) \cap \mathcal{M}$. 
How to find $\text{Ext}^n$-projectives?

Take inspiration from the classical case!
How to find $\text{Ext}^n$-projectives?

Take inspiration from the classical case!

Suppose $\mathcal{T} \subset \text{mod} \ A$ is a functorially finite torsion class:

1. Let $A \to T_0$ be the minimal left $\mathcal{T}$-approximation;
2. Take the cokernel to get $A \to T_0 \to T_1 \to 0$.
3. Auslander–Smalø showed that $T_1 \oplus T_0$ is an additive generator of the $\text{Ext}^1$-projectives in $\mathcal{T}$.
How to find $\text{Ext}^n$-projectives?

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Suppose $\mathcal{T} \subset \text{mod} \ A$ is a functorially finite torsion class:

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Suppose $\mathcal{M} \subset \text{mod} \ A$ is an $n$-cluster-tilting subcategory and $\mathcal{U} \subset \mathcal{M}$ is a functorially finite $n$-torsion class:

1. Let $A \to U_0$ be the minimal left $\mathcal{U}$-approximation;
2. Take the minimal $n$-cokernel to get $A \to U_0 \to U_1 \to \cdots \to U_n \to 0$. Note that the construction ensures $U_1, \ldots, U_n \in \mathcal{U}$.
3. We show $U_A := \bigoplus_{i=0}^{n} U_i$ is an additive generator of the $\text{Ext}^n$-projectives in $\mathcal{U}$. 
How to prove this?

The “easy” part is to show $U_A$ is $\text{Ext}^n$-projective in $\mathcal{U}$ with an inductive argument.
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The “easy” part is to show $U_A$ is Ext$^n$-projective in $U$ with an inductive argument. For the other direction, we consider $B = A/\text{Ann}(U)$ and the sequence

$$0 \to B \to U_0 \to U_1 \to \cdots \to U_n \to 0$$

in mod $B$. We can show Ext’s and $\tau_n$-rigidity are preserved moving from $U$ to mod $B$ and we get the following:
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in $\text{mod } B$. We can show $\text{Ext}$’s and $\tau_n$-rigidity are preserved moving from $\mathcal{U}$ to $\text{mod } B$ and we get the following:

**Theorem (AHJKPT)**

*With setup as above, suppose $M$ is $\text{Ext}^n$-projective in $\mathcal{U}$ and $\text{add}(U_A) \subset \text{add}(M)$. Then $M$ is an $n$-tilting $B$-module.*
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*With setup as above, suppose $M$ is Ext$^n$-projective in $\mathcal{U}$ and $\text{add}(U_A) \subset \text{add}(M)$. Then $M$ is an $n$-tilting $B$-module.*

Since tilting modules have a fixed number of summands, this proves $U_A$ generates all Ext$^n$-projectives!
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in mod $B$. We can show Ext’s and $\tau_n$-rigidity are preserved moving from $\mathcal{U}$ to mod $B$ and we get the following:

**Theorem (AHJKPT)**

*With setup as above, suppose $M$ is Ext$^n$-projective in $\mathcal{U}$ and add$(U_A) \subset$ add$(M)$. Then $M$ is an $n$-tilting $B$-module.*

Since tilting modules have a fixed number of summands, this proves $U_A$ generates all Ext$^n$-projectives!

Note, Martinez–Mendoza have similar results without considering the $n$-torsion class.
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It is easy to check that $(3 \oplus \frac{2}{3}, \frac{1}{2})$ is a $\tau_2$-rigid pair.

2-cluster-tilting subcategory $\mathcal{M}$ shown in yellow.
Example of non-surjectivity

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Let $A$ be the algebra with quiver $1 \overset{a}{\rightarrow} 2 \overset{b}{\rightarrow} 3$, with relation $ab = 0$.

- It is easy to check that $(3 \oplus 2, 1)$ is a $\tau_2$-rigid pair.
- $\text{Fac}(3 \oplus 2) \cap \mathcal{M}$ is shown on the left.
- But this is NOT a 2-torsion class!
Example of non-surjectivity

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Let $A$ be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$.

![Diagram of quiver with arrows from 1 to 2, 2 to 3, and 3 to 2.]

- It is easy to check that $(3 \oplus \frac{2}{3}, \frac{1}{2})$ is a $\tau_2$-rigid pair.
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- But this is NOT a 2-torsion class!

2-cluster-tilting subcategory $\mathcal{M}$ shown in yellow.

So in the higher setting, the injective map is the best we could hope for!
Thank you!

And Happy Birthday to Bill!