Higher torsion classes and τ_n-tilting theory

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Plan

GOAL: Generalise results about torsion classes and τ-tilting theory to a "higher setting".

1. Recall the classical definitions and results.
2. Introduce the "higher" setting $=$ n-abelian categories.
3. Characterise n-torsion classes within an n-abelian category.
4. Relate n-torsion classes with τ_n-rigid pairs.

Classical story - torsion pairs

Throughout, let A be a basic finite-dimensional algebra, so $\operatorname{mod} A$ is an abelian length category.

Definition (Dickson)

A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod} A$ is called a torsion pair if:

1. $\operatorname{Hom}_A(\mathcal{T}, \mathcal{F}) = 0$
2. *For all $M \in \operatorname{mod} A$, there is a short exact sequence*

$$0 \to tM \to M \to fM \to 0$$

   where $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$.

In this case, we call $\mathcal{T}$ a torsion class and $\mathcal{F}$ a torsion-free class.

Classical story - torsion pairs

Theorem (Dickson)

A full additive subcategory $\mathcal{T} \subset \operatorname{mod} A$ is a torsion class if and only if it is closed under quotients and extensions.

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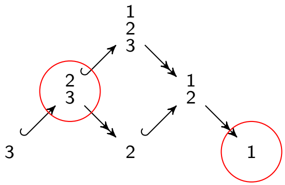

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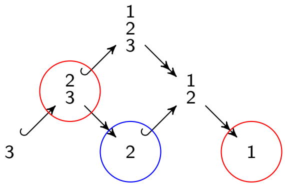

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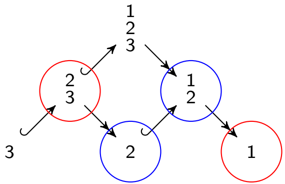

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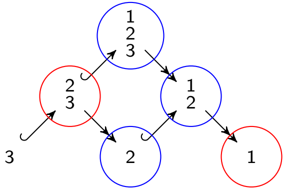

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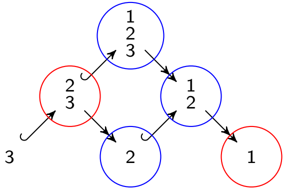


Very easy to describe all torsion classes - there are 14!

Classical story - τ-tilting theory

Let τ denote the Auslander–Reiten translation.

Definition (Adachi–Iyama–Reiten)

A pair (M, P) *with* $M \in \operatorname{mod} A$, $P \in \operatorname{proj} A$ *is called:*

1. τ*-rigid if* $\operatorname{Hom}_A(M, \tau M) = 0$ *and* $\operatorname{Hom}_A(P, M) = 0$.
2. *support* τ*-tilting if it is* τ*-rigid and* $|M| + |P| = |A|$.

In this case, AIR showed that being support τ-tilting is equivalent to being maximal with respect to the τ-rigid property.

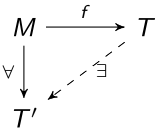
Classical story - relationship between torsion classes and τ-tilting theory

Theorem (Adachi–Iyama–Reiten)

In $\operatorname{mod} A$, there is a bijection

$$\{\ \textit{functorially finite torsion classes}\ \} \longleftrightarrow \{\ \textit{support } \tau\textit{-tilting pairs}\ \}.$$

1. $\mathcal{T} \subset \operatorname{mod} A$ is functorially finite if every $M \in \operatorname{mod} A$ has both a left and right approximation by $\mathcal{T}$.
2. $f : M \to T$ is a left $\mathcal{T}$-approximation if $T \in \mathcal{T}$ and



3. If A is representation finite (or even τ-tilting finite) then the functorial finiteness is automatic.

Classical story - relationship between torsion classes and τ-tilting theory

How does the bijection work?

Definition

An object $X \in \mathcal{T} \subset \operatorname{mod} A$ is called Ext^1-projective in $\mathcal{T}$ if $\operatorname{Ext}^1_A(X, T) = 0$ for all $T \in \mathcal{T}$.

Given a functorially finite torsion class $\mathcal{T} \subset \operatorname{mod} A$,

1. set T to be a basic additive generator of the Ext^1-projectives in $\mathcal{T}$;
2. set P to be the maximal basic projective module such that $\operatorname{Hom}_A(P, \mathcal{T}) = 0$.

Then (T, P) is a support τ-tilting pair.

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Given (T, P) support τ-tilting pair, $\operatorname{Fac}(T)$ is a torsion class.

Today

Goal: study n-torsion classes and τ_n-rigid pairs in n-abelian categories.

What is n-abelian?

Motivated by generalising the Auslander Correspondence, Iyama found the notion of n-cluster-tilting subcategories appearing naturally in representation theory.

Definition

A subcategory $\mathcal{M} \subset \operatorname{mod} A$ is called n-cluster-tilting if it is functorially finite and

$$\mathcal{M} = \{X \in \operatorname{mod} A \mid \operatorname{Ext}_A^{1 \leq i \leq n-1}(X, \mathcal{M}) = 0\}$$
$$= \{Y \in \operatorname{mod} A \mid \operatorname{Ext}_A^{1 \leq i \leq n-1}(\mathcal{M}, Y) = 0\}.$$

- $n = 1$; $\operatorname{mod} A$ is the only 1-cluster-tilting subcategory.
- $n = 2$; the "usual" notion of cluster-tilting.

Introduced by Jasso, n-abelian categories were designed to axiomatise the structure of n-cluster-tilting subcategories.

What is n-abelian?

Very loosely, the idea is to replace the importance of short exact sequences (and the kernels/cokernels that they consist of) with exact sequences of longer length.

Definition (Jasso)

1. *Given a morphism $f\colon X_0 \to X_1$, an n-cokernel of f is a sequence of morphisms*

$$X_1 \to X_2 \to \cdots \to X_{n+1}$$

such that the sequence

$$0 \to (X_{n+1}, Y) \to (X_n, Y) \to \cdots \to (X_1, Y) \to (X_0, Y)$$

is exact for all Y. (n-kernel is defined dually)

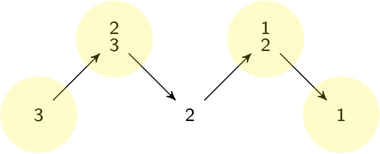
2. *A sequence of morphisms $X_0 \to X_1 \to X_2 \to \cdots \to X_{n+1}$ is called n-exact if*
    - $X_1 \to X_2 \to \cdots \to X_{n+1}$ *is an n-cokernel of* $X_0 \to X_1$*;*
    - $X_0 \to X_1 \to \cdots \to X_n$ *is an n-kernel of* $X_n \to X_{n+1}$.

What is n-abelian?

- Then $\mathcal{M} \subset \operatorname{mod} A$ is n-abelian if
  - every morphism has both an n-kernel and n-cokernel;
  - every monomorphism and its n-cokernel forms an n-exact sequence (plus dual requirement);
  - plus other technical things.
- Jasso showed n-cluster tilting subcategories of $\operatorname{mod} A$ are n-abelian. And the converse is also true (Kvamme, Ebrahimi–Nasr-Isfahani).
- In an n-cluster-tilting subcategory, n-exact sequences are exact! And vice versa!

Example

Let A be the algebra with quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$, with relation $ab = 0$.



- This has a 2-cluster-tilting subcategory shown in yellow.
- There is a 2-exact sequence

$$0 \to 3 \to \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to 1 \to 0.$$

- The 2-cokernel of $\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ is

$$\begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to 1 \to 0 \to 0.$$

n-torsion classes

Recall that for a torsion pair $(\mathcal{T}, \mathcal{F})$, any $M \in \operatorname{mod} A$ must have a short exact sequence

$$0 \to tM \to M \to fM \to 0$$

where $tM \in \mathcal{T}$, and $fM \in \mathcal{F}$.

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Notice that, since $\mathrm{Hom}_A(\mathcal{T}, X) = 0 \iff X \in \mathcal{F}$, this is the same as asking for a short exact sequence

$$0 \to tM \to M \to fM \to 0$$

where $tM \in \mathcal{T}$ and $0 \to \mathrm{Hom}_A(T, fM) \to 0$ is exact for all $T \in \mathcal{T}$.

n-torsion classes

Definition (Jørgensen)

Let $\mathcal{M} \subset \operatorname{mod} A$ be an n-cluster-tilting subcategory. A full subcategory $\mathcal{U} \subset \mathcal{M}$ is called an n-torsion class if, for each $M \in \mathcal{M}$, there exists an n-exact sequence

$$0 \to U_M \to M \to V_1 \to \cdots \to V_n \to 0$$

where $U_M \in \mathcal{U}$ and

$$0 \to \operatorname{Hom}_A(U, V_1) \to \cdots \to \operatorname{Hom}_A(U, V_n) \to 0$$

is exact for each $U \in \mathcal{U}$.

Torsion and n-torsion

Theorem (Asadollahi–Jørgensen–Schroll–Treffinger)

Let $\mathcal{M} \subset \mathrm{mod}\, A$ be an n-cluster-tilting subcategory. Any n-torsion class $\mathcal{U} \subset \mathcal{M}$ can be obtained as $\mathcal{T} \cap \mathcal{M}$ for a torsion class $\mathcal{T} \subset \mathrm{mod}\, A$.

Moreover, given $\mathcal{U}$, we can always choose $\mathcal{T}$ in such a way that:

1. $tM \in \mathcal{U}$ *for all* $M \in \mathcal{M}$;
2. $\mathrm{Ext}_A^{n-1}(tM, fM') = 0$ *for all* $M, M' \in \mathcal{M}$.
3. *For any* $M \in \mathcal{M}$, *the corresponding n-exact sequence is*

$$0 \to tM \to M \to V_1 \to \cdots \to V_n \to 0.$$

n-torsion - characterisation

Recall:

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Our result:

Theorem (A.–Haugland–Jacobsen–Kvamme–Palu–Treffinger)

Let $\mathcal{M} \subset \operatorname{mod} A$ be an n-cluster-tilting subcategory. Then a full additive subcategory $\mathcal{U} \subset \mathcal{M}$ is an n-torsion class if and only if it is contravariantly finite, closed under n-extensions and n-quotients.

Closure under n-quotients

Let $\mathcal{M} \subset \operatorname{mod} A$ be an n-cluster-tilting subcategory and let $\mathcal{C} \subset \mathcal{M}$. Suppose that

$$X \xrightarrow{f} Y \to Z_1 \to Z_2 \to \cdots \to Z_n \to 0$$

is an exact sequence in $\mathcal{M}$, so $Z_1 \to Z_2 \to \cdots \to Z_n$ is an n-cokernel of f.

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Example

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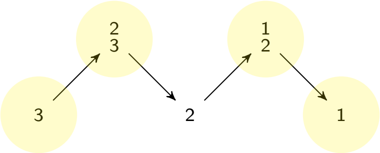

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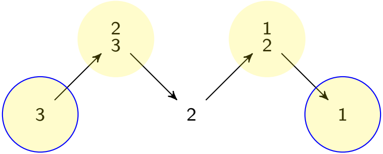

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- The 2-extension

$$0 \to 3 \to \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \to 1 \to 0$$

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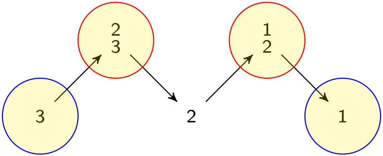

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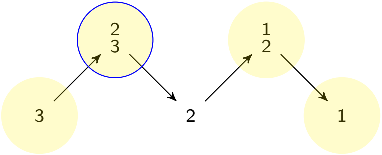

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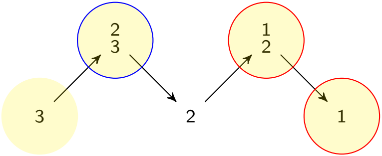

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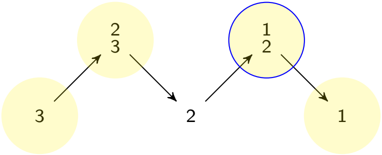

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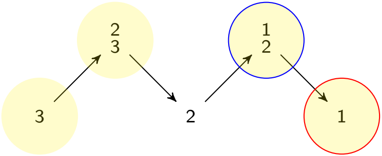

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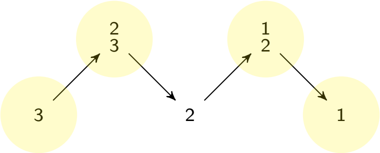

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So there are 6 2-torsion classes: $0, 3, 1, 1 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, 1 \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}, \mathcal{M}$.

τ_n-tilting theory

The n^{th}- Auslander–Reiten translation is defined as $\tau_n := \tau \circ \Omega^{n-1}$, where Ω denotes the syzygy functor.

Definition

Let $\mathcal{M} \subset \operatorname{mod} A$ be an n-cluster-tilting category. A pair (M, P), where $M \in \mathcal{M}$, $P \in \operatorname{proj} A$ is called τ_n-rigid if

$$\operatorname{Hom}_A(M, \tau_n M) = 0 \quad \textit{and} \quad \operatorname{Hom}(P, M) = 0.$$

Warning: there are definitions for support τ_n-tilting pairs, and maximal τ_n-rigid pairs but they are subtly different for $n > 1$.

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An object $X \in \mathcal{C} \subset \operatorname{mod} A$ is called Ext^n-projective in $\mathcal{C}$ if $\operatorname{Ext}^n_A(X, C) = 0$ for all $C \in \mathcal{C}$.

Proposition (AHJKPT, McMahon)

Let $\mathcal{M}$ be an n-cluster tilting subcategory of $\operatorname{mod} A$. Consider an object U in an n-torsion class $\mathcal{U} = \mathcal{T} \cap \mathcal{M}$ of $\mathcal{M}$ (where $\mathcal{T}$ is as in AJST Theorem). The following statements are equivalent:

1. *U is Ext^n-projective in $\mathcal{U}$.*
2. *$\tau_n U \in \mathcal{F}$ (where $\mathcal{F}$ is the torsion-free class associated to $\mathcal{T}$).*
3. *$\operatorname{Hom}(U', \tau_n U) = 0$ for all U' in $\mathcal{U}$.*

In particular, Ext^n-projective objects in an n-torsion class are τ_n-rigid.

Relationship between n-torsion and τ_n-tilting theory

Theorem (AHJKPT)

Let $\mathcal{M}$ be an n-cluster tilting subcategory of $\operatorname{mod} A$. In $\mathcal{M}$ there is an injective map

$$\{ \text{ ff } n\text{-torsion classes } \} \longrightarrow \{ \tau_n\text{-rigid pairs with } |A| \text{ summands } \},$$

which maps a ff n-torsion class to the pair (M, P) where

1. *M is a basic additive generator of the Ext^n-projectives in $\mathcal{U}$;*
2. *P is the maximal basic projective module such that $\mathrm{Hom}_A(P, \mathcal{U}) = 0$.*

Moreover, a partial inverse is given by $(M, P) \mapsto \mathrm{Fac}(M) \cap \mathcal{M}$.

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Suppose $\mathcal{T} \subset \operatorname{mod} A$ is a functorially finite torsion class:

1. Let $A \to T_0$ be the minimal left $\mathcal{T}$-approximation;
2. Take the cokernel to get $A \to T_0 \to T_1 \to 0$.
3. Auslander–Smalø showed that $T_1 \oplus T_0$ is an additive generator of the Ext^1-projectives in $\mathcal{T}$.

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Suppose $\mathcal{M} \subset \operatorname{mod} A$ is an n-cluster-tilting subcategory and $\mathcal{U} \subset \mathcal{M}$ is a functorially finite n-torsion class:

1. Let $A \to U_0$ be the minimal left $\mathcal{U}$-approximation;
2. Take the minimal n-cokernel to get $A \to U_0 \to U_1 \to \cdots \to U_n \to 0$. Note that the construction ensures $U_1, \ldots, U_n \in \mathcal{U}$.
3. We show $U_A := \bigoplus_{i=0}^{n} U_i$ is an additive generator of the Ext^n-projectives in $\mathcal{U}$.

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Note, Martinez–Mendoza have similar results without considering the n-torsion class.

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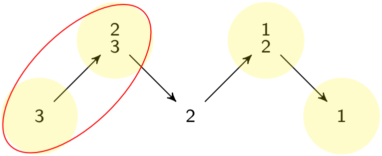

2-cluster-tilting subcategory $\mathcal{M}$ shown in yellow.

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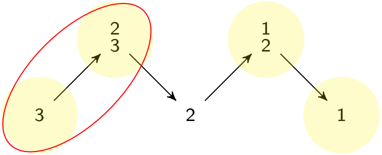

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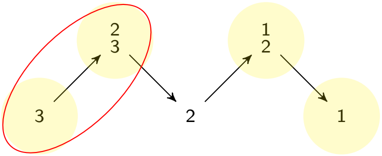

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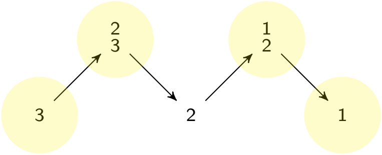

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So in the higher setting, the injective map is the best we could hope for!

Thank you!

And Happy Birthday to Bill!