Persistence in functional topology and data analysis

Ulrich Bauer

TUM

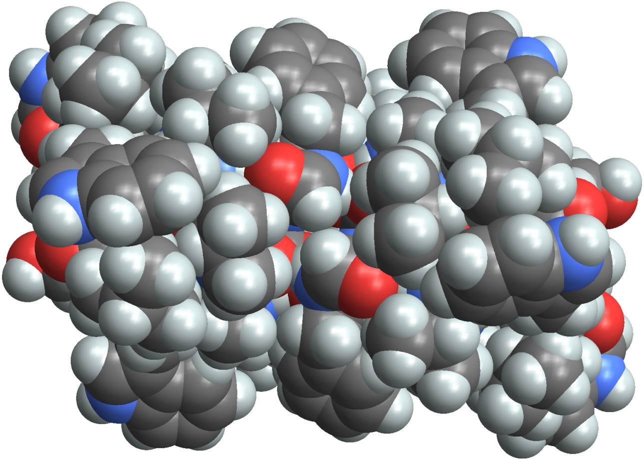
Sep 7, 2021

Conference in celebration of the work of Bill Crawley-Boevey

Northern Regional Meeting of the London Mathematical Society

Holes in data

Geometry and topology of biomolecules



Gramicidin (an antibiotic functioning as an ion channel)

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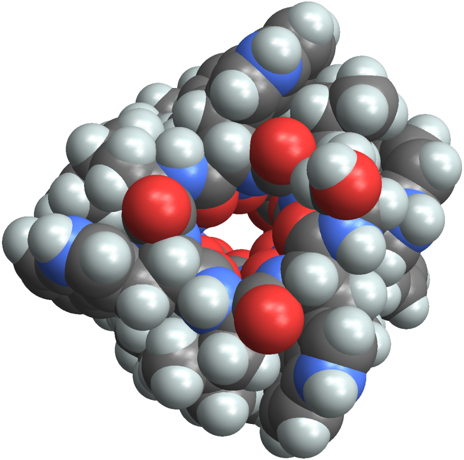

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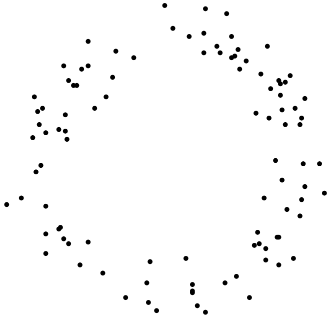

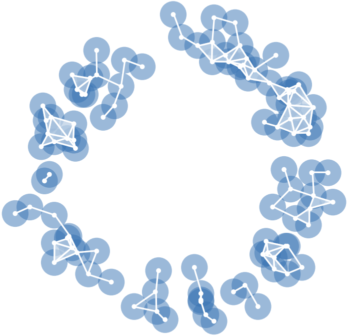

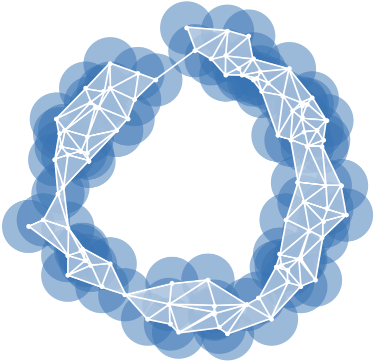

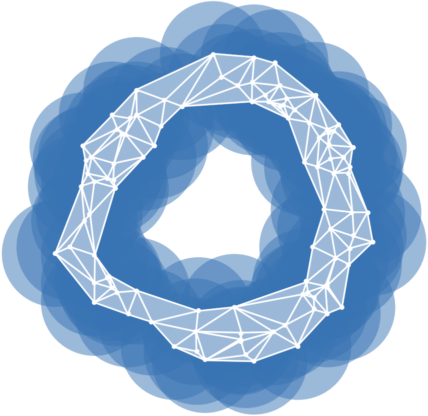

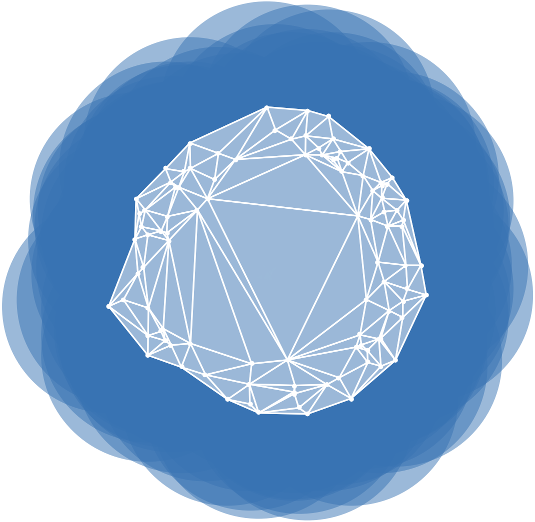

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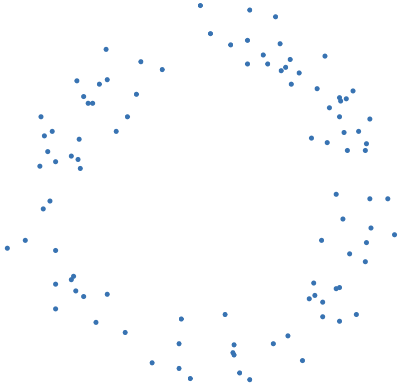

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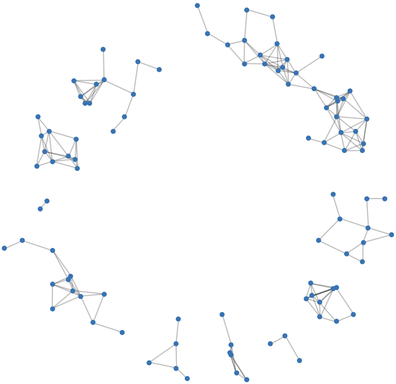

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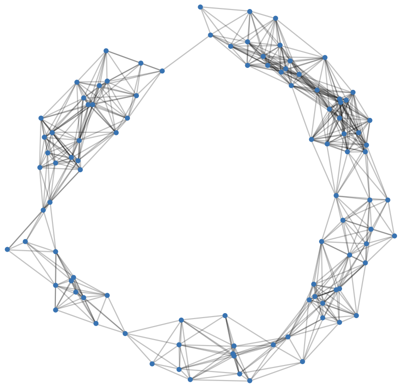

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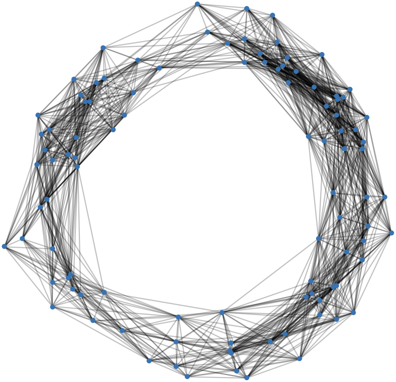

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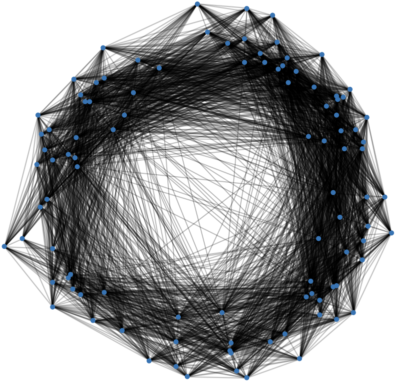

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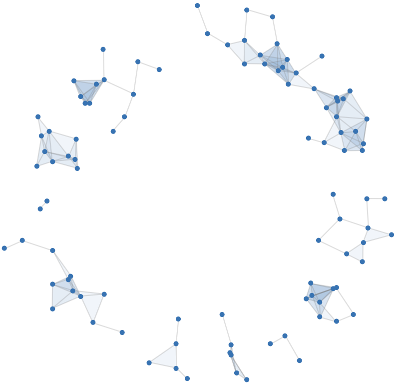

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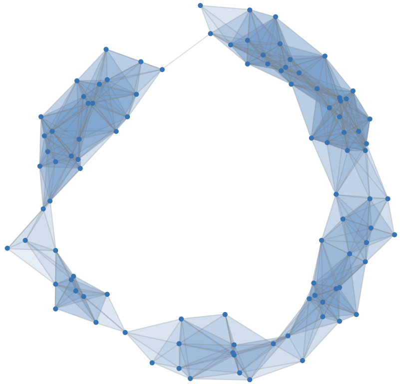

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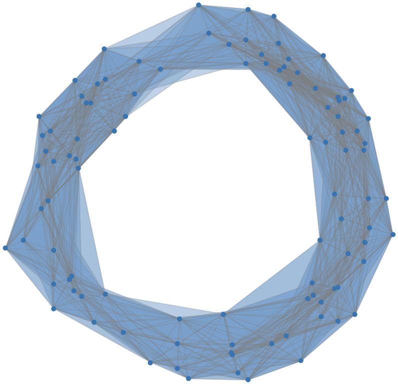

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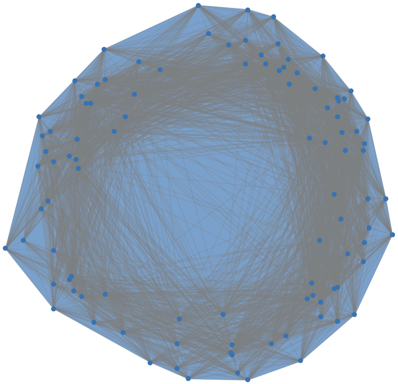

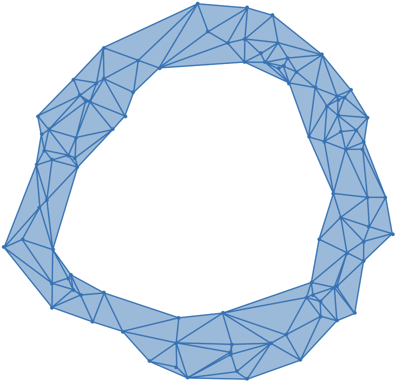

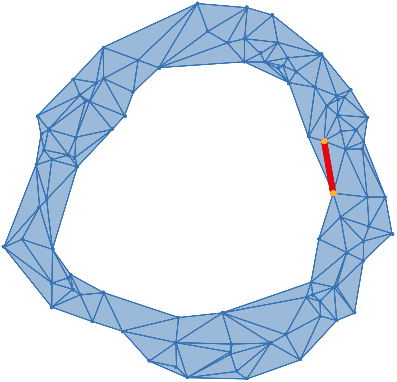

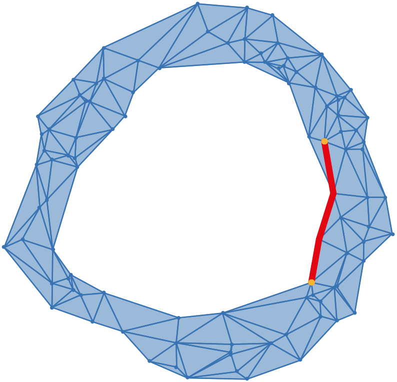

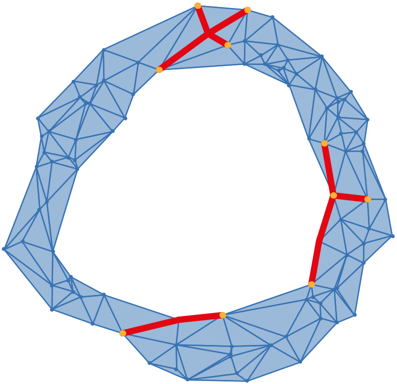

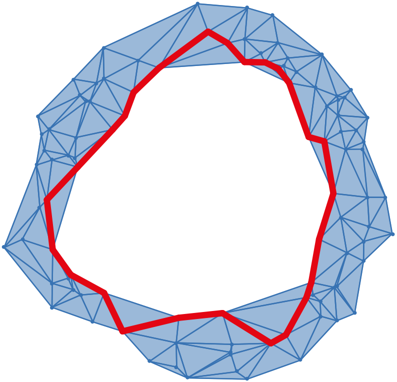

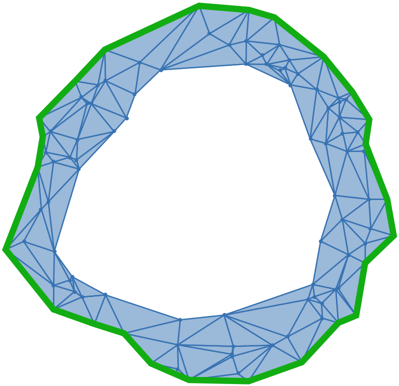

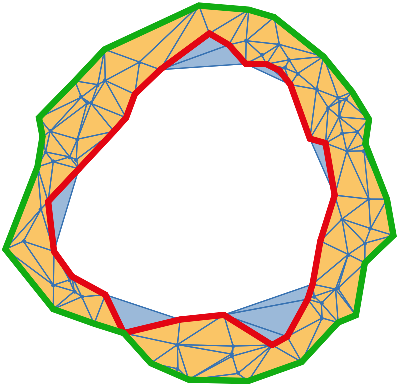

Homology inference

Inferring homology from samples

Given: finite sample $P \subset X$ of unknown shape $X \subset \mathbb{R}^d$

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This works, but requires strong assumptions:

Homology reconstruction by thickening

Theorem (Niyogi, Smale, Weinberger 2006)

Let X *be a submanifold of* $\mathbb{R}^d$. *Let* $P \subset X$ *and* $\delta > 0$ *be such that*

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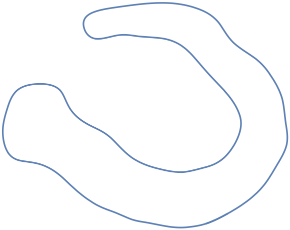

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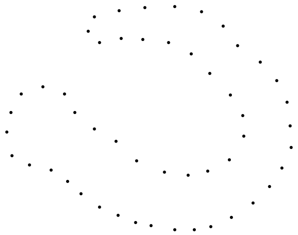

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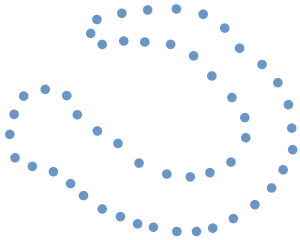

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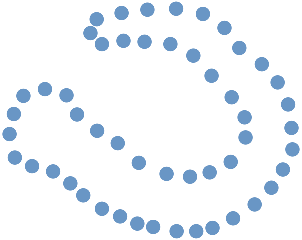

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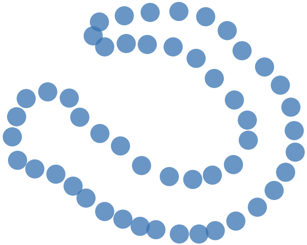

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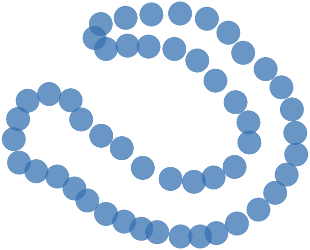

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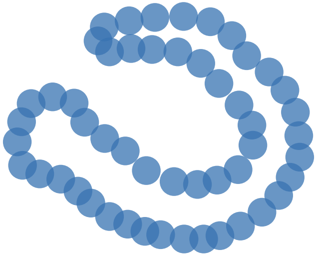

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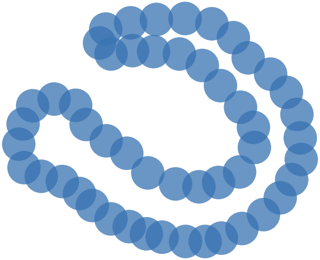

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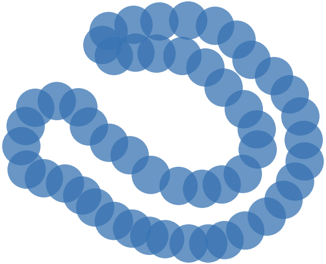

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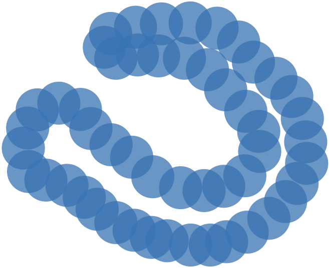

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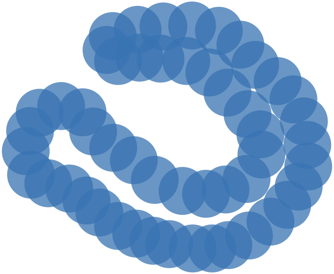

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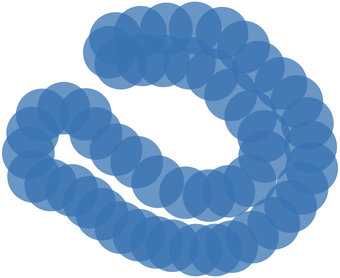

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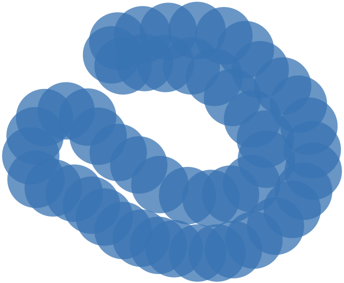

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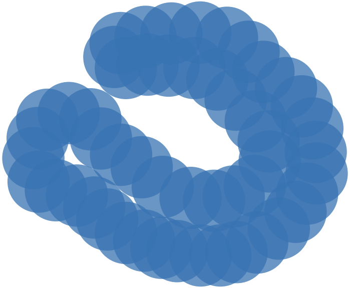

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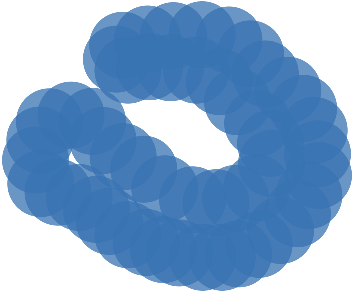

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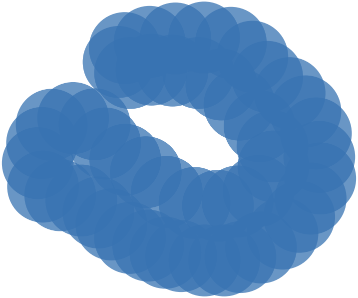

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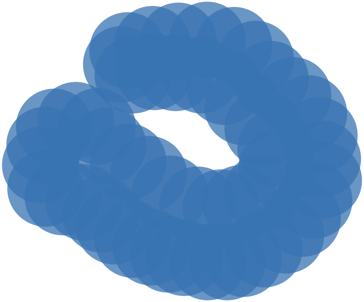

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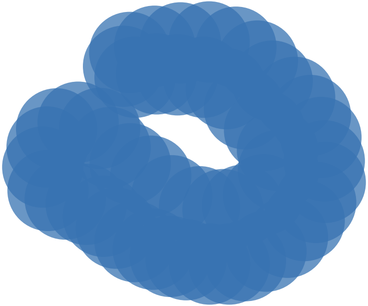

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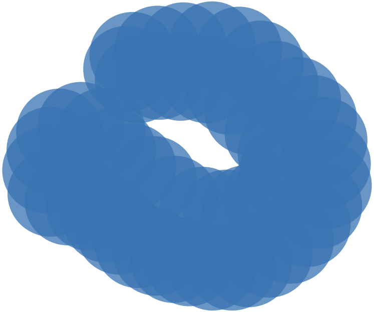

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Let $X \subset \mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

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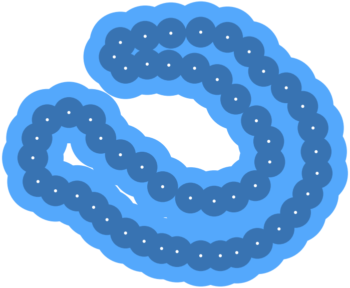

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Homology inference using persistence

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $X \subset \mathbb{R}^d$. Let $P \subset X$ and $\delta > 0$ be such that

- *P_δ covers X,*
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Then $H_*(X) \cong \operatorname{im} H_*(P_\delta \hookrightarrow P_{2\delta})$.

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$\square$

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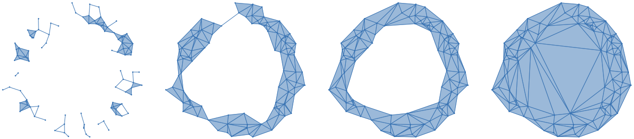

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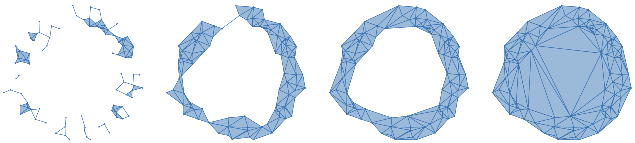

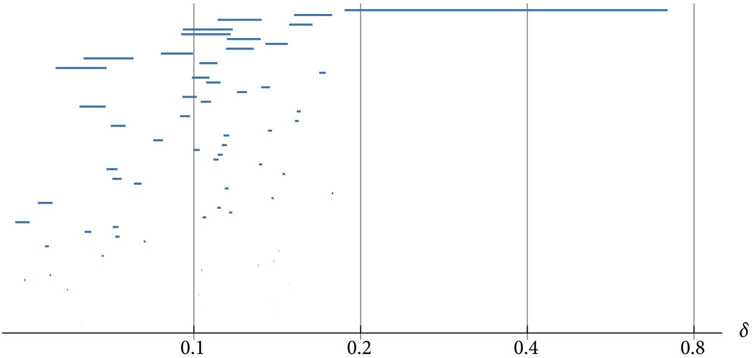

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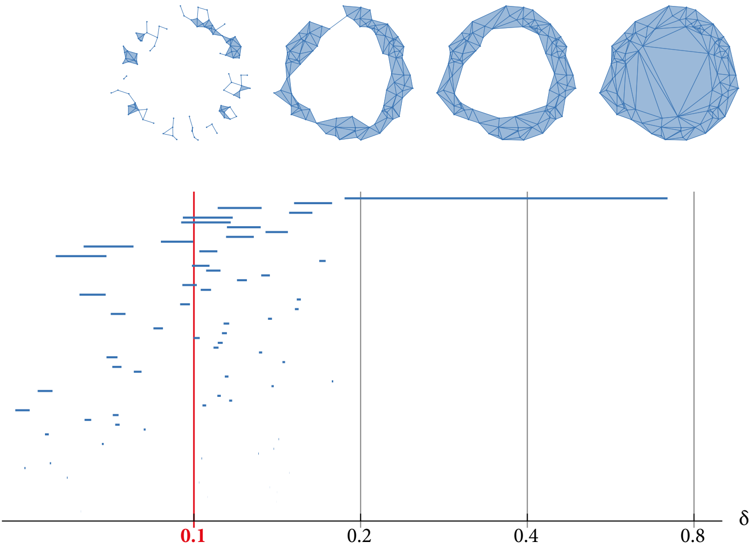

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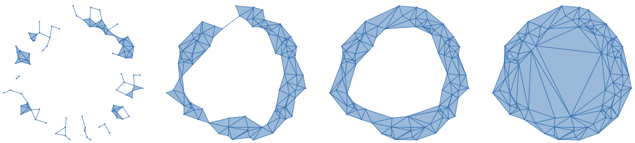

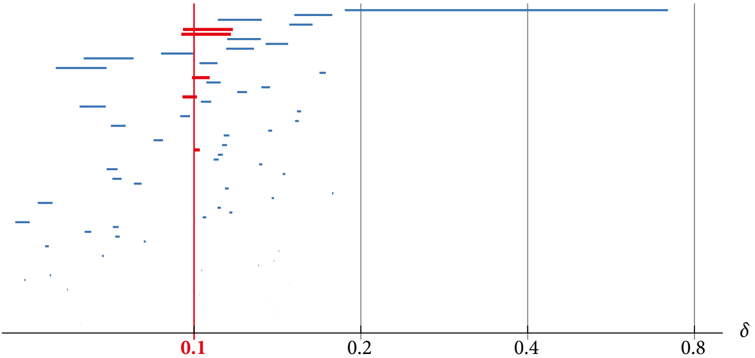

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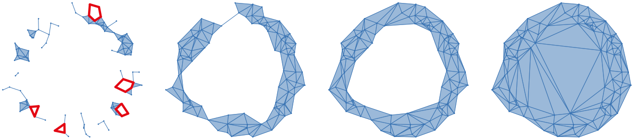

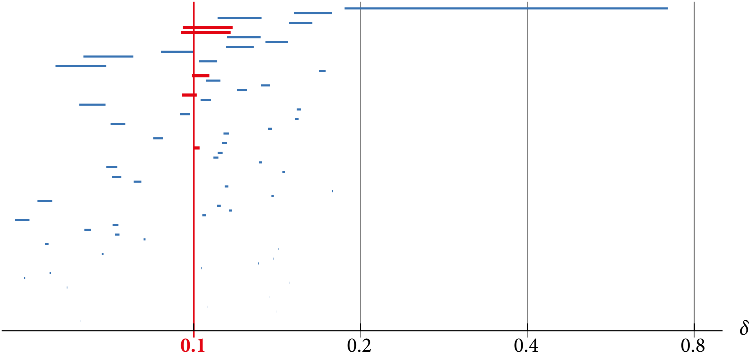

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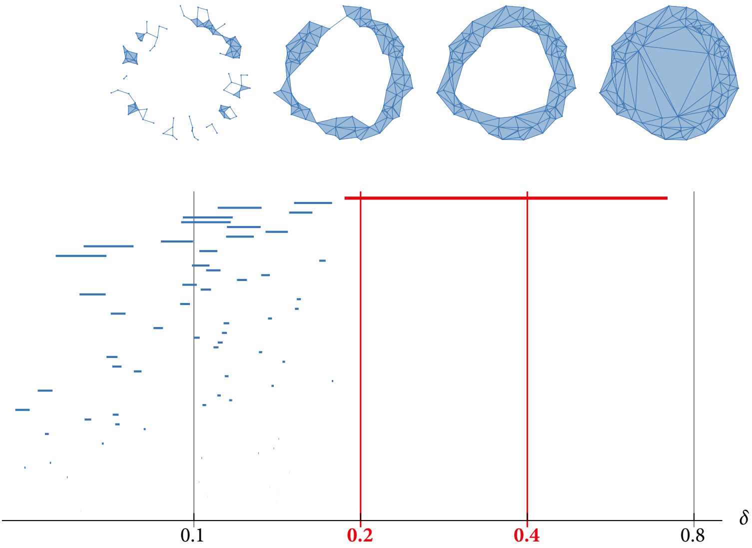

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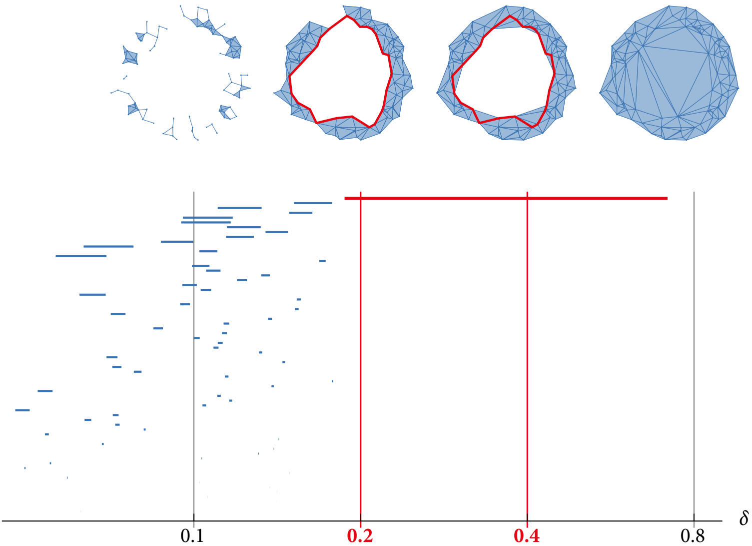

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- A filtration is a certain diagram $K : \mathbf{R} \to \mathbf{Top}$ of topological spaces, indexed over the poset of real numbers $\mathbf{R} := (\mathbb{R}, \leq)$

$$\cdots \dashrightarrow K_s \hookrightarrow K_t \dashrightarrow \cdots$$

  - a topological space K_t for each $t \in \mathbb{R}$
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- Persistent homology is a diagram $M = H_* \circ K : \mathbf{R} \to \mathbf{Vect}$ *(persistence module)*:

$$\cdots\cdots\to M_s \longrightarrow M_t \cdots\cdots\to$$

Barcodes: the structure of persistence modules

Theorem (Crawley-Boevey 2015)

Any persistence module $M : \mathbf{R} \to \mathbf{vect}$ *(of finite dim. vector spaces over some field* $\mathbb{F}$*) decomposes as a direct sum of* interval modules

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- The supporting intervals form the *persistence barcode*.
- We rarely have such a simple structure for other diagrams, like $\mathbf{R}^2 \to \mathbf{vect}$ (2-parameter persistence modules)

Two-parameter persistence

Consider grid-shaped commutative diagrams of vector spaces:

$$\begin{tikzcd}
V_{0,0} \arrow[r] \arrow[d] & V_{1,0} \arrow[r] \arrow[d] & V_{2,0} \arrow[r, dotted] \arrow[d] & V_{m,0} \arrow[d] \\
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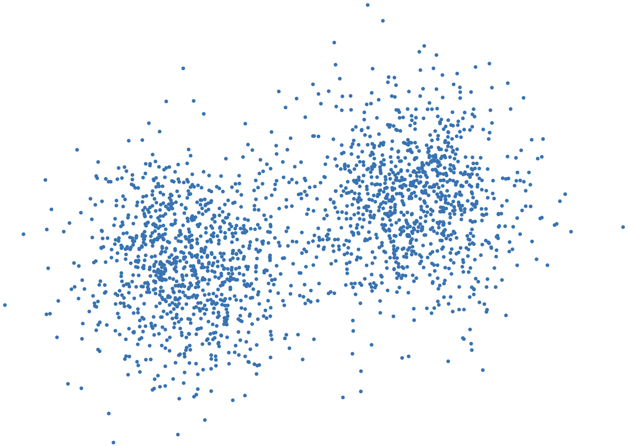
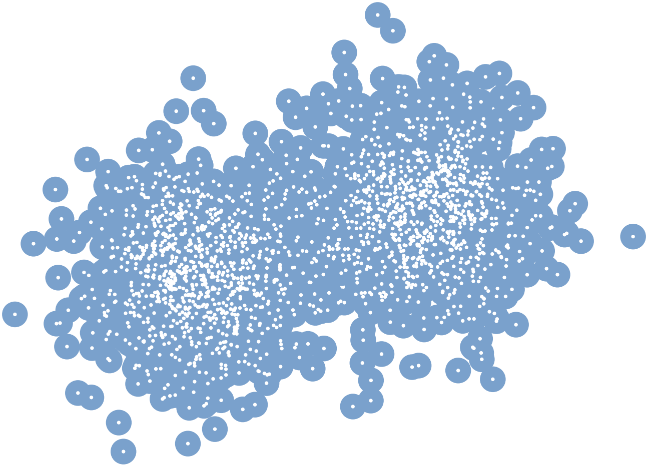
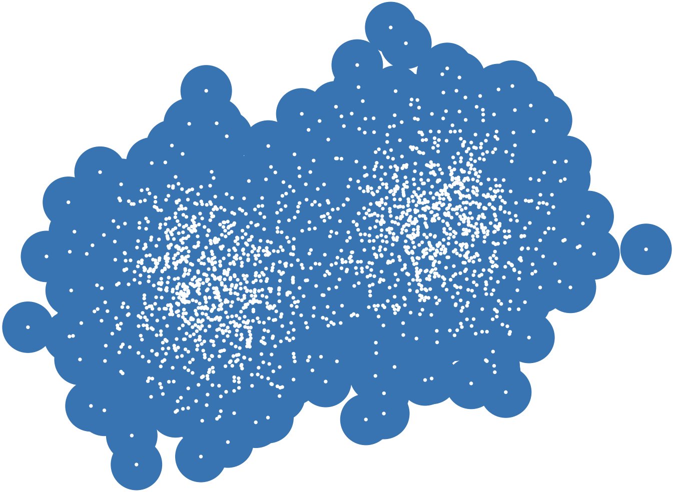
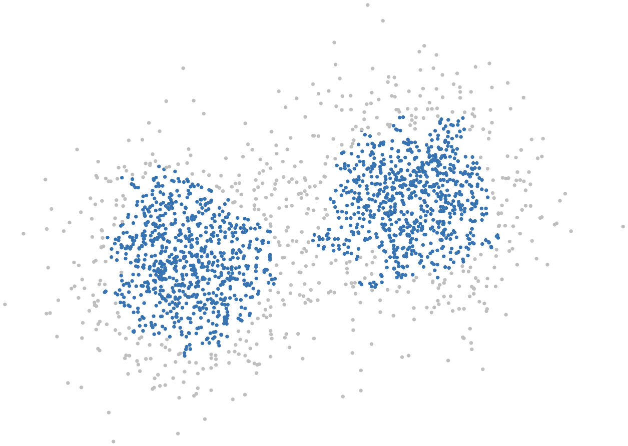
Classification of indecomposables [Drozd 77; Leszczyński 94]:

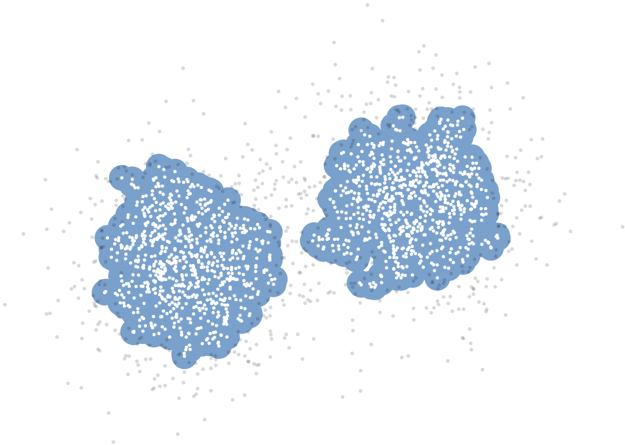
$$
m \cdot n \in \begin{cases} \{0,1,2,3\} & \textit{finite type} \text{ (finite classification)} \\ \{4\} & \textit{tame type} \text{ (1-parameter families)} \\ \{5,6,\dots\} & \textit{wild type} \text{ (undecidable theory)} \end{cases}
$$

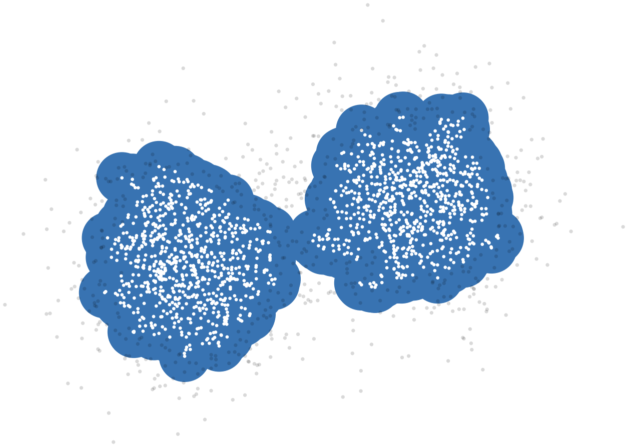


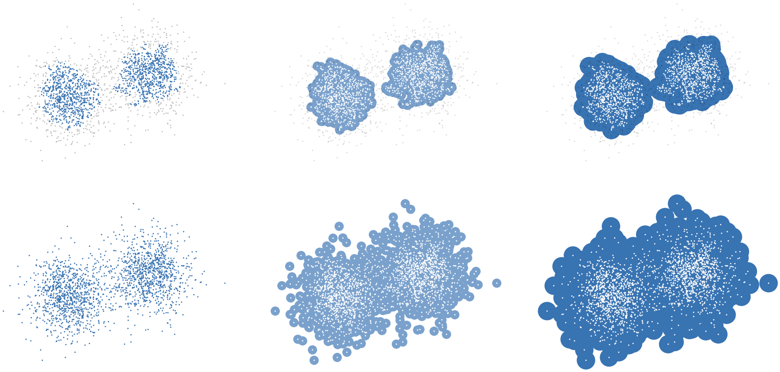












Two-parameter persistence with surjections

Common setup for 2-parameter persistence in degree 0:

- Merging components yields *surjective* horizontal maps

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Theorem (B., Botnan, Oppermann, Steen 2020)

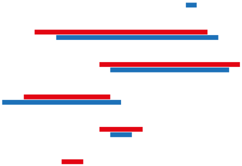
The representation type of $m \times n$ grids in which all horizontal maps are surjective is the same as that of general $m \times (n-1)$ grids.

Stability

Stability of persistence barcodes for functions

Theorem (Cohen-Steiner, Edelsbrunner, Harer 2005)

Let $f, g : X \to \mathbb{R}$ *with* $\|f - g\|_\infty = \delta$ *(and some regularity assumptions).*
Consider the persistence barcodes of (sublevel set filtrations of) f *and* g.
Then there exists a δ*-matching between their intervals, meaning that:*



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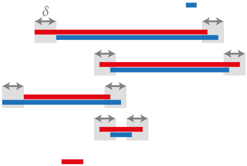

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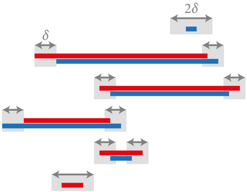

Persistence and stability: the big picture

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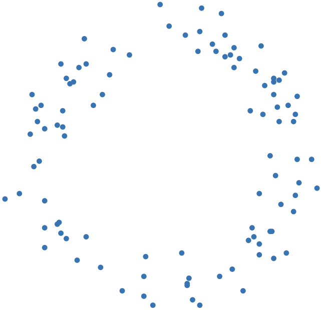
Geometric interleavings



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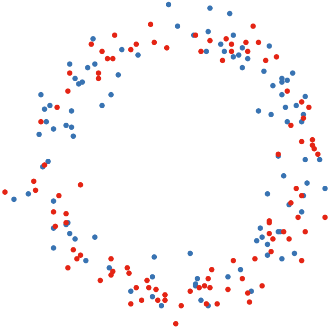

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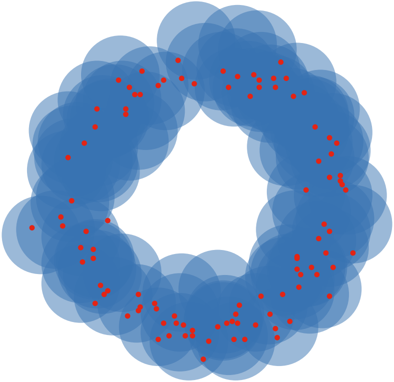

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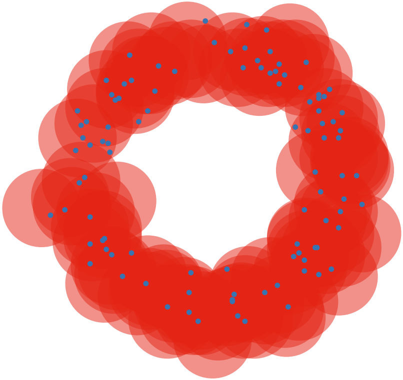

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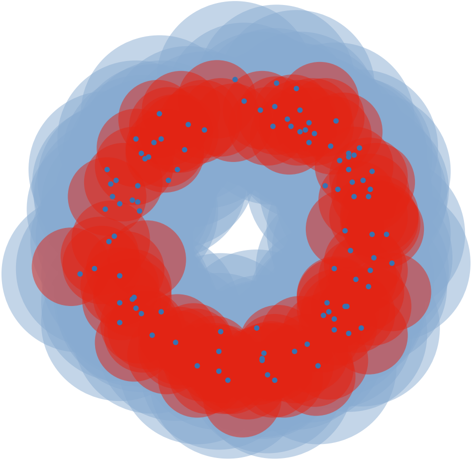

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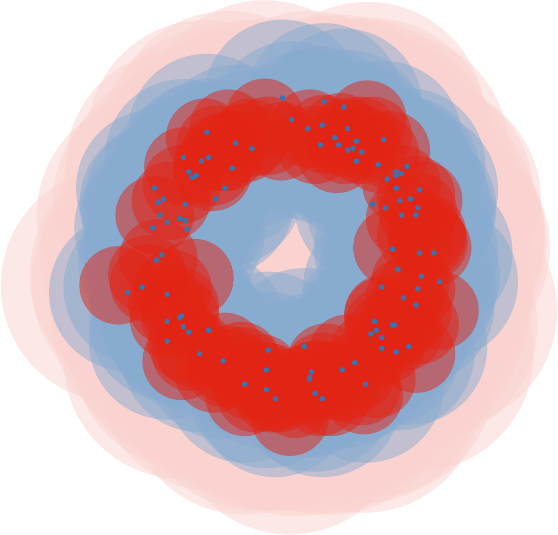

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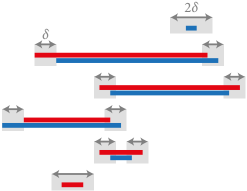
Applying homology (a functor) preserves commutativity

- persistent homology of f, g: δ-interleaved persistence modules

Algebraic stability of persistence barcodes

Theorem (Chazal et al. 2009, 2012; B, Lesnick 2015)

If two persistence modules are δ-interleaved, then there exists a δ-matching of their barcodes.



Structure of persistence sub-/quotient modules

Proposition (B, Lesnick 2015)

Let $M \twoheadrightarrow N$ be an epimorphism of persistence modules.
Then there is an injection of barcodes $B(N) \hookrightarrow B(M)$ with the following property:
if J is mapped to I, then

- *I and J are aligned below, and*
- *I bounds J above.*

This construction is functorial.

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Dually, there is an injection $B(M) \hookrightarrow B(N)$ for monomorphisms $M \hookrightarrow N$.

Induced matchings

For $f : M \to N$ a morphism of pfd persistence modules, the epi-mono factorization

$$M \twoheadrightarrow \operatorname{im} f \hookrightarrow N$$

gives an *induced matching* $\chi(f)$ between their barcodes:

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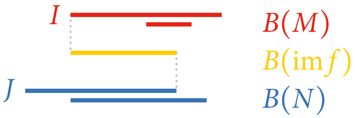
- compose the functorial injections $B(M) \hookleftarrow B(\operatorname{im} f) \hookrightarrow B(N)$ from before to a matching

$$\chi(f) : B(M) \nrightarrow B(N).$$



Algebraic stability via induced matchings

Consider an interleaving $f_t : M_t \to N_{t+\delta}$, $g_t : N_t \to M_{t+\delta}$ $(t \in \mathbb{R})$:

$$\begin{tikzcd}
& M_t \arrow[dr, "f_t"] & \\
N_{t-\delta} \arrow[ur, "g_{t-\delta}"] \arrow[rr, "N_{t-\delta,t+\delta}"] & & N_{t+\delta}
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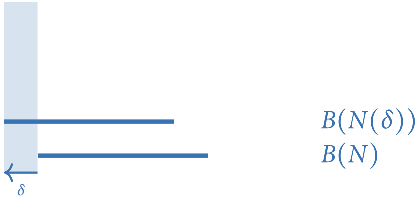

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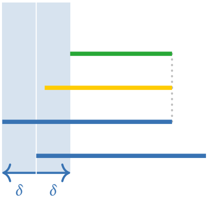

$B(\operatorname{im} N(-\delta, \delta))$
$B(\operatorname{im} f)$
$B(N(\delta))$
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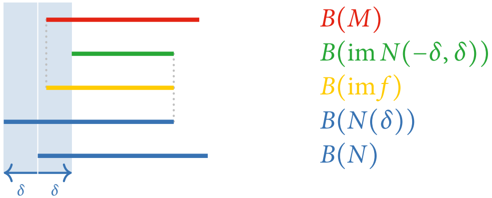
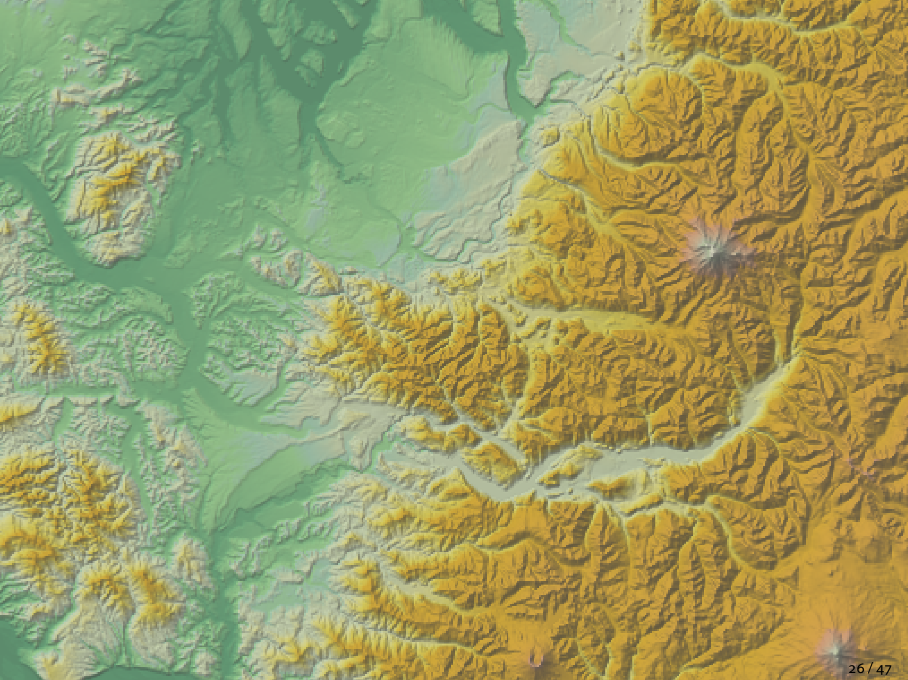
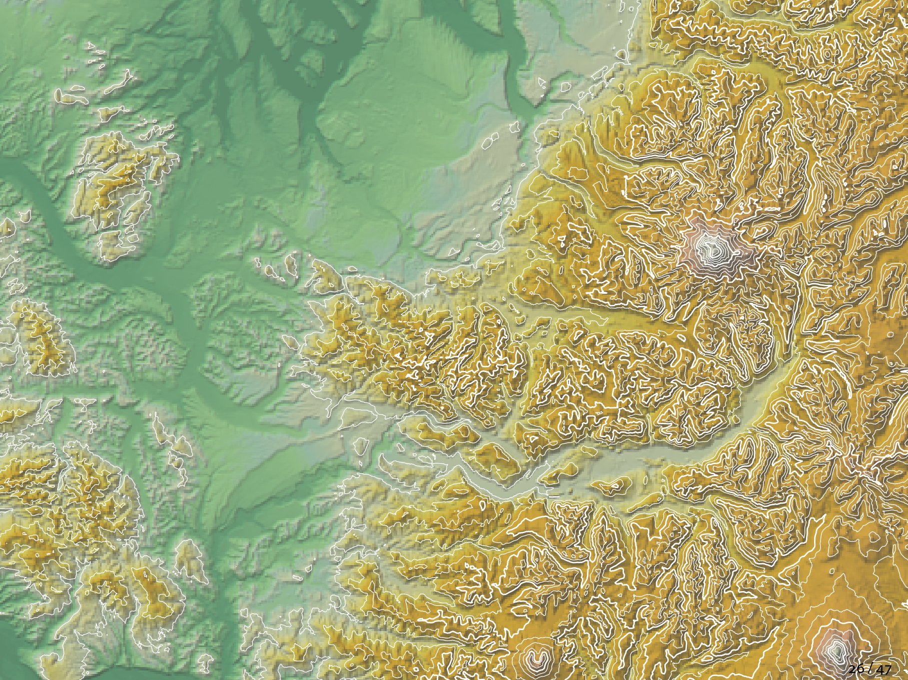
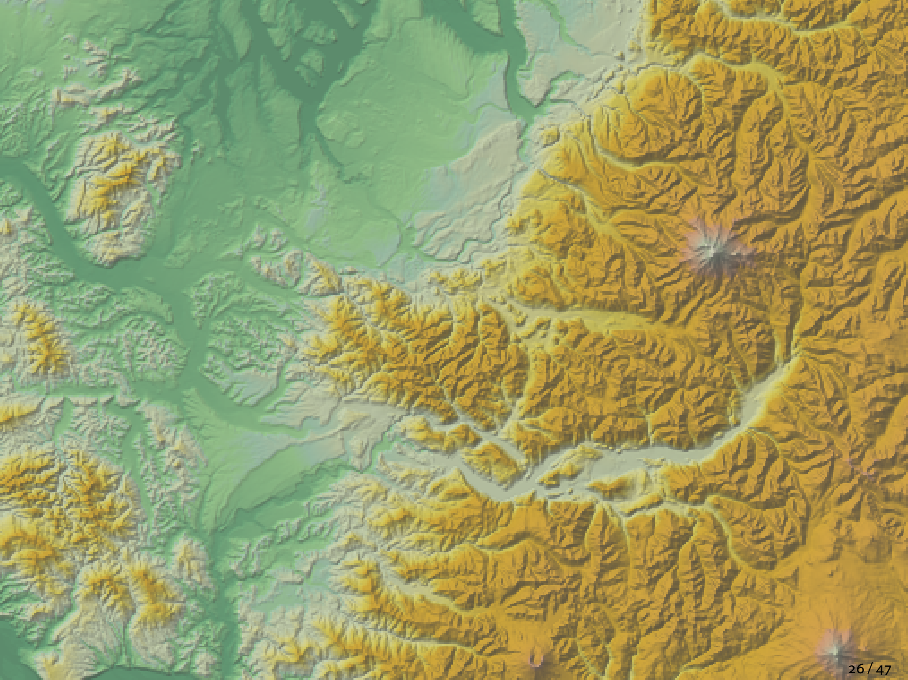
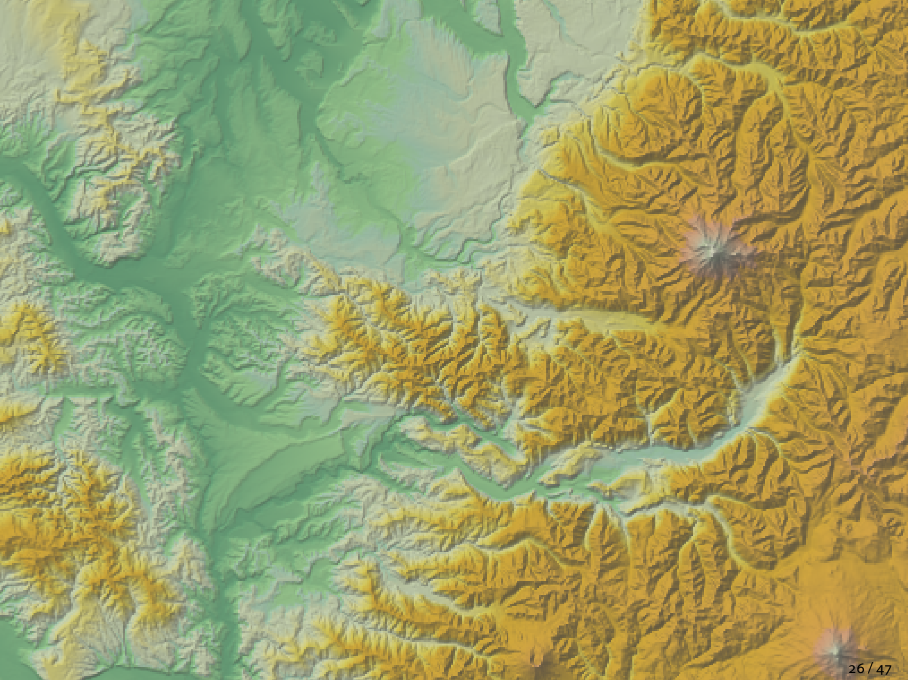


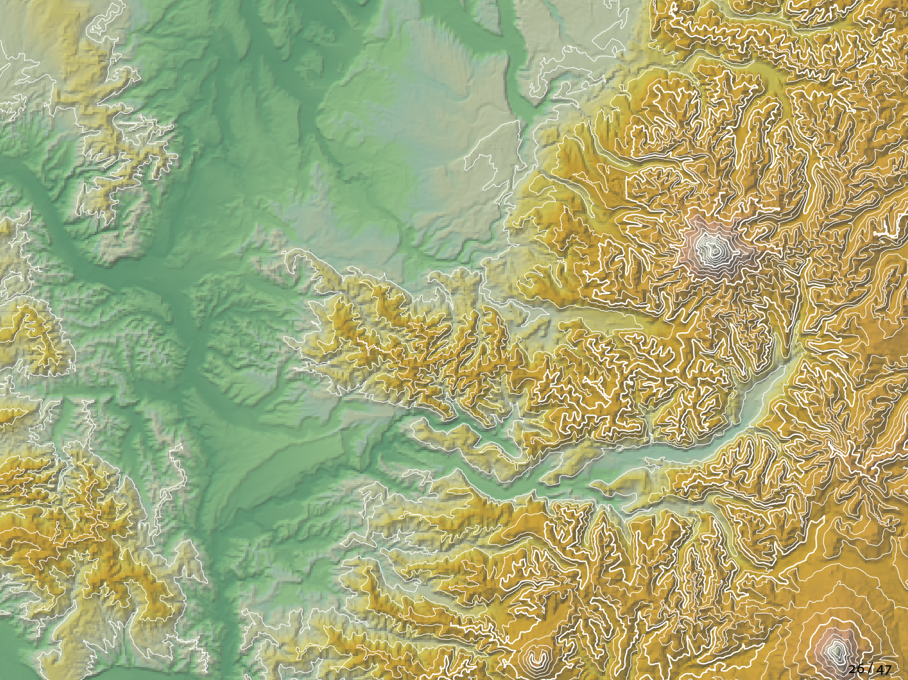
Simplification











Topological simplification of functions

Consider the following problem:

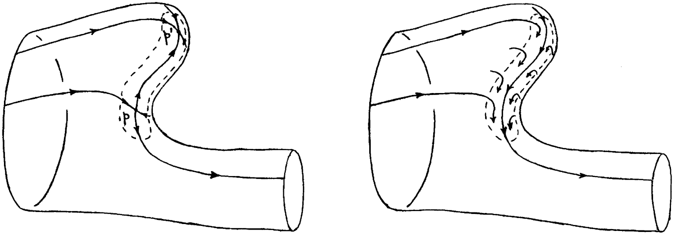
Problem (Topological simplification)

Given a function f and a real number $\delta \geq 0$, find a function f_δ with the minimal number of critical points subject to $\|f_\delta - f\|_\infty \leq \delta$.

Persistence and Morse theory

Morse theory (smooth or discrete):

- Relates critical points to homology of sublevel sets
- Provides a method for *canceling* pairs of critical points



(from Milnor: *Lectures on the h-cobordism theorem*, 1965)

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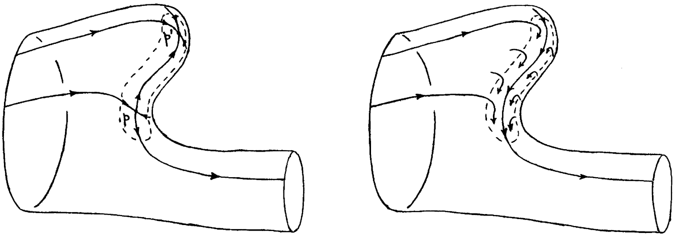

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Persistent homology:

- Relates homology of different sublevel set
- Identifies pairs of critical points (birth and death of homology) and quantifies their *persistence*

Persistence and discrete Morse theory

For a Morse fuction:

- critical points correspond to endpoints of barcode intervals

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Theorem (B, Lange, Wardetzky, 2011)

Let f be a function on a surface and let $\delta > 0$.
Canceling all pairs with persistence $\leq 2\delta$ yields a function f_δ

- *satisfying $\|f_\delta - f\|_\infty \leq \delta$ and*
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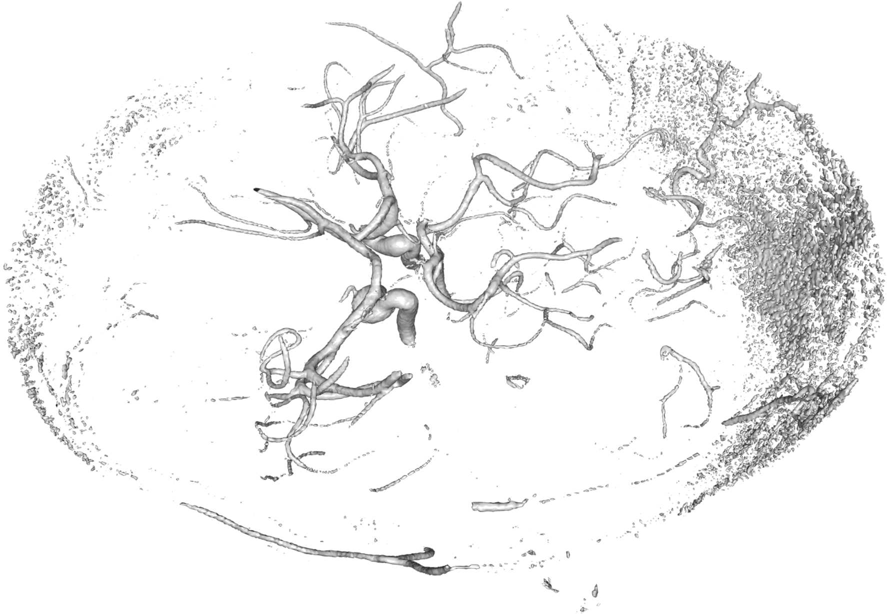
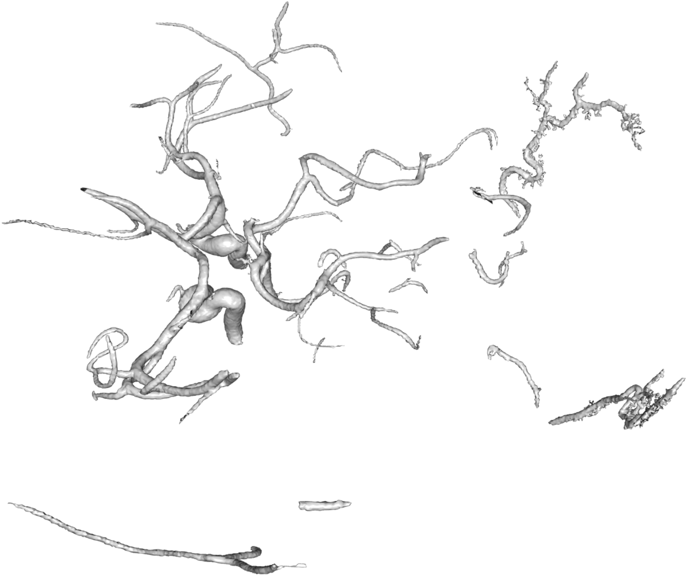
Does not generalize to higher-dimensional manifolds!





Sublevel set simplification

Let $F_t = f^{-1}(-\infty, t]$ denote the t-sublevel set of f.

Problem (Sublevel set simplification)

Given a function $f : X \subset \mathbb{R}^3 \to \mathbb{R}$ *and* $t \in \mathbb{R}$, $\delta > 0$,
find a function g *with* $\|g - f\|_\infty \leq \delta$ *minimizing* $\dim H_*(G_t)$.

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Theorem (Attali, B, Devillers, Glisse, Lieutier 2013)

Sublevel set simplification in $\mathbb{R}^3$ *is NP-hard.*

Functional topology

When was persistent homology invented?

H. Edelsbrunner, D. Letscher, and A. Zomorodian
Topological persistence and simplification
Foundations of Computer Science, 2000

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- P. Frosini
  A distance for similarity classes of submanifolds of a Euclidean space
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When was persistent homology invented first?

When was persistent homology invented first?

ANNALS OF MATHEMATICS
Vol. 41, No. 2, April, 1940

RANK AND SPAN IN FUNCTIONAL TOPOLOGY

BY MARSTON MORSE

(Received August 9, 1939)

1. Introduction.

The analysis of functions F on metric spaces M of the type which appear in variational theories is made difficult by the fact that the critical limits, such as absolute minima, relative minima, minimax values etc., are in general infinite in number. These limits are associated with relative k-cycles of various dimensions and are classified as 0-limits, 1-limits etc. The number of k-limits suitably counted is called the k^{th} type number m_k of F. The theory seeks to establish relations between the numbers m_k and the connectivities p_k of M. The numbers p_k are finite in the most important applications. It is otherwise with the numbers m_k.

The theory has been able to proceed provided one of the following hypotheses

When was persistent homology invented first?

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Exact homomorphism sequences in homology theory — ed.ac.uk [PDF]
JL Kelley, E Pitcher - Annals of Mathematics, 1947 - JSTOR
The developments of this paper stem from the attempts of one of the authors to deduce relations between homology groups of a complex and homology groups of a complex which is its image under a simplicial map. Certain relations were deduced (see [EP 1] and [EP 2] ...
Cited by 46 Related articles All 3 versions Cite Save More

Marston Morse and his mathematical works — ams.org [PDF]
R Bott - Bulletin of the American Mathematical Society, 1980 - ams.org
American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters2 as Poincaré, Veblen, LEJ Brouwer, GD Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to ...
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Unstable minimal surfaces of higher topological structure
M Morse, CB Tompkins - Duke Math. J, 1941 - projecteuclid.org
1. Introduction. We are concerned with extending the calculus of variations in the large to multiple integrals. Theproblem of the existenceof minimal surfaces of unstable type contains many of the typical difficulties, especially those ofa topological nature. Having studied this ...
Cited by 19 Related articles All 2 versions Cite Save

[PDF] **Persistence in discrete Morse theory** — psu.edu [PDF]
U Bauer - 2011 - Citeseer

When was persistent homology invented first?

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, November 1980

MARSTON MORSE AND HIS MATHEMATICAL WORKS

BY RAOUL BOTT[1]

1. Introduction. Marston Morse was born in 1892, so that he was 33 years old when in 1925 his paper *Relations between the critical points of a real-valued function of n independent variables* appeared in the Transactions of the American Mathematical Society. Thus Morse grew to maturity just at the time when the subject of Analysis Situs was being shaped by such masters[2] as Poincaré, Veblen, L. E. J. Brouwer, G. D. Birkhoff, Lefschetz and Alexander, and it was Morse's genius and destiny to discover one of the most beautiful and far-reaching relations between this fledgling and Analysis; a relation which is now known as *Morse Theory*.

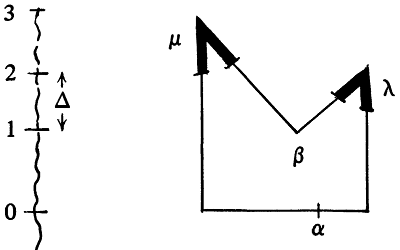
In retrospect all great ideas take on a certain simplicity and inevitability, partly because they shape the whole subsequent development of the subject. And so to us, today, Morse Theory seems natural and inevitable. However one only has to glance at these early papers to see what a tour de force it was

When was persistent homology invented first?

inequalities pertain between the dimensions of the A_i *and those of* $H(A_i)$. Thus the Morse inequalities already reflect a certain part of the "Spectral Sequence magic", and a modern and tremendously general account of Morse's work on rank and span in the framework of Leray's theory was developed by Deheuvels [D] in the 50's.

Unfortunately both Morse's and Deheuvel's papers are not easy reading. On the other hand there is no question in my mind that the papers [36] and [44] constitute another tour de force by Morse. Let me therefore illustrate rather than explain some of the ideas of the rank and span theory in a very simple and tame example.

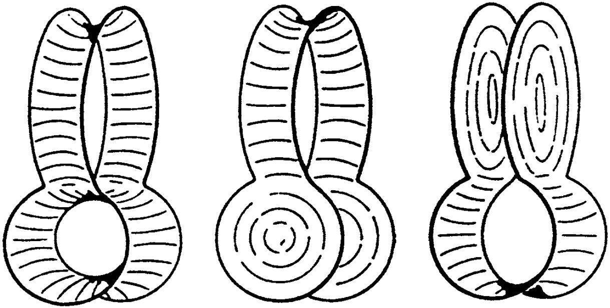
In the figure which follows I have drawn a homeomorph of $M = S^1$ in the plane, and I will be studying the height function $F = y$ on M.



Motivation and application: minimal surfaces

Problem (Plateau's problem)

Find a surface of least area spanned by a given closed Jordan curve.



(a) **(b)** **(c)**

(from Dierkes et al.: *Minimal Surfaces*, 2010)

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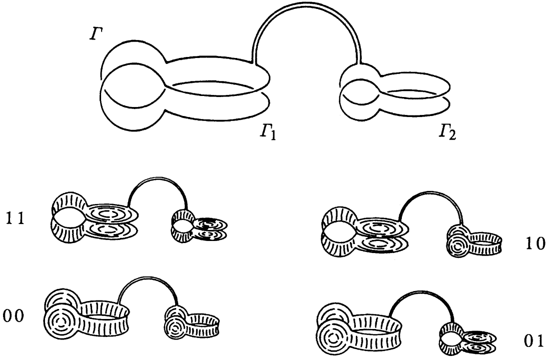

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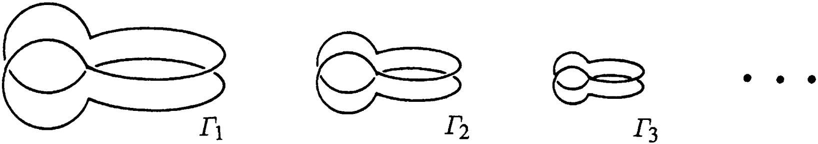

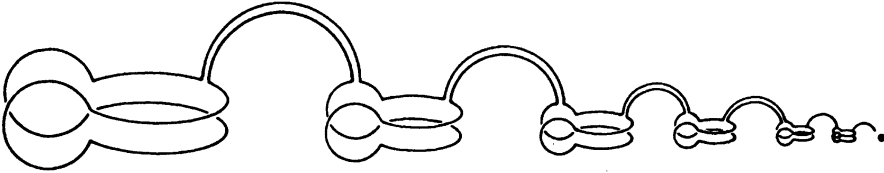

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The Douglas functional

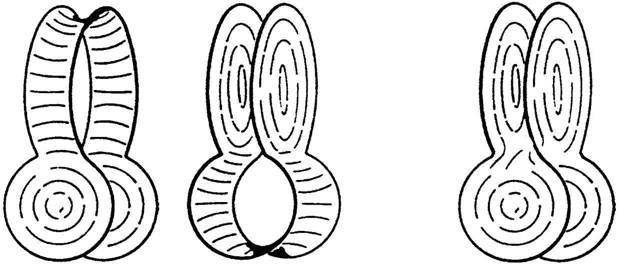
Theorem (Douglas 1930)

Given a Jordan curve $\Gamma : S^1 \to \mathbb{R}^3$, there is a functional on the space of reparametrizations $S^1 \to S^1$ fixing three arbitrary points $q_1, q_2, q_3 \in S^1$, whose critical points are in bijection with the minimal disks bounded by Γ.

Existence of unstable minimal surfaces

Theorem (Morse, Tompkins 1939; Shiffman 1939)

If there are two separate stable minimal surfaces with a given boundary curve, then there exists an unstable minimal surface (a critical point that is not a local minimum).



Morse inequalities

Theorem (Morse 1925)

Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold M. The Betti numbers β_i of M and the numbers m_j of index j critical points of f satisfy:

$$
\begin{aligned}
m_0 &\geq \beta_0 \\
m_1 - m_0 &\geq \beta_1 - \beta_0 \\
&\vdots \\
m_d - m_{d-1} + \cdots \pm m_0 &\geq \beta_d - \beta_{d-1} + \cdots \pm \beta_0
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$$

Corollary ("Mountain pass lemma")

If M is connected ($\beta_0 = 1$) and has two minima ($m_0 = 2$), then it also has a critical point of index 1 ($m_1 \geq \beta_1 - \beta_0 + m_0 = \beta_1 + 1$).

Q-tame persistence modules

Definition

A persistence module $M : \mathbf{R} \to \mathbf{vect}$ is *q-tame* (*ephemeral*) if for every $s < t$ the structure map $M_s \to M_t$ has finite (zero) rank.

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- Example: the Vietoris–Rips filtration of a compact metric space has q-tame persistent homology.
- Morse's goal, in modern language: sufficient conditions for q-tame persistent homology of sublevel sets of a function.

Structure of q-tame persistence modules

Theorem (Chazal, Crawley-Boevey, de Silva 2016)

The radical *of a q-tame persistence module M, defined by $(\operatorname{rad} M)_t = \sum_{s<t} \operatorname{im} M_{s,t}$, admits a barcode decomposition.*

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- *A* persistence diagram *describes the intervals in a barcode, modulo the endpoints.*
- *The* observable category *is the category of persistence modules, localized at the ephemerals.*

Generalized Morse inequalities

Assume that the sublevel sets of a bounded function $f : X \to \mathbb{R}$ are

- compact and
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Morse and Tompkins use this idea to show the existence of a minimal surface.

Weakly πLC filtrations

Definition (paraphrased from Morse)

The sublevel set filtration of a function $f\colon X \to \mathbb{R}$ is said to be *weakly homotopically locally connected*, or *weakly πLC*, if for any

- any point $x \in X$,
- any neighborhood V of x, and
- any value $t > f(x)$,

there is

- a value s with $f(x) < s < t$ and
- a neighborhood U of x with $U \subseteq V$

such that the inclusion $U \cap f_{\leq s} \to V \cap f_{\leq t}$ induces trivial maps on homotopy groups.

An example

Claim (Morse)

If $f\colon X \to \mathbb{R}$ is bounded below and the sublevel sets are compact, weakly πLC, and regular at infinity, then it has q-tame persistent homology.

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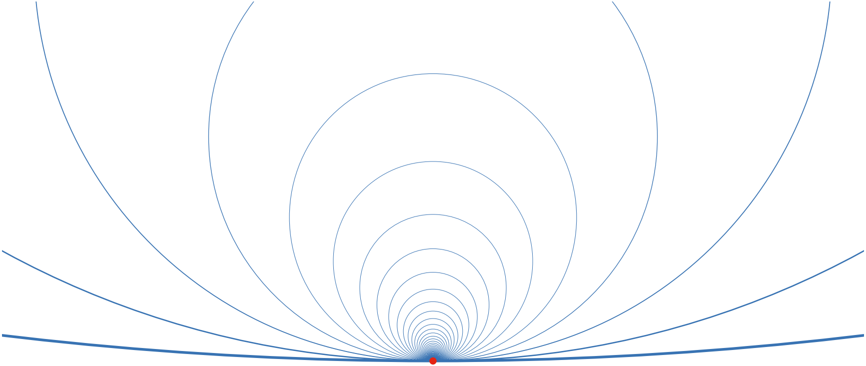


Locally homologically small filtrations

Definition (B, Medina-Mardones, Schmahl)

The sublevel set filtration of a function $f: X \to \mathbb{R}$ is called *locally homologically small* or *HLC* if for

- any point $x \in X$,
- any neighborhood V of x, and
- any pair of values s, t with $f(x) < s < t$

there is

- a neighborhood U of x with $U \subseteq V$

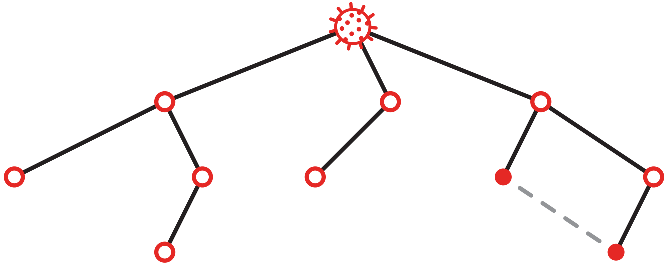
such that the inclusion $U \cap f_{\leq s} \to V \cap f_{\leq t}$ induces maps of finite rank on homology.

A sufficient condition for q-tame persistence

Theorem (B, Medina-Mardones, Schmahl 2021)

If the sublevel set filtration of a (not necessarily continuous) function $f: X \to \mathbb{R}$ is compact and HLS, then its persistent homology is q-tame.

Topology of viral evolution



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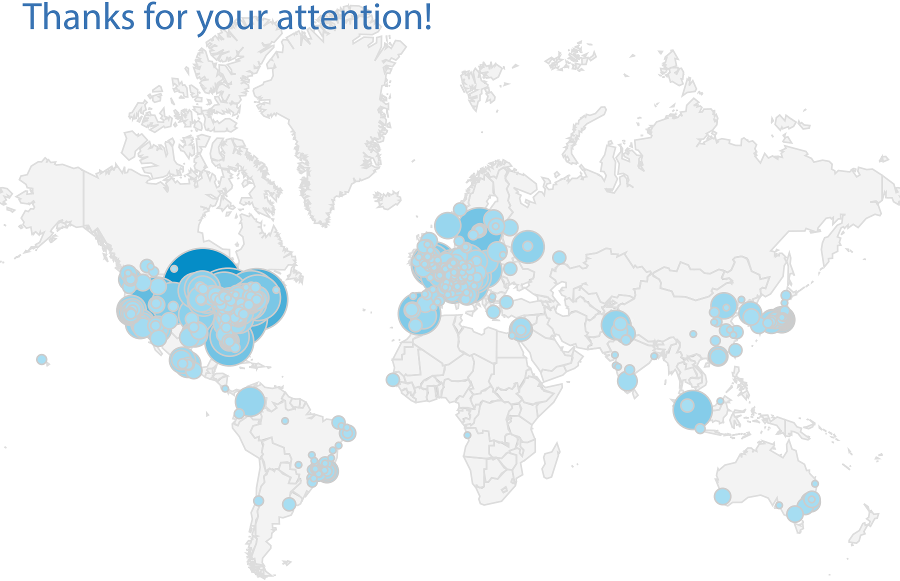
Thanks for your attention!



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