7. Sep. 2021

Surface combinatorics & module categories

Karin Baur

University of Leeds

Conference in celebration of the work of Bill Crawley-Boevey

https://www.math.uni-bielefeld.de/~wcrawley/quiver/thornsbg.jpg

1

1 Triangulations of surfaces

Set-up: S conn. oriented surface, $\emptyset \neq M \subset S$ a finite set of marked pts; at least one on each conn. comp. of ∂S

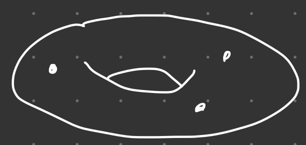
Marked pts in interior of S: punctures

(S,M) a marked surface (a ciliated surface) Fock-Goncharov

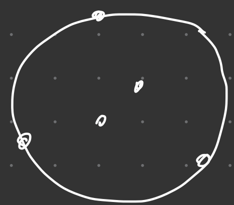
(S,M) is determined by

* g = genus
* b = # of conn. comp's of ∂S
* partition of marked pts on ∂S
* p = # of punctures

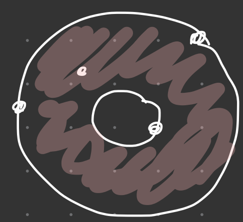
Examples



$b=0$, $g=1$, $p=3$



$b=1$, $g=0$, part (3), $p=2$

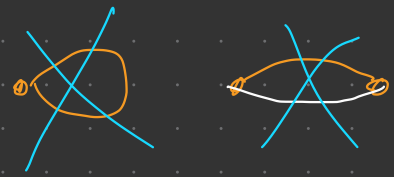


$b=2$, $g=0$, $(2,1)$, $p=1$

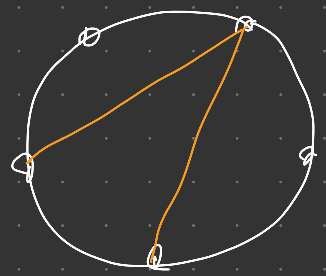
A <u>triangulation</u> of (S,M) is a max^l collection of pairwise non-crossing compatible arcs (together w. all bdy segments)

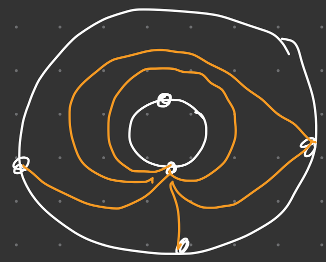
<u>arcs</u> : curves in (S,M) : endpts in M

interior of any arc : disjoint from M from ∂S



<u>Examples</u>





# of arcs in a triangul. of (S,M)

$$n = 6g + 3b + 3p + c$$

c ← $|M| - p$

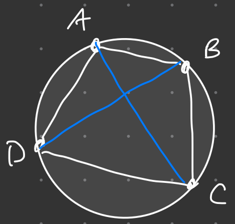
<u>rank</u> of (S,M)

(S, M) a polygon: Cluster algebras / categories (of type A) ← Dynkin type A

* **Cluster algebra** (convex n-gon: type A_{n-3}) Fomin–Zelevinsky 2003

The cluster alg. has cluster variables for each arc & for each bdy segment

$\rightsquigarrow \{p_{ab} \mid 1 \le a < b \le n\}$ Many relations between them; need only $2n-3$

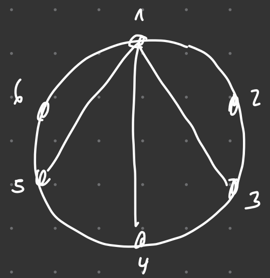


Ptolemy relation:

$$p_{ac} \cdot p_{bd} = p_{ab} \cdot p_{cd} + p_{bc}\, p_{ad}$$

$\rightsquigarrow \mathbb{C}[Gr(2,n)]$

eg



$\{p_{13}, p_{14}, p_{15}\} \cup \{p_{i,i+1} \mid 1 \le i \le 6\}$

"frozen variables" coefficients

Rem.:

Fomin–Shapiro–Thurston 2005: define cluster algebras $\mathcal{A}(S,M)$

* Cluster category (n-gon) Caldero – Chapoton – Schiffler 2005

Define quiver $\Gamma = \Gamma_{n-3}$ (n-gon) [stable translation quiver; Riedtmann]

Γ_0 vertices: $[i,j]$ $1 \leq i < j-1$, $j \leq n$ | Γ_1 arrows: $[i,j] \to [i,j+1]$, $[i,j] \to [i+1,j]$ (if defined)

translation: τ : $[i,j] \longrightarrow [i-1,j-1]$

$\mathcal{C}_{A_{n-3}}$: indec. obj. $M_{[i,j]}$ for $[i,j] \in \Gamma_0$ mesh category of (Γ, τ)

irred. morphisms from arrows of Γ_1

n=6

(Γ_3, τ)

$\tau : \leftarrow$

$$\begin{tikzcd}[column sep=small]
 & & {[1,5]} \arrow[dr] \arrow[rr, dash, dashed] & & {[2,6]} \arrow[dr] \arrow[rr, dash, dashed] & & {[1,3]} \arrow[dr] & & \\
 & {[1,4]} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & {[2,5]} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & {[3,6]} \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & {[1,4]} \arrow[dr] & \\
{[1,3]} \arrow[ur] \arrow[rr, dash, dashed] & & {[2,4]} \arrow[ur] \arrow[rr, dash, dashed] & & {[3,5]} \arrow[ur] \arrow[rr, dash, dashed] & & {[4,6]} \arrow[ur] \arrow[rr, dash, dashed] & & {[1,5]}
\end{tikzcd}$$

Thm $\mathcal{C}_Q := D^b(kQ)/\tau^{-1}[1]$ Buan – Marsh – Reineke – Reiten – Todorov

$Q: \overset{1}{\bullet} \to \bullet \to \bullet \cdots \bullet \to \overset{n-3}{\bullet}$

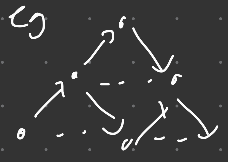
5

Q no oriented cycles

$\mathcal{C}_Q = D^b(\mathbb{C}Q)/\tau^{-1}\circ[1]$ cluster category:

$Q: \overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \circ \cdots \circ \longrightarrow \overset{n-3}{\circ}$

Remarks: • Quiver Γ is the Auslander-Reiten quiver of $\mathcal{C}_Q$. $\mathbb{C}Q$-mod

• $AR(D^b(kQ))$ has shape $\mathbb{Z}\times Q$ (triangular) $\rightsquigarrow$ 

built from copies of $AR(\mathbb{C}Q\text{-mod})$ in each degree

• $[1]$ sends to next copy of $\mathbb{C}Q$-mod

• τ to left

• $AR(\mathcal{C}_Q)$ is  (on Möbius-strip)

Geometric approach:

- irred maps ⟷ min. rotations ($[i,j] \to [i,j+1]$, $[i,j] \to [i+1,j]$)
- max'l rigid obj. ⟷ triangul of (S,M)
- extensions ⟷ crossings
- use triangul. to define a quiver w. relations to get to categories of modules

- Schiffler: cluster categories in type D_n
- Brüstle – Zhang $\mathcal{C}(S,M)$
- B-Marsh * m-cluster categories (via m-angulations) * tubes (annuli)
- B-Buan-Marsh torsion pairs & tubes → ∞ arcs for Prüfer/adic objects

---

- David-Roesler – Schiffler '11 cuts in internal triangles of a triang. surface

- Coelho Simoes – Parsons '16 'tiling algebras': ~~partial triang~~ tilings of disk
- B-Coelho Simoes '18 gentle algebras from tilings of surfaces

---

Quiver: * vertices ⟷ arcs of tiling * arrows ⟷ rotations in tiling

* relations from "inner angles" in tiles

String algebras

- B- Coelho Simoes : from labelled tilings of surfaces.

$Gr(k,n) = \{k\text{-dim}^l \text{ subspaces of } \mathbb{C}^n\}$ $\qquad 1 < k \leq n/2$

<u>Theorem</u> (Fomin-Zelevinsky for $k=2$, Scott for arbitrary k 2005)

The coordinate ring $\mathbb{C}[Gr(k,n)]$ has a cluster algebra structure where every Plücker coordinate (p_I, I a k-subset of $\{1,\dots,n\}$) is a cluster variable.

<u>$k=2$</u> : triangu

There exist clusters consisting of Plücker coordinates

<u>Rem</u> :

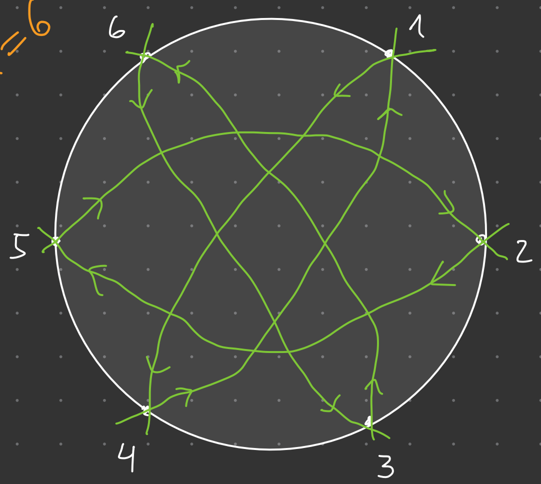
$k=2$ triangulations $\longleftrightarrow$ clusters (n-gon)

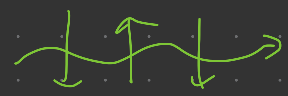
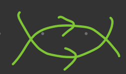
k arbitrary clusters arise from Postnikov's alternating strand diagrams on disk

Strand diagrams

* disk w. n marked pts
* oriented curves between points

$n=6$



* crossings alternate, mult. 2, transversal 
* no lenses  no 

[up to equivalences: ]

gives $\sigma \in S_n$; $\sigma = (14)(25)(36)$ here

* (k,n)-diagrams: strands induce permutation $\boxed{\sigma : i \mapsto i+k}$

Theorem (B King Marsh) (k,n)-diagrams give a combinatorial approach to the Grassmannian cluster categories of Jensen-King-Su

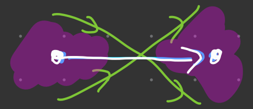
1. Q quiver from (k,n)-diagram
2. algebras from Q
3. diagram: gives a collection of k-subsets of $\{1, \dots, n\}$

① Quiver of strand diagram:

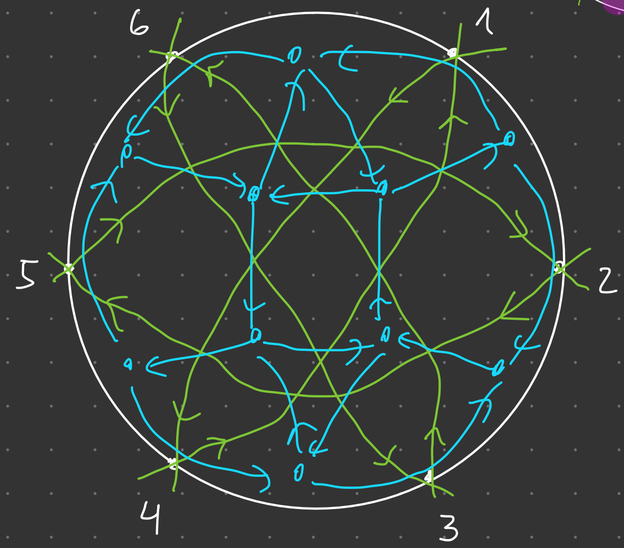
vertices of Q:

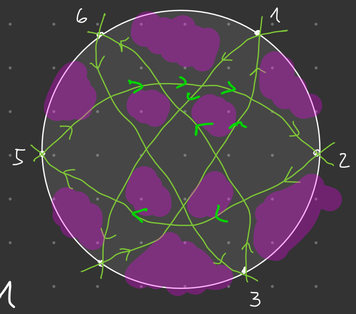
Q_0: alternating regions

Q_1: arrows



Ex.:

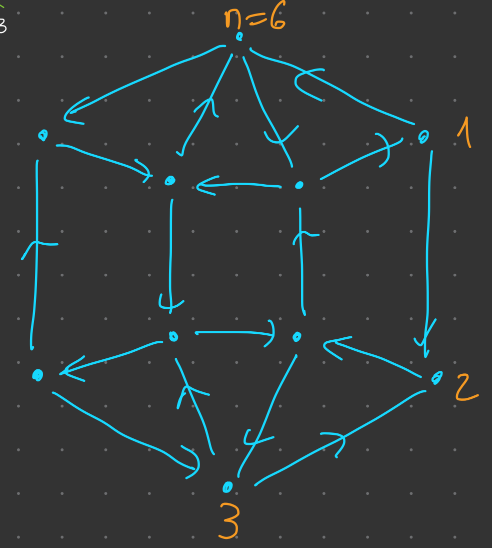




dimer model w. boundary (BKing Marsh)

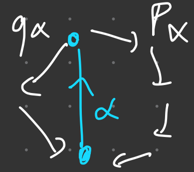
natural potential W

Q $\rightsquigarrow$



② Algebras from strand diagram:

" ∂W: "



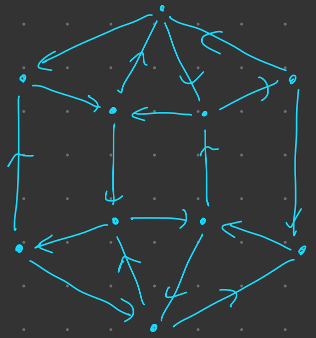
$p_\alpha = q_\alpha$

$A_Q := \mathbb{C}Q/\partial W$ algebra of Q (from a dimer model w. bdy)

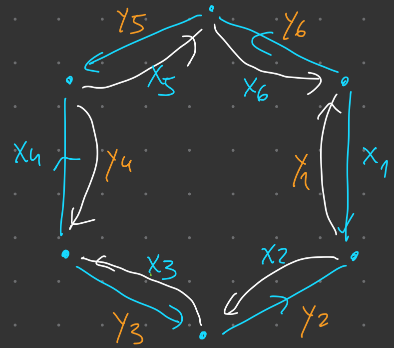
$B_Q := e A_Q e$ $\quad e = \sum$ idemp. of vertices on boundary

<u>boundary</u> algebra of Q

Example Q



For B_Q



Γ_n

2n relations

$$B_Q \cong B := \mathbb{C}\Gamma_n \big/ \left\langle \left\{ \begin{matrix} xy - yx \\ x^k - y^{n-k} \end{matrix} \right\} \right\rangle$$

algebra from Jensen–King–Su's defin. of $\mathcal{F}_{k,n}$

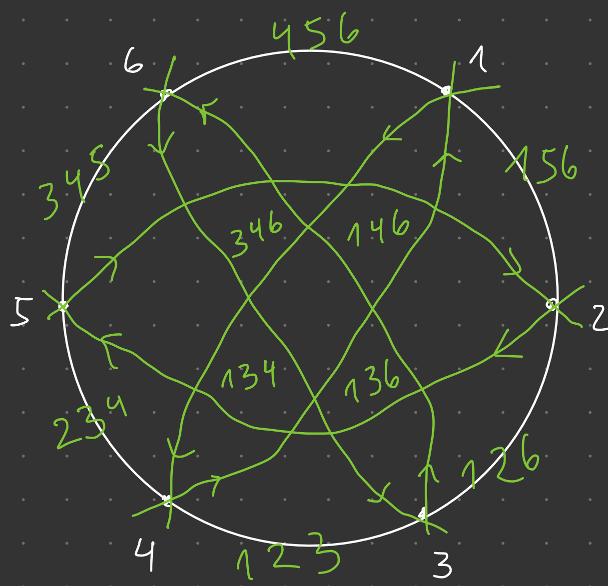
$Z = \mathbb{C}[[t]]$ $\qquad t = \sum_{i=1}^{n} x_i y_i$

③ k-subsets ; cat's of modules

$\mathcal{F}_{k,n} := CM(B) = \{$ B-modules free over $Z(B) \}$

Grassmannian cluster category (Jensen–King–Su)

Plücker coord's ⇝ indec. objects in $\mathcal{F}_{k,n}$



Every (k,n)-diagram P determines a coll. of k-subsets, rank 1-modules (Jensen–King–Su)

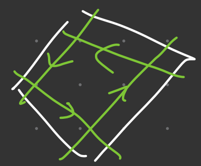
$$M := \bigoplus_{I \in P} M_I \in \mathcal{F}_{k,n} \text{ is cluster-tilting.}$$

• Serhiyenko – ShermanBennet – Williams
• Lam – Gashkin
• Pressland

Questions :

* How to construct strand diagrams for arbitrary permutations of $\{1, \dots, n\}$?

Approaches :



• tilings of disk $\rightsquigarrow$ Scott map B- Martin 2018
(not all permutations)
• tilings w. bicolored tiles Da Costa
(introduces internal pts in disk) Da Costa
• Start with Scott's rectangular arrangements = (k,n)-diagrams ; cut off connected regions
• 'degenerate' Scott's diagrams (similar as tiling approach)

* Strand diagrams for general surfaces : set-up?
associated algebras? Categories? (cf. BKM for $k=2$ (triangul.)

* Connections to root systems, higher rank (k,n) modules ;
characterise $\mathcal{F}_{k,n}$ outside finite / tame types ← (3,9) and (4,8)

eg

* $\mathcal{P}$ Postnikov diagram on disk w. 4 pts

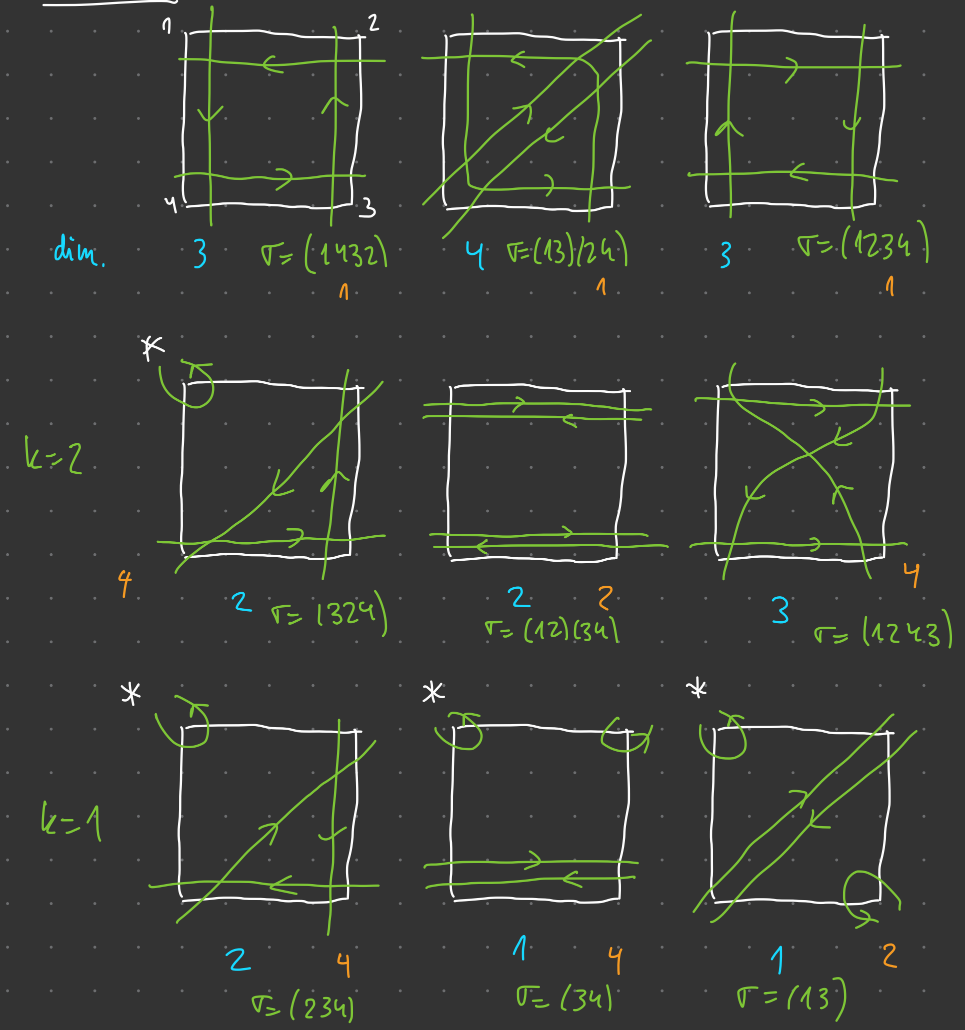
$\rightsquigarrow Q_{\mathcal{P}}$ cluster algebra $\mathcal{A}_{Q_{\mathcal{P}}}$; Pressland gives add. categorif. of it

quiver w. potential

Thm 4.5 $GP(B_{\mathcal{P}})$ is stably 2-CY Frobenius, strand diagr. gives a cl.-tilt. obj.

$$\{ X \in \operatorname{mod} B_{\mathcal{P}} \mid \operatorname{Ext}^i(X,B)=0 \ \forall i>0 \} \text{ “} GP(B_{\mathcal{P}}) \text{”}$$

* $\underline{n=4}$:



from tilings (use approach of B Marsh)

cells ⟷ 'decorated' permutations

* [two orientations of loop]

all others: from bicolored tilings (work in progress by Joel Da Costa, Leeds)

k=0