Characterisations of Σ-pure-injective objects in triangulated categories

2004.06854

Conference in celebration of the work of Bill Crawley-Boevey (Bielefeld/Manchester, 03/09/2021)

Motivation:

– to adapt theorem in [BT, Crawley-Boevey '18], says Σ-pure-injectives in Λ-Mod are sums of string & band modules, Λ a string algebra

1911.07691 – instead replace Λ-Mod with $K(\Lambda\text{-Proj})$, Λ a gentle algebra, required analogues of characterisations of Σ-pure-injective modules

Plan:

- recall ideas from model theory of modules
- change module category to compactly generated triangulated category
- consider canonical many-sorted language and definition of purity
- state main result and talk about proof
- define endoperfect objects and a result in work by Bill

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§1: Model theory of modules

For a ring R let $\mathcal{L}_R = \{0, +, r\times - \mid r \in R\}$, language for R-Mod

Positive-primite (=pp) formulas in $\mathcal{L}_R$ have the form

$$\varphi(x_0, \dots, x_{t-1}) = \exists\, x_t, \dots, x_d : \bigwedge_{i=1}^{k} \sum_{j=0}^{d} r_{ij} x_j = 0 \quad (r_{ij} \in R)$$

These are important — the quantifier elimination result of [Baur, '76] says any formula is equivalent to a Boolean combination of pp-formulas

$\wedge\ \vee\ \neg$

The pp-definable subgroup of M^t given by φ is the subgroup (+)

$$\varphi(M) = \{(m_0, \dots, m_{t-1}) \in M^t : \varphi(m_0, \dots, m_{t-1}) \text{ holds}\}$$

Take $t=1$. The subgroup $\varphi(M) \subseteq M$ is not (in general) an R-submodule, but it is an $\mathrm{End}_R(M)$-submodule.

For an embedding $f: M \hookrightarrow N$ of R-modules to reflect solutions of φ means that, $\forall\, l_1, \dots, l_t \in M$,

$$\exists (n_j) \in N^d : \bigwedge_i \textstyle\sum_j r_{ij} n_j = f(l_i) \implies \exists (m_j) \in M^d : \bigwedge_i \sum_j r_{ij} m_j = l_i$$

A pure embedding is one which reflects the solutions of any pp-formula, and M is pure-injective if any pure embedding $M \hookrightarrow N$ is a section.

[CB, '92, Modules of finite length over their endomorphism rings]

Lemma: if M is pure-injective then any cyclic $\mathrm{End}_R(M)$-submodule of M is an intersection of pp-definable subgroups of M.

We revisit this idea later. Our main result is analogous to the following

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[Gruson, Jensen '76]
[Zimmerman '77]
[Huisgen-Zimmerman '79]

Books by Jensen & Lenzing, or Prest

Explicit proofs of (iii $\Rightarrow$ i) & (iii $\Rightarrow$ iv) also in [CB, '92]

Theorem: for an R-module M, TFAE

(i) M is Σ-pure-injective, i.e. $M^{(I)} = \bigoplus_{i \in I} M$ is pure-injective $\forall I$

(ii) The canonical map $M^{(I)} \to M^I = \prod M$ is a section $\forall I$

(iii) Any descending chain $\varphi_1(M) \supseteq \varphi_2(M) \supseteq \dots$ of pp-definable subgroups of M must stabilise

(iv) M^I is a direct sum of indecomposables with local endomorphism rings, $\forall I$

(v) There is a cardinal κ such that M^I is a direct sum of modules each of which has cardinality at most κ

The equivalence of (i), (ii), (iii) and (iv) was generalised to locally finitely presented additive categories (and much more) in [CB, '94]

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§2: The canonical language for a triangulated category

For the remainder of the talk we use the notation...

- $\mathcal{T}$, triangulated category with small coproducts
- $\mathcal{T}^c$, full subcategory of compact objects
- $\mathrm{Mod}\,\mathcal{T}^c$, category of contravariant additive functors $\mathcal{T}^c \to Ab$
- $Y: \mathcal{T} \to \mathrm{Mod}\,\mathcal{T}^c$, restricted yoneda functor, $M \mapsto \mathcal{T}(-,M)|_{\mathcal{T}^c}$

For an analogue of the theorem from §1, we need appropriate definition of a language $\mathcal{L}_{\mathcal{T}}$. Formulas in $\mathcal{L}_{\mathcal{T}}$ will be sorted by some set. Hence we assume that $\mathcal{T}$ is compactly generated, and that $\mathcal{S}$ is a set of compact objects giving a skeleton of $\mathcal{T}^c$.

[Garkusha, Prest '05] Definition: for each $G \in \mathcal{S}$ introduce:
- a countable set of variables $x_1, x_2, \dots$ of sort $G \in \mathcal{S}^1$
- a predicate/constant 0_G of sort $G \in \mathcal{S}^1$
- a function $+_G$ of sort $(G,G,G) \in \mathcal{S}^3$
- $\forall H \in \mathcal{S}$, $\alpha \in \mathcal{T}(G,H)$ a function $-\circ\alpha$ of sort $(H,G) \in \mathcal{S}^2$
- an equality symbol $\underset{G}{=}$ between terms of sort G

A pp-formula in the $\mathcal{S}$-sorted language $\mathcal{L}_{\mathcal{T}}$ has the form

$$\varphi(x_0,\dots,x_{t-1}) = \exists\, x_t,\dots,x_d : \bigwedge_{i=1}^{k} \psi_i(x_0,\dots,x_d)$$

where each ψ_i is atomic, and the variables x_j have the appropriate sort $G_j \in \mathcal{S}$. We indicate the sort of a variable with subscripts

For example, $\forall F, G, H \in \mathcal{S}$, $\beta \in \mathcal{T}(F,G)$, $\alpha \in \mathcal{T}(G,H)$, letting
$\rho(x_G) = \exists y_H : y_H \circ \alpha \underset{G}{=} x_G$, $\rho'(x_G) = x_G \beta \underset{F}{=} 0_F$

both ρ & ρ' are pp-formulas in one free variable of sort G.

[Garkusha, Prest '05] Proposition: if φ is any pp-formula in one free variable of sort $G \in \mathcal{S}$ then φ is equivalent to $\exists y_H : y_H \circ \alpha \underset{G}{=} x_G$ for some $H \in \mathcal{S}$ and $\alpha \in \mathcal{T}(G,H)$

For example, suppose $\varphi(x_G) = x_G \beta \underset{F}{=} 0_F$ for $\beta \in \mathcal{T}(F,G)$. As $\mathcal{S}$ is a skeleton for the triangulated subcategory $\mathcal{T}^c$, we can complete β to a triangle $F \xrightarrow{\beta} G \xrightarrow{\alpha} H \longrightarrow F[1]$ where $H \in \mathcal{S}$. Here α is a weak cokernel of β inside $\mathcal{T}^c$.

For a ring R any R-module M defines an $\mathcal{L}_R$-structure, and any R-linear map defines a morphism of $\mathcal{L}_R$-structures. So what about $\mathcal{L}_\mathcal{T}$ in our new context?

Any object M of $\mathcal{T}$ defines an $\mathcal{L}_\mathcal{T}$-structure $\bar{M}$:
- the underlying set of $\bar{M}$ is $\bigsqcup_{G \in \mathcal{S}} \mathcal{T}(G,M)$
- the constant symbols are interpreted as follows;

$$0_G \rightsquigarrow \text{zero morphism } 0 \in \mathcal{T}(G,M)$$
$$+_G \rightsquigarrow \text{map } \mathcal{T}(G,M) \times \mathcal{T}(G,M) \xrightarrow{+} \mathcal{T}(G,M)$$
$$-\circ\alpha \longrightarrow \text{map } \mathcal{T}(H,M) \xrightarrow{-\circ\alpha} \mathcal{T}(G,M)$$

For $\mathcal{L}_\mathcal{T}$-structures X, Y with underlying sets $\bigsqcup_G X_G$, $\bigsqcup_G Y_G$ a morphism $X \to Y$ of $\mathcal{L}_\mathcal{T}$-structures is given by maps $X_G \to Y_G$ which are (in some sense) compatible with 0_G, $+_G$ & $-\circ\alpha$.

Fact: any morphism $\theta \in \mathcal{T}(M,N)$ defines a morphism $\bar{M} \xrightarrow{\bar{\theta}} \bar{N}$ of $\mathcal{L}_{\mathcal{T}}$-structures

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[Garkusha, Prest '05] Definition: let $G \in \mathcal{S}$ and M be an object in $\mathcal{T}$. A pp-definable subgroup of M of sort G is a subgroup of $\mathcal{T}(G,M)$ of the form $M\alpha := \{\gamma\alpha \mid \gamma \in \mathcal{T}(H,M)\}$ $(\alpha : G \to H \in \mathcal{S})$

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§3: Purity and the restricted yoneda functor

Recall $Y : \mathcal{T} \to \mathrm{Mod}\,\mathcal{T}^c$ is defined by $Y(M) = \mathcal{T}(-,M)$ considered as a functor $\mathcal{T}^c \to \mathrm{Ab}$. Since $\mathcal{S}$ is a skeleton, morphisms $Y(M) \to Y(N)$ correspond to morphisms $\bar{M} \to \bar{N}$ of $\mathcal{L}_{\mathcal{T}}$-structures

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[Krause '00] Definition: a morphism $\theta \in \mathcal{T}(M,N)$ is a pure-monomorphism provided, for each object G in $\mathcal{T}^c$, the map

$$Y(\theta)_G : \mathcal{T}(G,M) \to \mathcal{T}(G,N), \quad \sigma \mapsto \theta\sigma$$

is injective.

Warning: pure-monomorphisms are not categorical monomorphisms.

Remark: after unpacking definitions, it is straightforward to show that $\theta \in \mathcal{T}(M,N)$ is a pure-monomorphism if and only if the morphism $\bar{\theta} : \bar{M} \to \bar{N}$ of $\mathcal{L}_{\mathcal{T}}$-structures reflects solutions to pp-formulas

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[Krause, '00] Definition: an object M in $\mathcal{T}$ is pure-injective provided any pure-monomorphism $M \to N$ is a section.

[Harada, '73] Definition: Let $\mathcal{A}$ be a Grothendieck category with a set of finitely generated generators. Fix objects B, C in $\mathcal{A}$. By a *C-annihilator of B* we mean a subobject A of B, of the form

$$A = \bigcap_{\lambda \in X} \operatorname{Ker}(\lambda) \quad \text{for some subset } X \subseteq \mathcal{A}(B, C).$$

Remark: The category $\mathcal{A} = \operatorname{Mod} \mathcal{T}^c$ is locally coherent Grothendieck: one may take $\{Y(G) : G \in \mathcal{S}\}$ as a set of finitely presented generators, and the full subcategory $\operatorname{mod} \mathcal{T}^c$ of finitely presented functors is abelian. Hence we can consider Haradas definition here.

[Garcia, Dung '94] Theorem: for an object M of $\mathcal{T}$, TFAE:

[Krause, '00]

[Krause, Reichenbach '00]

[BT, '20]

+ all citations for theorem in {1...

(i) M is *Σ-pure-injective*, i.e. $M^{(I)} = \bigoplus_{i \in I} M$ is pure-injective $\forall I$

(ii) The countable coproduct $M^{(\mathbb{N})}$ is pure-injective

(iii) Any ascending chain of $Y(M)$-annihilators of $Y(G)$ must stabilise, $\forall G \in \mathcal{S}$

(iv) The canonical map $M^{(I)} \to M^I = \prod M$ is a section $\forall I$

(v) Any descending chain of pp-definable subgroups of M of sort G must stabilise, $\forall G \in \mathcal{S}$

(vi) M^I is a direct sum of indecomposables with local endomorphism rings, $\forall I$

The equivalence of parts (i), (ii), (iii) & (iv) follow from the two theorems below...

[Krause, '00]

Theorem: for an object M of $\mathcal{T}$, TFAE

(i) M is pure-injective
(ii) $Y(M)$ is an injective object in $\mathrm{Mod}\,\mathcal{T}^c$
(iii) $\forall I$ there is a morphism $M^I \to M$ such that

top and left arrows defined by universal properties

$$\begin{tikzcd} M^{(I)} \arrow[r] \arrow[dr] & M \\ & M^I \arrow[u] \end{tikzcd}$$

commutes

[Garcia, Dung '94]

c.f [Faith '66] for modules

Theorem: for an injective object A in a Grothendieck category $\mathcal{A}$ with a set X of finitely generated generators, TFAE:

(i) A is Σ-injective, i.e. $A^{(I)}$ is injective $\forall I$.
(ii) The countable coproduct $A^{(\mathbb{N})}$ is injective.
(iii) $\forall B \in X$, any ascending chain of A-annihilator subobjects of B must stabilise

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§4: A proof, and endocoperfect objects

The following was inspired by an argument of [Huisgen-Zimmermann '98], used to give a short proof of Faith's theorem (above when $\mathcal{A} = R\text{-Mod}$).

Lemma: Fix $G \in \mathcal{S}$ and an object M of $\mathcal{T}$. Then any strictly ascending chain of $Y(M)$-annihilator subobjects of $Y(G)$ gives a strictly descending chain of pp-definable subgroups of M of sort G.

Proof: any such ascending chain $P_1 \subsetneq P_2 \subsetneq \dots$ gives a strictly descending chain of subsets

$$\mathrm{Hom}_{\mathrm{Mod}\,\mathcal{T}^c}(Y(G), Y(M)) \supseteq \mathcal{K}(1) \supsetneq \mathcal{K}(2) \supsetneq \dots$$

where $P_i = \bigcap_{\lambda \in \mathcal{K}(i)} \mathrm{Ker}(\lambda)$.

By assumption, $\forall n > 0$ there is an object $F_n \in \mathcal{S}$ and a morphism $\beta_n : F_n \to G$ lying in $P_{n+1}(F_n) \setminus P_n(F_n)$

Hence $\lambda_G(1_G)\beta_n = 0$ for all $\lambda \in \mathcal{K}(n+1)$.
Similarly $\mu^n_G(1_G)\beta_n \neq 0$ for some $\mu^n \in \mathcal{K}(n)$.

Now complete β_n to a triangle

$$F_n \xrightarrow{\beta_n} G \xrightarrow{\alpha_n} H_n \longrightarrow F_n[1]$$

Let $\gamma_n = \mu^n_G(1_G)$, so that $\gamma_{n+1}\beta_n = 0 \neq \gamma_n \beta_n$.

As in the Proposition above from [Garkusha, Prest '05], this means γ_{n+1} factors through α_n, and γ_n does not factor through α_n. Thus

$$\gamma_1 \notin M\alpha_1 \ni \gamma_2 \notin M\alpha_2 \ni \gamma_3 \notin \dots$$

which gives a strict descending chain

$$M\alpha_1 \supsetneq M\alpha_1 \cap M\alpha_2 \supsetneq \dots$$

Again, using the Proposition, each $\bigcap_{i=1}^n M\alpha_i$ is a pp-definable subgroup of sort G $\square$

Definition: we say an object M of $\mathcal{T}$ is endocoperfect if, for all $G \in S$, any descending chain of cyclic $\mathrm{End}_{\mathcal{T}}(M)$-submodules of $\mathcal{T}(G,M)$ must stabilise.

Remark: this is equivalent to saying that $\mathcal{T}(G,M)$ is perfect as a module over $\mathrm{End}_{\mathcal{T}}(M)$, in the sense of [Björk, '69], in which it is shown that any such module has the descending chain condition on finitely generated submodules

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Corollary: endonoetherian endocoperfect objects in $\mathcal{T}$ must be Σ-pure injective

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Proof: any pp-definable subgroup of M of sort G is also an $\mathrm{End}_{\mathcal{T}}(M)$-submodule, and as M is endonoetherian, finitely generated as an $\mathrm{End}_{\mathcal{T}}(M)$-module. By the remark any descending chain of such modules must stabilise, hence M is Σ-pure-injective.

Lemma: any Σ-pure-injective object is endocoperfect.

Proof: adapting the first lemma we saw, from [CB, '92], one can show that any cyclic $\mathrm{End}_{\mathcal{T}}(M)$ submodule of $\mathcal{T}(G,M)$ is an intersection of pp-definable subgroups of sort G, provided M is pure-injective. Assuming M is Σ-pure injective, these intersections must be finite $\square$

Thank you for your attention, and
happy birthday Bill!!