Tame non-commutative nodal curves and related finite dimensional algebras

Igor Burban

University of Paderborn, Germany

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Theorem (Geigle & Lenzing, 1985)

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where T_λ is the tubular algebra of Ringel of type $(2, 2, 2, 2; \lambda)$

$$\begin{tikzcd}
 & & & \circ \arrow[dlll, "a_1"'] \arrow[dl, "a_2"] \arrow[dr, "a_3"'] \arrow[drrr, "a_4"] & & & \\
\circ \arrow[drrr, "b_1"'] & & \circ \arrow[dr, "b_2"] & & \circ \arrow[dl, "b_3"'] & & \circ \arrow[dlll, "b_4"] \\
 & & & \circ & & &
\end{tikzcd}$$

$$\begin{aligned} b_1a_1 - b_2a_2 &= b_3a_3 \\ b_1a_1 - \lambda b_2a_2 &= b_4a_4 \end{aligned}$$

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we have: gl.dim$(T - \mathrm{mod}) = 2$.

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- Hence, $D^b\big(\mathsf{Coh}^{\mathbb{Z}_2}(E)\big)$ and $D^b(T-\mathrm{mod})$ *are not equivalent*.

Question: What is a relation between $D^b\big(\mathsf{Coh}^{\mathbb{Z}_2}(E)\big)$ and $D^b(T-\mathrm{mod})$?

Degenerate tubular algebra II

Proposition (Burban & Drozd, 2018)

There exists a commutative diagram of categories and functors

$$\begin{tikzcd} D^b(T - \mathsf{mod}) \arrow[rr, two heads, "\mathsf{P}"] & & D^b\big(\mathsf{Coh}^{\mathbb{Z}_2}(E)\big) \\ & \mathsf{Perf}^{\mathbb{Z}_2}(E) \arrow[ul, hook, "\mathsf{J}"] \arrow[ur, hook', "\mathsf{I}"'] & \end{tikzcd}$$

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First step of the proof. As $E/\mathbb{Z}_2 \cong \mathbb{P}^1$, we have: $\mathsf{Coh}^{\mathbb{Z}_2}(E) \simeq \mathsf{Coh}(\mathbb{E})$, where $\mathbb{E} := (\mathbb{P}^1, \mathcal{A})$ and $\mathcal{A} \subset Mat_2(\mathcal{O}_{\mathbb{P}^1})$ is a certain *sheaf of orders*.

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- $\widetilde{D} \otimes (D/J) \cong \widetilde{D}/J \cong \Bbbk \times \Bbbk$.

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- We have: $$\begin{tikzcd} A/J \arrow[r, "\cong"] \arrow[d, hook] & \Bbbk \times \Bbbk \arrow[d, hook] \\ H/J \arrow[r, "\cong"] & \mathrm{Mat}_2(\Bbbk) \end{tikzcd}$$

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Remark

In fact, $A \cong D * \mathbb{Z}_2$, where $D = \mathbb{k}[\![x, y]\!]/(xy)$ and $\mathbb{Z}_2$ acts on D by $x \leftrightarrow y$.

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For any $0 < c < n$ such that $\gcd(n, c) = 1$, consider the action of the cyclic group $G = \langle \rho \,|\, \rho^n = e \rangle$ on $D = \mathbb{k}[\![u, v]\!]/(uv)$, given by the rule

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Then $A := D * G$ is nodal. For example, for $c = n - 1$ we have:

$$A \cong \begin{tikzcd}
 & \overset{2}{\circ} \arrow[dl, bend right, "a_1"'] \arrow[r, bend right, "b_2"'] & \circ \arrow[l, bend right, "a_2"'] \arrow[dr, bend left] & \\
\overset{1}{\circ} \arrow[ur, bend right, "b_1"'] \arrow[dr, bend right, "a_n"'] & & \vdots & \circ \arrow[ul, bend left] \arrow[dl, bend left] \\
 & \overset{n}{\circ} \arrow[ul, bend right, "b_n"'] \arrow[r, bend right] & \circ \arrow[l, bend right] \arrow[ur, bend left] &
\end{tikzcd}$$

modulo the relations $a_k b_k = 0 = b_k a_k$ for all $1 \leq k \leq n$.

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A is isomorphic to the arrow ideal completion of the path algebra

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Example (Zhelobenko order, Zhelobenko 1958)

$$H := R \times \begin{pmatrix} R & \mathfrak{m} \\ R & R \end{pmatrix} \supset A := \left\{ \left(a, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \middle| \, a(0) = b_{11}(0) \right\}$$

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A is the completion of $\begin{tikzcd} \bullet \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \bullet \arrow[l, bend left, "c"] \end{tikzcd}$ $\qquad ba = 0,\ ac = 0.$

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Example (Gelfand 1970, representations of $\mathrm{SL}_2(\mathbb{R})$)

$$A := \begin{pmatrix} R & \mathfrak{m} & \mathfrak{m} \\ R & R & \mathfrak{m} \\ R & \mathfrak{m} & R \end{pmatrix} \subset H = \begin{pmatrix} R & \mathfrak{m} & \mathfrak{m} \\ R & R & R \\ R & R & R \end{pmatrix} \overset{M}{\sim} \begin{pmatrix} R & \mathfrak{m} \\ R & R \end{pmatrix}.$$

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Example (Zhelobenko 1958, Gelfand & Ponomarev 1968, on $\mathrm{SL}_2(\mathbb{C})$)

$$H := R \times \begin{pmatrix} R & \mathfrak{m} \\ R & R \end{pmatrix} \supset A := \left\{ \left(a, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right) \middle| \, a(0) = b_{11}(0) \right\}$$

A is the completion of $\begin{tikzcd} \bullet \arrow[loop left, "a"] \arrow[r, bend left, "b"] & \bullet \arrow[l, bend left, "c"] \end{tikzcd}$ $\qquad ba = 0, \; ac = 0.$

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- For $\mathbb{k} \neq \overline{\mathbb{k}}$, the picture becomes more involved as $\mathrm{Br}\big(\mathbb{k}((t))\big) \neq 0$.

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$$\begin{tikzcd}
\mathsf{Perf}(A) \arrow[d, hook] \arrow[dr, hook, "P \overset{\mathbb{L}}{\otimes}_A -"] & \\
D^b(A-\mathsf{mod}) & D^b(B-\mathsf{mod}) \arrow[l, two heads, "\mathrm{Hom}_B(P, -)"]
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Then we have a recollement diagram (after Parshall, Cline and Scott):

$$\begin{tikzcd}[column sep=huge] D^b(\bar{A} - \mathsf{mod}) \arrow[r, "\mathsf{J}" description] & D^b(B - \mathsf{mod}) \arrow[l, bend right, "\mathsf{J}^*"'] \arrow[l, bend left, "\mathsf{J}^!"] \arrow[r, "\mathsf{DG}" description] & D^b(H - \mathsf{mod}) \arrow[l, bend right, "\mathsf{LF}"'] \arrow[l, bend left, "\mathsf{RH}"] \end{tikzcd}$$

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where $\mathsf{G} = \mathrm{Hom}_B(Q, -)$, $\mathsf{F} = Q \otimes_H -$, $\mathsf{H} = \mathrm{Hom}_H(Q^\vee, -)$ and

$$0 \longrightarrow \begin{pmatrix} C \\ C \end{pmatrix} \longrightarrow \begin{pmatrix} A \\ C \end{pmatrix} \longrightarrow \mathsf{J}(\bar{A}) \longrightarrow 0.$$

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- Global version of nodal orders: non-commutative nodal projective curves.

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$$\Gamma\big(X, \mathcal{K} \otimes_{\mathcal{O}} \mathcal{A}\big) \cong \mathrm{Mat}_n\big(\mathbb{k}(X)\big)$$

for some $n \in \mathbb{N}$.

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Observation. If $\mathbb{X}$ is projective and rational then $D^b\big(\mathrm{Coh}(\widetilde{\mathbb{X}})\big)$ and $D^b\big(\mathrm{Coh}(\mathbb{Y})\big)$ have tilting objects!

Geigle-Lenzing tilting for weighted projective lines

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Fact: A rational hereditary curve $\widetilde{\mathbb{X}} = (\mathbb{P}^1, \mathcal{H})$ (a weighted projective line) is specified by its ramification points $(\lambda_1 : \mu_1), \dots, (\lambda_r : \mu_r) \in \mathbb{P}^1$ and the corresponding weights $l_1, \dots, l_r \in \mathbb{N}$.

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Theorem (Geigle & Lenzing, 1985)

$D^b\big(\mathsf{Coh}(\widetilde{\mathbb{X}})\big) \simeq D^b(\Gamma - \mathsf{mod})$, *where* Γ *is the path algebra of*

$$
\begin{tikzcd}
& \circ \arrow[r, "u_{12}"] & \circ \arrow[r] & \dots \arrow[r] & \circ \arrow[ddr, "u_{1l_1}"] & \\
& \circ \arrow[r, "u_{22}"] & \circ \arrow[r] & \dots \arrow[r] & \circ \arrow[dr, "u_{2l_2}"'] & \\
\circ \arrow[uur, "u_{11}"] \arrow[ur, "u_{21}"'] \arrow[rrrrr, bend left, "z"] \arrow[rrrrr, bend right, "w"'] \arrow[ddr, "u_{r1}"'] & & & & & \circ \\
& \vdots & \vdots & & \vdots & \\
& \circ \arrow[r, "u_{r2}"] & \circ \arrow[r] & \dots \arrow[r] & \circ \arrow[uur, "u_{rl_r}"'] &
\end{tikzcd}
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modulo the relations $u_{il_i} \dots u_{i1} = \lambda_i w - \mu_i z$ *for* $1 \le i \le r$.

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Theorem (Burban & Drozd, 2018)

Let $\mathcal{E}$ be the GL tilting bundle on $\widetilde{\mathbb{X}} = (\widetilde{X}, \mathcal{H})$

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Let $\mathcal{E}$ be the GL tilting bundle on $\widetilde{\mathbb{X}} = (\widetilde{X}, \mathcal{H})$ and $\mathcal{S} \in \mathsf{Tor}(\mathbb{X})$ be given by

$$0 \longrightarrow \begin{pmatrix} \mathcal{C} \\ \mathcal{C} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{A} \\ \mathcal{C} \end{pmatrix} \longrightarrow \mathcal{S} \longrightarrow 0.$$

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Fact. Let Γ be a finite dimensional $\mathbb{k}$-algebra for which there exists a non-commutative projective curve $\mathbb{T}$ such that

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If Γ is **derived tame** then $\mathbb{T}$ is a **tame** projective rational **nodal** curve.

Combinatorics of tame non-commutative nodal curves

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Let $r, s \in \mathbb{N}_0$ and

$$\vec{p} = \bigl((p_1^+, p_1^-), \dots, (p_r^+, p_r^-)\bigr) \in \bigl(\mathbb{N}^2\bigr)^r \quad \text{and} \quad \vec{q} = (q_1, \dots, q_s) \in \mathbb{N}^s.$$

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$$\Xi_i^\pm := \bigl\{x_{i,1}^\pm, \dots, x_{i,p_i^\pm}^\pm\bigr\} \quad \text{and} \quad \Xi_j^\circ := \bigl\{w_{j,1}, \dots, w_{j,q_j}\bigr\}.$$

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Theorem (Drozd-Voloshyn 2013, Burban-Drozd 2021)

Up to a Morita equivalence, tame non-commutative nodal curves are parameterized by the data $\big(\Xi(\vec{p}, \vec{q}), \approx\big)$.

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Remark

Stacky chains/cycles of projective lines of are special classes of non-commutative tame nodal curves.

Tame nodal curves and derived-tame algebras

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Let $(\vec{p}, \vec{q}, \approx)$ be a datum defining a tame non-commutative nodal curve.

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Coming next: explicit description of derived-tame algebras $\Lambda(\vec{p}, \vec{q}, \approx)$.

Let $\vec{p} = \big((p_1^+, p_1^-), \dots, (p_r^+, p_r^-)\big) \in \big(\mathbb{N}^2\big)^r$ and $\vec{q} = (q_1, \dots, q_s) \in \mathbb{N}^s$.

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$$
\Gamma\big(p_i^+, p_i^-\big) =
\begin{tikzcd}
 & \circ \arrow[r, "x^+_{i,2}"] & \circ & \ldots & \circ \arrow[r] & \circ \arrow[dr, "x^+_{i,p_i^+}"] & \\
\circ \arrow[ur, "x^+_{i,1}"] \arrow[d, "x^-_{i,1}"'] & & & & & & \circ \\
\circ \arrow[r, "x^-_{i,2}"'] & \circ \arrow[r] & \circ & \ldots & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[u, "x^-_{i,p_i^-}"']
\end{tikzcd}
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\circ \arrow[r, "x^-_{i,2}"'] & \circ \arrow[r] & \circ & \ldots & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[u, "x^-_{i,p_i^-}"']
\end{tikzcd}$$

$$\Gamma\left(2, 2, q_j\right) = \begin{tikzcd}
 & & & \circ \arrow[drrr, bend left, "v_j^+"] & & & \\
\circ \arrow[urrr, bend left, "u_j^+"] \arrow[drrr, bend right, "u_j^-"'] \arrow[r, "w_{j,1}"] & \circ \arrow[r, "w_{j,2}"] & {} & \ldots & \circ \arrow[r] & \circ \arrow[r, "w_{j,q_j}"] & \circ \\
 & & & \circ \arrow[urrr, bend right, "v_j^-"'] & & &
\end{tikzcd}$$

modulo the relation $v_j^+ u_j^+ + v_j^- u_j^- + w_{j,q_j} \ldots w_{j,1} = 0$.

Derived tame algebras $\Lambda(\vec{p}, \vec{q}, \approx)$

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$\approx$ is a symmetric (but not necessarily reflexive) relation on the set

$$\Xi := \left\{ x_{i,k}^{\pm} \middle| \; 1 \le i \le r, 1 \le k \le p_i^{\pm} \; \right\} \cup \left\{ w_{j,l}^{\pm} \middle| \; 1 \le j \le s, 1 \le l \le q_j \; \right\}.$$

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The algebra $\Lambda(\vec{p}, \vec{q}, \approx)$ is obtained from

$$\big(\Gamma\big(p_1^+, p_1^-\big) \times \cdots \times \Gamma\big(p_r^+, p_r^-\big)\big) \times \big(\Gamma\big(2, 2, q_1\big) \times \cdots \times \Gamma\big(2, 2, q_s\big)\big)$$

by the following procedure of adding new vertices and arrows.

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by the following procedure of adding new vertices and arrows.
Case 1. Let $\varrho' \ne \varrho'' \in \Xi$ be such that $\varrho' \approx \varrho''$

$$\circ \xrightarrow{\;\;\varrho'\;\;} \circ$$

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Case 1. Let $\varrho' \ne \varrho'' \in \Xi$ be such that $\varrho' \approx \varrho''$

$$\begin{tikzcd}
\circ \arrow[r, "\varrho'"] & \circ \arrow[dr, "\vartheta'"] & \\
 & & \bullet \\
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\qquad\qquad \vartheta' \varrho' = 0 = \vartheta'' \varrho''$$

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$$\begin{tikzcd} & & \bullet \\ \circ \arrow[r, "\varrho"] & \circ \arrow[ur, "\vartheta_+"] \arrow[dr, "\vartheta_-"'] & \\ & & \bullet \end{tikzcd} \qquad \vartheta_{\pm}\varrho = 0$$

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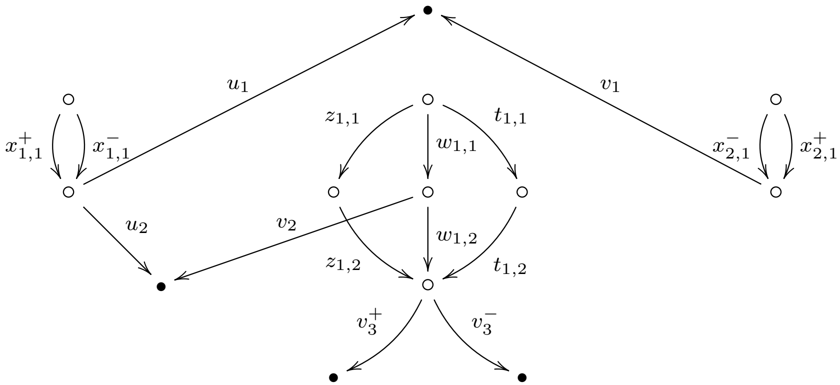
Example. Let $\vec{p} = (3, 2)$, $\vec{q}$ be void (i.e. $r = 1$ and $s = 0$) and $\approx$ be given by the rule $x^+_{1,1} \approx x^-_{1,1}$ and $x^+_{1,3} \approx x^-_{1,2}$. Then the corresponding gentle algebra $\Lambda(\vec{p}, \approx)$ is the path algebra of the following quiver

$$\begin{tikzcd}
 & \circ \arrow[rr, "x^+_{1,2}"] \arrow[dr, "u_1"] & & \circ \arrow[dr, "x^+_{1,3}"] & & \\
\circ \arrow[ur, "x^+_{1,1}"] \arrow[drr, "x^-_{1,1}"'] & & \bullet & & \circ \arrow[r, bend left, "u_2"] \arrow[r, bend right, "v_2"'] & \bullet \\
 & & \circ \arrow[u, "v_1"'] \arrow[urr, "x^-_{1,2}"'] & & &
\end{tikzcd}$$

subject to the relations: $u_1 x^+_{1,1} = 0 = v_1 x^-_{1,1}$ and $u_2 x^+_{1,3} = 0 = v_2 x^-_{1,2}$.

Example. Let $\vec{p} = \big((1,1),(1,1)\big)$, $\vec{q} = (2)$ and $\approx$ be given by: $x_{1,1}^{+} \approx w_{1,1}$, $x_{1,1}^{-} \approx x_{2,1}^{+}$ and $w_{1,2} \approx w_{1,2}$. Then $\Lambda(\vec{p}, \vec{q}, \approx)$ is the path algebra



$$\begin{cases} z_{1,2}z_{1,1} + w_{1,2}w_{1,1} + t_{1,2}t_{1,1} = 0 \\ v_3^{\pm} w_{1,2} = 0 \\ u_1 x_{1,1}^{-} = 0 = v_1 x_{2,1}^{+} \\ u_2 x_{1,1}^{+} = 0 = v_2 w_{1,1} \end{cases}$$

For any $n \in \mathbb{N}$, let Υ_n be the path algebra of the following quiver

$$
\begin{tikzcd}
\bullet \arrow[d, bend right, "a_1^+"'] \arrow[d, bend left, "a_1^-"] & \bullet \arrow[d, bend right, "a_2^+"'] \arrow[d, bend left, "a_2^-"] & & \bullet \arrow[d, bend right, "a_n^+"'] \arrow[d, bend left, "a_n^-"] \\
\bullet \arrow[d, "b_1^+"'] \arrow[dr, "b_1^-"] & \bullet \arrow[d, "b_2^+"'] \arrow[dr, "b_2^-"] & \cdots & \bullet \arrow[d, "b_n^+"] \arrow[dlll, bend left=10, "b_n^-"] \\
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\bullet & \bullet & \cdots & \bullet
\end{tikzcd}
$$

modulo the relations

$$b_i^{\pm} a_i^{\mp} = 0, c_i^- b_i^+ = 0 \quad \text{and} \quad c_{i+1}^+ b_i^- = 0 \quad \text{for} \quad 1 \le i \le n.$$

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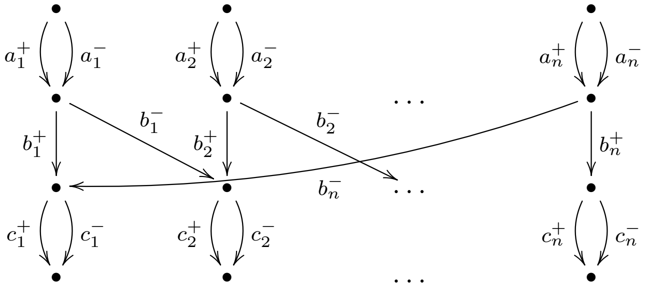

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Since $\mathsf{HH}^3(\Upsilon_n) \neq 0$, the algebra Υ_n can not be derived equivalent to any algebra of the form $\Lambda(\vec{p}, \vec{q}, \approx)$.
On the other hand, $D^b(\Upsilon_n - \mathsf{mod})$ is equivalent to the derived category of coherent sheaves an an appropriate non-commutative projective tame nodal curve, whose central curve is a cycle of $2n$ projective lines.

Derived tame algebras and non-commutative nodal curves

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- Tame non-commutative nodal curves recently appeared in a work of Polishchuk on trigonometric solutions of the associative Yang–Baxter equation.

Happy Birthday, Bill!