Chern classes of quantizable sheaves

V. Baranovsky & V. G.

X smooth quasi-projective variety $/\mathbb{C}$

$S \subseteq X$ closed subvariety

$$H^i_S(X) := H^i_{dR}(X, X \setminus S)$$

de Rham cohomology w/ supp S

$$H^i_S(X) = 0 \qquad i < 2\,\mathrm{codim}\, S$$

$\mathcal{F}$ coherent sheaf on X s.t.

$\mathrm{supp}\,\mathcal{F} \subseteq S \rightsquigarrow$ Chern character with support

$$ch(\mathcal{F}) = \sum_i ch_i(\mathcal{F})$$

$$ch_i(\mathcal{F}) \in H^{2i}_S(X)$$

Always assume $\dim(\mathrm{supp}\,\mathcal{F}) = \dim S$

$\{S_j\}$ irred. components of S of max dim

$$[\operatorname{Supp}\mathcal{F}] = \sum_j \operatorname{mult}(\mathcal{F}; S_j)\cdot[S_j]$$

support cycle of $\mathcal{F}$

$$\boxed{\operatorname{ch}_{\operatorname{codim} S}(\mathcal{F}) \frown [S] = [\operatorname{Supp}\mathcal{F}]}$$

Poincaré duality

Now X algebraic symplectic

$\mathcal{O}_\hbar$ deformation quantization of $\mathcal{O}_X$:= structure sheaf:

- $\mathcal{O}_\hbar$ $\hbar$-adically complete flat $\mathbb{C}[[\hbar]]$-algebra
- $\mathcal{O}_\hbar / \hbar\cdot\mathcal{O}_\hbar = \mathcal{O}_X$

Quantizations are parametrized by Deligne-Fedosov-Kontsevich class:

$c(\mathcal{O}_\hbar) \in H^2_{dR}(X)[[\hbar]]$

Let $\mathcal{M}$ be a coherent $\mathcal{O}_\hbar$-module

$\rightsquigarrow \mathcal{M}/\hbar\mathcal{M}$ coherent $\mathcal{O}_X$-module

$SS(\mathcal{M}) := \mathrm{supp}(\mathcal{M}/\hbar\mathcal{M})$ singular support of $\mathcal{M}$

Integrability of characteristics theorem (Gabber)

Every irreducible component S_j of $SS(\mathcal{M})$ is coisotropic, so

$$\boxed{\dim S_j \geq \frac{1}{2}\dim X}$$

Extreme case: $SS(\mathcal{M})$ is Lagrangian

$$\Longleftrightarrow \quad \dim SS(\mathcal{M}) = \frac{1}{2}\dim X$$

Define

$$\boxed{\tau(\mathcal{M}) := ch(\mathcal{M}/\hbar\,\mathcal{M}) \cdot e^{c(\mathcal{O}_\hbar)} \cdot \hat{A}(X)}$$

$\hat{A}$ Hirzebruch hat-genus
($\sim$ Todd class of X)

$$\tau(\mathcal{M}) = \sum_i \tau_i(\mathcal{M})$$

$$\tau_i(\mathcal{M}) \in H^{2i}_S(X)[\hbar]$$

S contains $\mathrm{supp}(\mathcal{M}/\hbar\,\mathcal{M})$ as above.

Vanishing thm (Baranovsky - G)

$\tau_i(\mathcal{M}) = 0$ unless $\boxed{\mathrm{codim}\, S \leq i \leq \frac{1}{2}\dim X}$

Remarks

- Gabber $\Rightarrow$ $\mathrm{codim}\, S \leq \frac{1}{2}\dim X$
- A priori possible

$\tau_i(\mathcal{M}) \neq 0$ also for $\frac{1}{2}\dim X \leq i \leq \dim X$

Corollary (Lagrangian case)

If $SS(\mathcal{M})$ is Lagrangian then

- $\tau_i(\mathcal{M}) = 0 \quad \forall\, i \neq d := \frac{1}{2}\dim X$
- $\tau_d(\mathcal{M}) \cap [S] = ch_d(\mathcal{M}/\hbar\mathcal{M}) \cap [S]$

$$= [\mathrm{supp}(\mathcal{M}/\hbar\mathcal{M})]$$

Heuristic motivation:

Assume $S \subset X$ smooth Lagrangian

$\mathcal{F}$ line bundle on S. Then:

- Quantization of $\mathcal{F}$ mod $(\hbar^2)$

  $\rightsquigarrow$ connection on $\mathcal{F} \otimes K_S^{-1/2}$

- $\exists$ Quantization of $\mathcal{F}$

$\mod (\hbar^3) \Rightarrow$ connection is flat

$\Rightarrow$ chern classes vanish

Proofs: use cyclic homology

$K = \mathbb{C}((\hbar))$, X smooth quasiprojective

$A := K \otimes_{\mathbb{C}[[\hbar]]} \mathcal{O}_\hbar$ (invert $\hbar$)

$=$

A sheaf of K-algebras on X

Define:

$HC^-_S(A) := HC^-_{S,0}(X, A)$

degree zero negative cyclic homology of A/K with support in S.

Mimic Keller to show:

$$\boxed{R\Gamma_S\left(X, \begin{matrix}\text{sheafified}\\ \text{mixed cmx}\\ \text{of } A\end{matrix}\right) \simeq HC^-(\mathrm{Perf}_S A)}$$

similarly define

$HP_S(A)$ = degree zero periodic cyclic ...

$d := \dim X$

Main Thm. Have a commutative diagram

$$\begin{tikzcd} HC^-_S(A) \arrow[r, "\sim"] \arrow[d, "\text{injective}"'] & \bigoplus_{0 \le i \le \frac{d}{2}} K \otimes H^{2i}_S(X) \arrow[d] \\ HP_S(A) \arrow[r, "\sim"] & K \otimes H^{even}_S(X) \end{tikzcd}$$

s.t. for any coherent $\mathcal{O}_X$-module M

$$HC^-_S(A) \ni ch(K \otimes M) \longmapsto \tau(M)$$

Pf of the vanishing thm:

$ch(K \otimes M)$ comes from HC^- $\Rightarrow$

$$\tau(M) \in \bigoplus_{0 \le i \le d/2} K \otimes H^{2i}_S(X)$$

$\Rightarrow$ $\tau_i(M) = 0$ for $i > d/2$. $\square$

Key point:

$$\mathrm{Im}\left[HC^-_S(\mathcal{O}_X) \to H^{ev}_S(X) \right]$$

may be nonzero for all $0 \le i \le d$.

$$HP_S(\mathcal{O}_X) \approx HP_S(A)$$

$$HC^-_S(\mathcal{O}_X) \rightsquigarrow HC^-_S(A)$$

kills degrees $i > \frac{1}{2} d$

<u>Corollary</u> (for Hochschild homology)

$$HH_{S,i}(A) \simeq K \otimes H^{d+i}_S(X)$$

An application to <u>symplectic resolutions</u>

$$\pi : X \longrightarrow Y$$

$\mathbb{G}_m \curvearrowright$ on X and Y.

π birational, projective,
$\mathbb{G}_m$-equivariant.

<u>Ex</u> Resolution of Kleinian singularities

$$X = \widetilde{\mathbb{C}^2/\Gamma} \longrightarrow Y = \mathbb{C}^2/\Gamma$$

$\mathbb{G}_m$-action $\rightsquigarrow$ can set $\hbar = 1$

$\rightsquigarrow$ sheaf on X of filtered algebras $\mathcal{B}$

$$\operatorname{gr} \mathcal{B} = \mathcal{O}_X$$

$$B := \Gamma(X, \mathcal{B}) \qquad \mathbb{C}\text{-algebra}$$

$$\operatorname{gr} B = \mathbb{C}[X]$$

M a $\mathcal{B}$-module $\rightsquigarrow$ <u>characteristic</u>

cycle $CC(M) := [\operatorname{Supp} gr(M)]$

Thm Characteristic cycles of finite dimensional B-modules are linearly independent (under mild conditions)