Silting theory under change of rings

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Setup and question

Main setup:

- R commutative Noetherian complete local ring, $\mathfrak{a} \subseteq \mathfrak{m} \subset \mathrm{R}$.
- Λ Noetherian R-algebra, that is, $\Lambda_{\mathrm{R}} \in \operatorname{mod}\mathrm{R}$. Set $\overline{\Lambda} = \Lambda/\mathfrak{a}\Lambda$.

$$\begin{tikzcd} \mathrm{R} \arrow[r] \arrow[d, two heads] & \Lambda \arrow[d, two heads] \\ \mathrm{R}/\mathfrak{a} \arrow[r] & \overline{\Lambda} \cong \Lambda \underset{\mathrm{R}}{\otimes} \mathrm{R}/\mathfrak{a} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{D}^{-}(\operatorname{mod}\Lambda) \arrow[d, "\mathbb{F}"] \\ \mathrm{D}^{-}(\operatorname{mod}\overline{\Lambda}) \end{tikzcd} \qquad \begin{tikzcd} M^{\bullet} \arrow[d, mapsto] \\ \overline{M^{\bullet}} = M^{\bullet} \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} \overline{\Lambda} \end{tikzcd}$$

Motivating question:

How is the derived representation theory of Λ different from that of $\overline{\Lambda}$?

General problem:

Usually, the functor $\mathbb{F}$ is not dense and does not respect isomorphism classes.

Main assumption:

$$\mathrm{Tor}_{+}^{\mathrm{R}}(\Lambda, \mathrm{R}/\mathfrak{a}) = 0, \text{ that is, } \mathrm{Tor}_{n}^{\mathrm{R}}(\Lambda, \mathrm{R}/\mathfrak{a}) = 0 \text{ for any } n > 0. \qquad (\star)$$

Example 1 (A gentle quotient of the Gelfand quiver)

$$(Q, I): \begin{tikzcd} \bullet \arrow[r, bend right, "b"'] & \bullet \arrow[l, bend right, "a"'] \arrow[r, bend left, "c"] & \bullet \arrow[l, bend left, "d"] \end{tikzcd} \qquad ab = cd$$

$$(Q, J): \begin{tikzcd} \bullet \arrow[r, bend right, "b"'] & \bullet \arrow[l, bend right, "a"'] \arrow[r, bend left, "c"] & \bullet \arrow[l, bend left, "d"] \end{tikzcd}$$

$$x \longmapsto ba + ab + dc$$

$$\begin{tikzcd} \mathrm{R} = \Bbbk[\![x]\!] \arrow[r, hook] \arrow[d] & \Lambda = \widehat{\Bbbk Q/I} \arrow[d, two heads] \\ \mathrm{R}/\mathfrak{m} \cong \Bbbk \arrow[r, hook] & \overline{\Lambda} \cong \Bbbk Q/J \end{tikzcd}$$

- $(\, P_1 \xrightarrow{a\cdot} P_2 \xrightarrow{b\cdot} P_1 \,)$ has no lift to $\mathrm{D}^-(\mathrm{mod}\,\Lambda)$.
- Set $B_\lambda^\bullet = (\, L_1 \oplus L_3 \xrightarrow{\begin{bmatrix} \lambda ba\cdot & da\cdot \\ bc\cdot & dc\cdot \end{bmatrix}} L_1 \oplus L_3 \,)$. Then $B_\lambda^\bullet \ncong B_0^\bullet$ but $\overline{B_\lambda^\bullet} \cong \overline{B_0^\bullet}$ for any $\lambda \neq 0$.
- Λ is skew-gentle, while $\overline{\Lambda}$ is gentle.

Structure of the talk

1. Lifting problems
2. Silting bijections
3. Applications and variations

1.1 An incomplete history of lifting problems

$$\begin{tikzcd} \mathfrak{a} \subseteq \mathfrak{m} \subset \mathrm{R} \arrow[r] \arrow[d, two heads] & \Lambda \arrow[d, two heads] \\ \mathrm{R}/\mathfrak{a} \arrow[r] & \bar{\Lambda} = \Lambda/\mathfrak{a}\Lambda \end{tikzcd} \qquad \begin{tikzcd} \exists L^{\bullet}\textcolor{red}{?} \arrow[d, dashed, mapsto] & \mathrm{D}^{-}(\mathrm{mod}\,\Lambda) \arrow[d, "\mathbb{F}"'] \\ P^{\bullet} \in & \mathrm{D}^{-}(\mathrm{mod}\,\bar{\Lambda}) \end{tikzcd}$$

Derived lifting problem: Does $P^{\bullet}$ lift to $L^{\bullet}$?

Abelian lifting problem: Does a proj. resolution $P^{\bullet}$ lift to a proj. resolution $L^{\bullet}$?

Yes – if $\mathrm{Hom}_{\mathrm{D}(\bar{\Lambda})}(P^{\bullet}, P^{\bullet}[2]) = 0$ assuming any of the setups:

setup	abelian lifting problem	derived lifting problem
"representation theory of finite groups"	R regular, $\dim \mathrm{R} = 1$, $\Lambda = \mathrm{R}G$ $\mathfrak{a} = \mathfrak{m}$ [J.A. Green (1959)]	Λ_{R} is free $\mathfrak{a} = \mathfrak{m}$ [Rickard (1991a)]
commutative algebra	$\mathrm{R} = \Lambda$, $\mathfrak{a} = (\underbrace{x_1, x_2, \ldots, x_\ell}_{\mathrm{R}\text{-regular}})$	
	[Eisenbud (1980)]	[Yoshino (1997)]
non-commutative generalizations	$\mathfrak{a} = (\underbrace{x_1, x_2, \ldots, x_\ell}_{\mathrm{R}\text{- and }\Lambda\text{-regular}})$ [Auslander-Ding-Solberg (1990)]	common denominator of setups?

1.2 A criterion to lift complexes

Next goal: lift $P^\bullet \in \mathrm{D}^-(\mathrm{mod}\,\bar{\Lambda})\mathrm{K}^-(\mathrm{proj}\,\bar{\Lambda})$

Basic observations: For any $n > 0$ set $\Lambda_n = \Lambda/\mathfrak{a}^n\Lambda$. Then

$$\Lambda \cong \varprojlim \Big(\dots \Lambda_{n+1} \twoheadrightarrow \Lambda_n \twoheadrightarrow \dots \Lambda_2 \twoheadrightarrow \Lambda_1 = \bar{\Lambda} \Big)$$

For any $n > 0$ set $\mathcal{C}_n = \mathrm{K}^-(\mathrm{proj}\,\Lambda_n)$. Try to lift by induction:

$$\begin{array}{ccccccc}
\dots \mathcal{C}_{n+1} & \longrightarrow & \mathcal{C}_n & \longrightarrow & \dots \mathcal{C}_2 & \longrightarrow & \mathcal{C}_1 \\
\dots \exists L^\bullet_{n+1}? & \mapsto & \exists L^\bullet_n? & \mapsto & \dots \exists L^\bullet_2? & \mapsto & L^\bullet_1 = P^\bullet
\end{array}$$

If there is a sequence of iterated lifts, then $L^\bullet = \varprojlim L^\bullet_n$ yields a lift of $P^\bullet$.

n-th lifting step: Assume that $P^\bullet$ has a lift $L^\bullet_n$ for $n > 0$.

A construction of Eisenbud yields a certain morphism

$$\triangle_n : P^\bullet \longrightarrow P^\bullet \underset{\mathrm{R}/\mathfrak{a}}{\otimes} \mathfrak{a}^n/\mathfrak{a}^{n+1}[2] = \alpha_n(P^\bullet)[2]$$

It holds that: $\triangle_n \sim 0 \quad \Rightarrow \quad P^\bullet$ has a lift $L^\bullet_{n+1}$.

Proposition 1

If $\mathrm{Hom}_{\mathrm{D}(\bar{\Lambda})}(P^\bullet, \alpha_n(P^\bullet)[2]) = 0$ for any $n > 0$, then $P^\bullet$ has a lift $L^\bullet \in \mathrm{D}^-(\mathrm{mod}\,\Lambda)$.

1.3 Lifting complexes without higher self-extensions

Proposition 1

If $\mathrm{Hom}(P^{\bullet}, \alpha_n(P^{\bullet})[2]) = 0$ *for any* $n > 0$, *then* $P^{\bullet}$ *has a lift* $L^{\bullet} \in \mathrm{D}^{-}(\mathrm{mod}\,\Lambda)$.

Lemma 2

Let $(\mathfrak{a}^n/\mathfrak{a}^{n+1})_{\mathrm{R}/\mathfrak{a}}$ *be free for any* $n > 0$. *Then* $\alpha_n(P^{\bullet}) = P^{\bullet} \underset{\mathrm{R}/\mathfrak{a}}{\otimes} \mathfrak{a}^n/\mathfrak{a}^{n+1} \cong P^{\bullet}$ *and*

$$\mathrm{Tor}_{+}^{\mathrm{R}}(\Lambda, \mathrm{R}/\mathfrak{a}^n) = 0 \quad \textit{for any } n > 0. \tag{$\star\star$}$$

In particular, $P^{\bullet}$ *lifts if* $\mathrm{Hom}(P^{\bullet}, P^{\bullet}[2]) = 0$.

For any $Q^{\bullet} \in \mathrm{D}^{-}(\mathrm{mod}\,\bar{\Lambda})$ denote

$$P^{\bullet} \geq Q^{\bullet} \quad \text{if } \mathrm{Hom}(P^{\bullet}, Q^{\bullet}[n]) = 0 \text{ for any } n > 0.$$

Proposition 3

Assume $(\star\star)$ *and* $P^{\bullet} \in \mathrm{per}\,\bar{\Lambda}$ *such that* $P^{\bullet} \geq Q^{\bullet}$. *Then* $P^{\bullet} \geq \alpha_n(Q^{\bullet})$ *for any* $n > 0$.
In particular, $P^{\bullet}$ *lifts if* $P^{\bullet} \geq P^{\bullet}[1]$.

The idea of the proof is inspired by recent work of Nasseh, Ono and Yoshino (2021).

1.4 Summary on lifting problems

Recall of **main setup** and **derived lifting problem**:

$$
\begin{tikzcd}
\mathfrak{a} \subseteq \mathfrak{m} \subset \mathrm{R} \arrow[r] \arrow[d, two heads] & \Lambda \arrow[d, two heads] \\
\mathrm{R}/\mathfrak{a} \arrow[r] & \overline{\Lambda} = \Lambda/\mathfrak{a}\Lambda
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\exists L^{\bullet}{\color{red}?} \arrow[d, dashed, mapsto] & \mathrm{D}^{-}(\operatorname{mod}\Lambda) \arrow[d, "\mathbb{F}"'] \\
P^{\bullet} \in & \mathrm{D}^{-}(\operatorname{mod}\overline{\Lambda})
\end{tikzcd}
$$

Common denominators of setups:

$$
\begin{tikzcd}[column sep=small]
\begin{array}{c}\text{“\textit{complete intersections}”} \\ \mathfrak{a} = (\underbrace{x_1, x_2, \ldots, x_\ell}_{\mathrm{R}\text{- and }\Lambda\text{-regular}})\end{array} \arrow[dr, Rightarrow] & \begin{array}{c}\text{“\textit{rep. theory of groups}”} \\ \mathfrak{a} = \mathfrak{m},\ \ \Lambda_{\mathrm{R}}\ \text{free}\end{array} \arrow[d, Rightarrow] \arrow[r, Rightarrow] & \Lambda_{\mathrm{R}}\ \text{free} \arrow[d, Rightarrow] \\
 & \begin{array}{c}\mathfrak{a}^n/\mathfrak{a}^{n+1}\ \text{free}\ \ \forall n > 0 \\ \operatorname{Tor}^{\mathrm{R}}_{+}(\Lambda, \mathrm{R}/\mathfrak{a}) = 0\end{array} \arrow[r, Rightarrow, "\text{Lemma 2}"] \arrow[d, Rightarrow, "\text{Lemma 2}"] & \operatorname{Tor}^{\mathrm{R}}_{+}(\Lambda, \mathrm{R}/\mathfrak{a}^n) = 0\ \ \forall n > 0 \arrow[d, Rightarrow, "\text{Proposition 3}"] \\
 & \begin{array}{c}P^{\bullet}\ \ \text{lifts if}\ \ \operatorname{Hom}(P^{\bullet}, P^{\bullet}[2]) = 0 \\ \text{if } P^{\bullet} \text{ is a projective resolution, so is } L^{\bullet}\end{array} \arrow[r, Rightarrow] & \begin{array}{c}P^{\bullet}\ \ \text{lifts if}\ P^{\bullet} \geq P^{\bullet}[1] \\ \text{and } P^{\bullet} \in \operatorname{per}\overline{\Lambda}.\end{array}
\end{tikzcd}
$$

2.1 Recall of silting and tilting complexes

Notation for $L^{\bullet}, M^{\bullet}, T^{\bullet} \in \operatorname{per}\Lambda$:

$$L^{\bullet} \geq M^{\bullet} \quad \text{if } \operatorname{Hom}(L^{\bullet}, M^{\bullet}[n]) = 0 \text{ for any } n > 0$$
$$T^{\bullet} \trianglerighteq T^{\bullet} \quad \text{if } \operatorname{Hom}(T^{\bullet}, T^{\bullet}[i]) = 0 \text{ for any } i \in \mathbb{Z}\backslash\{0\}$$

Definition (Keller and Vossieck (1988), Rickard (1988))

1. $L^{\bullet}$ *is silting if* $\langle L^{\bullet}\rangle = \operatorname{per}\Lambda$ *and* $L^{\bullet}$ *is presilting, that is,* $L^{\bullet} \geq L^{\bullet}$.
2. $T^{\bullet}$ *is tilting if* $\langle T^{\bullet}\rangle = \operatorname{per}\Lambda$ *and* $T^{\bullet}$ *is pretilting, that is,* $T^{\bullet} \trianglerighteq T^{\bullet}$.

- silt Λ = isomorphism classes of basic silting complexes of Λ.

Theorem (Aihara and Iyama (2012))

1. (silt $\Lambda, \geq$) *is a partially ordered set.*
2. *For any* $L^{\bullet} \in$ silt Λ *and any* $D^{\bullet} \in \operatorname{smd} L^{\bullet}$ *there exists* $\mu_{D^{\bullet}}(L^{\bullet}) \in$ silt Λ.

Theorem (Rickard (1988))

Λ *is derived equivalent to a ring* Γ $\Leftrightarrow$ *there exists* $T^{\bullet} \in$ tilt Λ : $\operatorname{End}_{\mathrm{D}(\Lambda)}(T^{\bullet}) \cong \Gamma$.

In this case: (silt $\Lambda, \geq$) $\xrightarrow{\sim}$ (silt $\Gamma, \geq$)

Proposition 4 (G.; cf. [Iyama and Kimura (2021)])

Any $L^\bullet, M^\bullet \in \operatorname{per}\Lambda$ *satisfy:*

$$L^\bullet \geq M^\bullet \quad \Leftrightarrow \quad \overline{L^\bullet} \geq \overline{M^\bullet} \quad \Rightarrow \quad \operatorname{Hom}(L^\bullet, M^\bullet) \xrightarrow{\mathbb{F}} \operatorname{Hom}(\overline{L^\bullet}, \overline{M^\bullet})$$

Proof. (of first " $\Rightarrow$ ", based on Rickard [1991b]): Set $K^\bullet = \mathbb{R}\operatorname{Hom}_\Lambda(L^\bullet, M^\bullet)$.

$$\operatorname{Tor}_+(\Lambda, \mathrm{R}/\mathfrak{a}) = 0 \quad \Rightarrow \quad \overline{K^\bullet} = K^\bullet \overset{\mathbb{L}}{\underset{\mathrm{R}}{\otimes}} \mathrm{R}/\mathfrak{a} \cong \mathbb{R}\operatorname{Hom}_{\bar{\Lambda}}(\overline{L^\bullet}, \overline{M^\bullet})$$

Using the Künneth spectral sequence

$$E_2^{pq} = \operatorname{Tor}^{\mathrm{R}}_{-p}(\mathrm{H}^q(K^\bullet), \mathrm{R}/\mathfrak{a}) \ \Rightarrow \ E^{p+q} = \mathrm{H}^{p+q}(\overline{K^\bullet})$$

it follows that

$$L^\bullet \geq M^\bullet \Leftrightarrow \mathrm{H}^+(K^\bullet) = 0 \Rightarrow E^{+q} = E^{p+} = 0 \Rightarrow E^+ = 0 \Leftrightarrow \mathrm{H}^+(\overline{K^\bullet}) = 0 \Leftrightarrow \overline{L^\bullet} \geq \overline{M^\bullet}$$

□

Corollary 5

There are embeddings $f_{ps}\colon$ presilt $\Lambda \hookrightarrow$ presilt $\bar{\Lambda}$ *and* $f_s\colon$ silt $\Lambda \hookrightarrow$ silt $\bar{\Lambda}$ *.*

2.3 Pretilting complexes under change of rings

Proposition (Rickard (1991b))

For any $T^\bullet \in \operatorname{per}\Lambda$ *such that* $T^\bullet \trianglerighteq T^\bullet$: $\operatorname{Tor}_+^{\mathrm{R}}(\operatorname{End}(T^\bullet), \mathrm{R}/\mathfrak{a}) = 0 \;\Rightarrow\; \overline{T^\bullet} \trianglerighteq \overline{T^\bullet}$.

So the embedding f_s restricts:

$$f_t\colon \mathsf{tilt}^{\mathrm{R}/\mathfrak{a}}\Lambda = \{T^\bullet \in \mathsf{tilt}\,\Lambda \mid \operatorname{Tor}_+^{\mathrm{R}}(\operatorname{End}(T^\bullet), \mathrm{R}/\mathfrak{a}) = 0\} \longrightarrow \mathsf{tilt}\,\overline{\Lambda}$$

Next question: Are lifts of pretilting complexes pretilting?

Definition

$\mathrm{R}/\mathfrak{a}$ *is Tor-rigid if for any* $M \in \operatorname{mod}\mathrm{R}$:

$$\operatorname{Tor}_1^{\mathrm{R}}(M, \mathrm{R}/\mathfrak{a}) = 0 \quad \Rightarrow \quad \operatorname{Tor}_+^{\mathrm{R}}(M, \mathrm{R}/\mathfrak{a}) = 0.$$

Proposition 6

Let $\mathrm{R}/\mathfrak{a}$ *be Tor-rigid. For any* $T^\bullet \in \operatorname{per}\Lambda$, $L^\bullet \in \mathrm{D}^-(\operatorname{mod}\Lambda)$ *and* $i \in \mathbb{Z}$:

$$\operatorname{Hom}(\overline{T^\bullet}, \overline{L^\bullet}[i]) = 0 \quad \Rightarrow \quad \operatorname{Hom}(T^\bullet, L^\bullet[i]) = 0$$

In particular:

(a) *If* $\overline{T^\bullet} \trianglerighteq \overline{T^\bullet}$, *then* $T^\bullet \trianglerighteq T^\bullet$. (b) *If* $\overline{L^\bullet}$ *is a projective resolution, so is* $L^\bullet$.

2.4 Main result: silting and tilting bijections

Recall: Λ Noetherian R-algebra, R complete local, $\mathfrak{a} \subseteq \mathfrak{m} \subset \mathrm{R}$ and $\overline{\Lambda} = \Lambda/\mathfrak{a}\Lambda$.

Theorem 7

Assume that $\mathrm{R}/\mathfrak{a}$ *be Tor-rigid and* $\mathrm{Tor}_+^{\mathrm{R}}(\Lambda, \mathrm{R}/\mathfrak{a}^n) = 0$ *for any* $n > 0$, *for example,*

"complete intersections"

$$\mathfrak{a} = (\underbrace{x_1, x_2, \ldots, x_\ell}_{\mathrm{R}\text{- and }\Lambda\text{-regular}})$$

or

"rep. theory of groups"

$\mathfrak{a} = \mathfrak{m}$, Λ_{R} *free*

or

"rep. theory of orders"

R *regular*, Λ_{R} *free*

Then there are bijections:

$$\begin{tikzcd}
L^\bullet \arrow[d, "\mathbb{F}"'] & \mathsf{silt}\,\Lambda \arrow[d, hook, "f_s"', "\wr"] & \mathsf{tilt}^{\mathrm{R}/\mathfrak{a}}\,\Lambda \arrow[l, hook'] \arrow[d, hook, "f_t"', "\wr"] & \mathsf{tilt}^{\mathrm{R}/\mathfrak{m}}\,\Lambda \arrow[l, hook', "\text{equal if } \mathfrak{a}=\mathfrak{m}"] \arrow[d, dashed, "f_t^*"', "\wr"] \\
\overline{L}^\bullet & \mathsf{silt}\,\overline{\Lambda} & \mathsf{tilt}\,\overline{\Lambda} \arrow[l, hook'] & \mathsf{tilt}^{\mathrm{R}/\mathfrak{m}}\,\overline{\Lambda} \arrow[l, hook', "\text{equal if } \mathfrak{a}=\mathfrak{m}"']
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{tilt}^{\mathrm{R}/\mathfrak{a}}\,\Lambda &= \{T^\bullet \in \mathsf{tilt}\,\Lambda \mid \mathrm{Tor}_+^{\mathrm{R}}(\mathrm{End}(T^\bullet), \mathrm{R}/\mathfrak{a}) = 0\} \\
\mathsf{tilt}^{\mathrm{R}/\mathfrak{m}}\,\Lambda &= \{\quad \text{"} \quad \mid \mathrm{End}(T^\bullet)_{\mathrm{R}} \text{ free}\} \\
\mathsf{tilt}^{\mathrm{R}/\mathfrak{m}}\,\overline{\Lambda} &= \{P^\bullet \in \mathsf{tilt}\,\overline{\Lambda} \mid \mathrm{End}(P^\bullet)_{\mathrm{R}/\mathfrak{a}} \text{ free}\}
\end{aligned}$$

2.5 Remarks on silting bijections

$$\begin{tikzcd}
\mathsf{silt}\,\Lambda \arrow[d, dashed, "f_s"', "\wr"] & \mathsf{tilt}^{R/\mathfrak{a}}\,\Lambda \arrow[l, hook'] \arrow[d, dashed, "f_t"', "\wr"] & \mathsf{tilt}^{R/\mathfrak{m}}\,\Lambda \arrow[l, hook'] \arrow[d, dashed, "f_t^*"', "\wr"] \\
\mathsf{silt}\,\overline{\Lambda} & \mathsf{tilt}\,\overline{\Lambda} \arrow[l, hook'] & \mathsf{tilt}^{R/\mathfrak{m}}\,\overline{\Lambda} \arrow[l, hook']
\end{tikzcd}$$

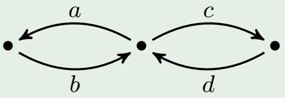
1. In case $\mathfrak{a} = \mathfrak{m}$ and Λ_R is free, $f_t = f_t^*$ is bijective by [Rickard (1991a)].
   - To show that " $\overline{T^\bullet} \in \mathsf{tilt}\,\overline{\Lambda} \Rightarrow T^\bullet \in \mathsf{tilt}\,\Lambda$ " Rickard proved that:

$$\langle T^\bullet \rangle = \mathrm{per}\,\Lambda \quad \underset{T^\bullet \trianglerighteq T^\bullet}{\Leftrightarrow} \quad \langle T^\bullet \rangle^\perp \cap \mathrm{D}^-(\mathrm{mod}\,\Lambda) = 0. \tag{1}$$

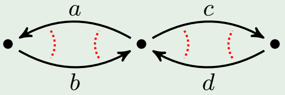
2. The proof of Theorem 7 follows Rickard's approach.
   - **Main difficulty:** to extend characterization (1) assuming $T^\bullet \geq T^\bullet$.

     This extension uses dg-categorical arguments due to Keller.
3. Eisele showed independently that f_s is bijective if Λ and $\overline{\Lambda}$ are quotients of a common $\Bbbk[\![x]\!]$-order [Eisele (2021)].
   In this context, Eisele studied also derived Picard groups.

Example 2 (A gentle quotient of the Gelfand quiver)

(Q, I) : 

$ab = cd$

(Q, J) : 

$$\begin{tikzcd} \mathrm{R} = \Bbbk[\![x]\!] \arrow[r, hook, "x \longmapsto ba + ab + dc"] \arrow[d, two heads] & \Lambda = \widehat{\Bbbk Q/I} \arrow[d, two heads] \\ \mathrm{R}/\mathfrak{m} \cong \Bbbk \arrow[r, hook] & \bar{\Lambda} \cong \Bbbk Q/J \end{tikzcd}$$

$$(ba)^m = (ab)^m = (cd^m) = (dc)^m = 0$$

$$\begin{tikzcd} & \text{silt}\,\Lambda \arrow[dl, "\sim"'] \arrow[dd, "\wr"] \\ \text{silt}\,\Lambda_m \arrow[dr, dashed, "\sim"'] & \\ & \text{silt}\,\bar{\Lambda} \end{tikzcd}$$

$(m > 1)$

Although a quotient Λ_m is derived-wild and the order Λ is skew-gentle, both have "gentle silting theory".

3.2 Quotients of a preprojective algebra of type $\widetilde{\mathbb{A}}$

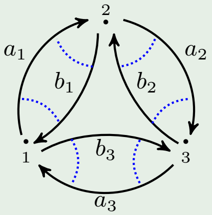
Example 3

Let $\Lambda = \widehat{\prod}_Q$ with $Q = \widetilde{\mathbb{A}}_2$. Then Λ_{R} is free via:

$$\mathfrak{m} \subset \mathrm{R} = \Bbbk[\![x, y]\!] \longrightarrow \Lambda \qquad x \longmapsto \text{sum of all 3-cycles} \qquad y \longmapsto \sum_{i=1}^{3} a_i b_i$$

Theorem 7 yields bijections for the family of quotients $(\overline{\Lambda}_{\mathfrak{a}})_{\mathfrak{a} \subseteq \mathfrak{m}}$ with $\overline{\Lambda}_{\mathfrak{a}} = \Lambda / \mathfrak{a}\Lambda$:

$$\begin{tikzcd}
 & \text{silt}\,\Lambda \arrow[dl, "\sim"'] \arrow[dd, "\wr"] \arrow[dr, "\sim"] & \\
\text{silt}\,\overline{\Lambda}_{\mathfrak{a}} \arrow[dr, dashed, "\sim"'] & & \text{silt}\,\overline{\Lambda}_y \arrow[dl, dashed, "\sim"] \\
 & \text{silt}\,\overline{\Lambda}_{\mathfrak{m}} &
\end{tikzcd}$$



$$\begin{aligned}
a_1 b_1 &= b_3 a_3 = 0 \\
a_2 b_2 &= b_1 a_1 = 0 \\
a_3 b_3 &= b_2 a_2 = 0 \\
a_1 a_2 a_3 &= b_3 b_2 b_1 \\
a_2 a_3 a_1 &= b_1 b_3 b_2 \\
a_3 a_1 a_2 &= b_2 b_1 b_3
\end{aligned}$$

$\Rightarrow$ the $\Bbbk[\![x]\!]$-order $\overline{\Lambda}_y$ is nodal, and thus derived-tame by [Burban, Drozd (2004)]

$\Rightarrow$ there is hope to classify silting complexes over derived-wild quotients $\overline{\Lambda}_{\mathfrak{a}}$ and Λ

3.3 Silting embeddings and descent in a more general setup

Proposition 8

Let S *be a commutative ring,* Λ *a Noetherian* S*-algebra and* Γ *an* S*-algebra such that:*

- $\mathrm{Tor}_{+}^{\mathrm{S}}(\Lambda, \Gamma) = 0$,
- *for any* $M \in \operatorname{mod} \mathrm{S}$*:* $\; M \otimes \Gamma \cong 0 \Rightarrow M = 0$.

Then there are well-defined injective maps, and for any $L^{\bullet} \in \operatorname{per} \Lambda$ *it holds that:*

$$
\begin{tikzcd}
\operatorname{add} L^{\bullet} \arrow[d, "\mathbb{F}"'] \\
\operatorname{add} \mathbb{F}(L^{\bullet})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{silt}_{\mathcal{C}}\, \Lambda \arrow[d, hook, dashed, "f_s"'] & \mathsf{tilt}_{\mathcal{C}}^{\Gamma}\, \Lambda \arrow[l, hook'] \arrow[d, hook, dashed, "f_t"'] \\
\mathsf{silt}_{\mathcal{C}}\, \Lambda \otimes \Gamma & \mathsf{tilt}_{\mathcal{C}}\, \Lambda \otimes \Gamma \arrow[l, hook']
\end{tikzcd}
\qquad
\begin{tikzcd}
L^{\bullet} \in \mathsf{silt}\, \Lambda \\
\mathbb{F}(L^{\bullet}) \in \mathsf{silt}\, \Lambda \otimes \Gamma \arrow[u, Rightarrow, dashed, "\text{if } {}_{\mathrm{S}}\Gamma \in \operatorname{mod} \mathrm{S}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
L^{\bullet} \in \mathsf{tilt}^{\Gamma}\, \Lambda \\
\mathbb{F}(L^{\bullet}) \in \mathsf{tilt}\, \Lambda \otimes \Gamma \arrow[u, Rightarrow, dashed, "\text{if } {}_{\mathrm{S}}\Gamma \text{ is Tor-rigid}"]
\end{tikzcd}
$$

Corollary 9

If ${}_{\mathrm{S}}\Gamma$ *is faithfully flat, the maps* f_s *and* f_t *are well-defined and injective.*

Theorem (Iyama and Kimura (2021))

For any $N^{\bullet} \in \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ *it holds that*

$$N^{\bullet} \in \mathsf{silt}\, \Lambda \quad \Leftrightarrow \quad N^{\bullet}_{\mathfrak{p}} \in \mathsf{silt}\, \Lambda_{\mathfrak{p}} \text{ for any prime ideal } \mathfrak{p} \text{ of } \mathrm{S}.$$

3.4 A variation of the silting bijection for skew-central quotients

As before, Λ is a Noetherian $\mathbb{R}$-algebra. Let $s \in \operatorname{rad}\Lambda$ be *regular* and *normal*, that is:

$$\Lambda \xhookrightarrow{\cdot s} \Lambda, \qquad \Lambda \xhookrightarrow{s\cdot} \Lambda, \qquad s\Lambda = \Lambda s.$$

- There is an automorphism $\sigma = \sigma_s$ of Λ such that $sa = \sigma(a)s$ for any $a \in \Lambda$.
- Redefine $\overline{\Lambda}$ by $\Lambda/s\Lambda$. The automorphism σ induces an automorphism $\alpha = \bar{\sigma}$ of $\overline{\Lambda}$.

Theorem 10

In the setup above, the functor $\mathbb{F}$ *induces bijections*

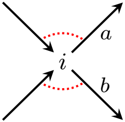
$$\begin{tikzcd}
\text{tilt}^{\sigma,s}\Lambda = \{T^{\bullet} \in \text{tilt}^{\sigma}\Lambda \mid \operatorname{Hom}_{\mathrm{D}(\bar{\Lambda})}(\overline{T^{\bullet}}, \overline{T^{\bullet}}[-1]) = 0\} \arrow[d, dashed, "f_t^{\sigma}"', "\wr"] \arrow[dr, hook] & \\
\text{tilt}^{\alpha}\overline{\Lambda} \arrow[dr, hook] & \text{silt}^{\sigma}\Lambda = \{L^{\bullet} \in \text{silt}\Lambda \mid L^{\bullet}_{\sigma} = L^{\bullet} \overset{\mathbb{L}}{\underset{\Lambda}{\otimes}} \Lambda_{\sigma} \cong L^{\bullet}\} \arrow[d, dashed, "f_s^{\sigma}"', "\wr"] \\
& \text{silt}^{\alpha}\overline{\Lambda}
\end{tikzcd}$$

- If s is central, then $\text{silt}^{\sigma}\Lambda = \text{silt}\,\Lambda$ and $\text{silt}^{\alpha}\overline{\Lambda} = \text{silt}\,\overline{\Lambda}$.
- The converse is not true, fortunately!

3.5 Ribbon graph orders and Brauer graph algebras

Definition of Λ and $\overline{\Lambda}$:

$\mathbf{\Lambda}$: Let (Q, I) be *2-regular gentle*, that is, at any $i \in Q_0$:



The arrow ideal completion Λ of its path algebra $\Bbbk Q/I$ is a *ribbon graph order*.

Remark: The ring Λ has a *central* element

$$x = \text{sum of repetition-free cycles} = \sum_{\substack{\{a,b\} \subseteq Q_1 \\ s(a)=s(b)}} c_a + c_b$$

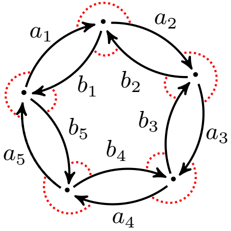
$\overline{\mathbf{\Lambda}}$: Choose positive integers $m = (m_a)_{a \in Q_1}$ which do not differ along cycles.

This yields a *normal* element:

$$s_m = \sum_{\substack{\{a,b\} \subseteq Q_1 \\ s(a)=s(b)}} c_a^{m_a} - c_b^{m_b}$$

Then $\overline{\Lambda} = \Lambda / s_m \Lambda$ is a *Brauer graph algebra*.

Example:



$$x = a_2 b_2 + b_1 a_1 + \dots$$

Choose $m_1, \dots, m_5 \in \mathbb{N}$ and add relations:

$$\begin{aligned} (a_2 b_2)^{m_2} &= (b_1 a_1)^{m_1} \\ (a_1 b_1)^{m_1} &= (b_4 a_4)^{m_4} \\ &\vdots \\ (a_3 b_3)^{m_3} &= (b_2 a_2)^{m_2} \end{aligned}$$

Corollary 11

Let B *be a Brauer graph algebra, so* B $\cong \Lambda/s\Lambda$ *for a ribbon graph order* Λ *and normal regular element* $s \in$ rad Λ. *Then there are bijections*

$$\begin{tikzcd}
\text{tilt}^{s,\sigma}\Lambda \arrow[d, dashed, "f_t^\sigma"', "\wr"] \arrow[dr, equal, "\Lambda \textit{ not always symmetric!}"] & & \\
\text{tilt}^{\alpha}\,\mathrm{B} \arrow[dr, equal, "\mathrm{B} \textit{ is symmetric}"'] & \text{silt}\,\Lambda \arrow[d, dashed, "f_s"', "\wr"] \arrow[r, phantom, "="] & \text{silt}^{\sigma}\Lambda \arrow[d, dashed, "f_s^\sigma"', "\wr"] \\
 & \text{silt}\,\mathrm{B} \arrow[r, phantom, "="] & \text{silt}^{\alpha}\,\mathrm{B}
\end{tikzcd}$$

Proof. s induces an involution σ of $\Lambda \cong \widehat{\Bbbk Q/I}$ such that

$$e_i \mapsto e_i \text{ for any } i \in Q_0, \quad a \mapsto \varepsilon_a a = \pm a \quad \text{for any } a \in Q_1.$$

and an involution $\alpha = \bar{\sigma}$ of B. Theorem 10 yields f_s^σ and f_t^σ.

Burban and Drozd (2004) gave a description of $\mathrm{ind}[\mathrm{per}\,\Lambda]$.

It can be shown that $L^\bullet_\sigma \cong L^\bullet$ for any $L^\bullet \in \mathrm{ind}[\mathrm{per}\,\Lambda]$, and that $[\alpha] \in \mathrm{Out}_0$ B.

The latter implies $P^\bullet_\alpha \cong P^\bullet$ for any $P^\bullet \in \mathrm{per}\,\Lambda$ with $\mathrm{Hom}_{\mathrm{D(B)}}(P^\bullet, P^\bullet[1]) = 0$ by a result of Huisgen-Zimmermann and Saorin (2003). □

Thank You for listening!

Happy birthday, Bill !