The Deligne – Simpson Problem

A guide to the work of WCB.

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Given conjugacy classes $\mathcal{C}_i$ in $\mathrm{GL}_n(\mathbb{C})$ $(i = 1, \ldots, k)$,

can we find matrices $M_i \in \mathcal{C}_i$ such that

$$M_1 \cdots M_k = 1?$$

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A solution is called **irreducible** provided the M_i have no common invariant subspace.

Can we find an irreducible solution?

Motivation

Fix distinct points $D = (a_1, \ldots, a_k)$ in $\mathbb{P}^1$.

The fundamental group of the complement is

$$\pi_1 := \pi_1(\mathbb{P}^1 - D) = \langle g_1, \ldots, g_k \mid g_1 \cdots g_k = 1 \rangle.$$

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The Riemann – Hilbert Correspondence says that taking monodromy yields an equivalence between

π_1-representations (of dimension n), and

locally-free sheaves E (of rank n) on $\mathbb{P}^1$ equipped with a logarithmic connection

$$\nabla \colon E \to E \otimes \Omega^1(D) \cong E(k-2)$$

so a morphism of sheaves of $\mathbb{C}$-modules satisfying a Leibniz-type rule, e.g.

$$\nabla(fs) = f\nabla(s) + (x-a_1)\cdots(x-a_k)df\, s \quad \text{on } \mathbb{A}^1$$

Motivation

Let U be affine open, intersecting D at $a_j = 0$.

Assume E is trivial over U, with fibre V.

Then we have a holomorphic map $N\colon U \to \mathrm{End}(V)$, such that flat connections ($\nabla(s) = 0$) correspond to functions

$$s\colon U - \{0\} \to V, \quad s(u)' = -N(u)(s(u))/u.$$

Moreover, all flat connections are meromorphic at the origin.

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Moreover, all flat connections are meromorphic at the origin.

The **residue** is $N(0)$, and the monodromy is conjugate to $\exp(-2\pi i N(0))$.

Thus, given some conjugacy classes, the DSP asks whether there exists a sheaf with logarithmic connection (regular system of ODEs) having these monodromies.

Survey: `http://math.stanford.edu/~conrad/papers/rhtalk.pdf`

Conjugacy classes of matrices

Let $\mathcal{C}$ be a conjugacy class in $M_n(\mathbb{C})$. For a judicious choice of scalars $\xi_1, \ldots, \xi_w$ and integers $n = d_0, d_1, \ldots, d_w = 0$, we have

$$M \in \mathcal{C} \quad \Longleftrightarrow \quad \operatorname{rank}(M - \xi_1) \cdots (M - \xi_t) = d_t \quad \forall t.$$

Conjugacy classes of matrices

Proposition

Fix $\mathcal{C}$, corresponding to (ξ_t, d_t). T.f.a.e.

1. $M \in \bar{\mathcal{C}}$ $\quad$ $M \in \mathcal{C}$

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Proposition

Fix $\mathcal{C}$, corresponding to (ξ_t, d_t). T.f.a.e.

1. $M \in \bar{\mathcal{C}}$ $\quad M \in \mathcal{C}$
2. *there exists a flag* $\mathbb{C}^n = V_0 \supset V_1 \supset \cdots \supset V_w = 0$ *with*
   $\dim V_t = d_t$ *and*
   $(M - \xi_t)(V_{t-1}) \subseteq V_t$ $\quad (M - \xi_t)(V_{t-1}) = V_t$

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Fix $\mathcal{C}$, corresponding to (ξ_t, d_t). T.f.a.e.

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2. *there exists a flag* $\mathbb{C}^n = V_0 \supset V_1 \supset \cdots \supset V_w = 0$ *with* $\dim V_t = d_t$ *and* $(M - \xi_t)(V_{t-1}) \subseteq V_t$ $\quad$ $(M - \xi_t)(V_{t-1}) = V_t$
3. *there exist vector spaces and maps*

$$\mathbb{C}^n = V_0 \underset{b_1}{\overset{a_1}{\leftrightarrows}} V_1 \underset{b_2}{\overset{a_2}{\leftrightarrows}} V_2 \text{ - - - - - - } V_{w-1} \underset{b_w}{\overset{a_w}{\leftrightarrows}} V_w = 0$$

with $\dim V_t = d_t$, *a_t injective, b_t surjective and*

$$M - a_1 b_1 = \xi_1, \quad a_{t+1} b_{t+1} - a_t b_t = \xi_t - \xi_{t+1}$$

Star-shaped quiver

Given conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_k$ in $M_n(\mathbb{C})$.

Let $\mathcal{C}_i$ correspond to scalars ξ_{it} and integers d_{it} $(t \leq w_i)$

We define the star quiver Q

$$
\begin{tikzcd}
 & (1,1) \arrow[dl] & (1,2) \arrow[l] \arrow[r, dash, dashed] & {} & (1, w_1 - 1) \arrow[l] \\
0 & (2,1) \arrow[l] \arrow[d, dash, dashed] & (2,2) \arrow[l] \arrow[r, dash, dashed] \arrow[d, dash, dashed] & {} & (2, w_2 - 1) \arrow[l] \arrow[d, dash, dashed] \\
 & (k,1) \arrow[ul] & (k,2) \arrow[l] \arrow[r, dash, dashed] & {} & (k, w_k - 1) \arrow[l]
\end{tikzcd}
$$

This comes with an associated lattice $\mathbb{Z}^{Q_0}$, root system Φ, and symmetric bilinear form $(-,-)$.

We set $p(f) = 1 - \frac{1}{2}(f,f)$ for $f \in \mathbb{Z}^{Q_0}$.

Additive DSP

The additive DSP asks whether, given $\mathcal{C}_i$, there exist $M_i \in \mathcal{C}_i$ with

$$M_1 + \cdots + M_k = 0.$$

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Can we find an irreducible solution?

We define $\lambda \in \mathbb{C}^{Q_0}$ via

$$\lambda_0 = \sum_i \xi_{i1}, \quad \lambda_{it} = \xi_{it} - \xi_{it+1}.$$

Deformed preprojective algebra

The **deformed preprojective algebra** $\Pi^\lambda Q$ is the path algebra of the doubled quiver

$$
\begin{tikzcd}
 & (1,1) \arrow[dl, shift right, "a_{11}"'] \arrow[r, shift right, "b_{12}"'] & (1,2) \arrow[l, shift right, "a_{12}"'] \arrow[r, dashed, no head] & \cdot \arrow[r, shift right] & (1,w_1-1) \arrow[l, shift right] \\
0 \arrow[ur, shift right, "b_{11}"'] \arrow[r, shift right, "b_{21}"'] \arrow[dr, shift right, "b_{k1}"'] & (2,1) \arrow[l, shift right, "a_{21}"'] \arrow[r, shift right, "b_{21}"'] \arrow[d, dashed, no head] & (2,2) \arrow[l, shift right, "a_{22}"'] \arrow[r, dashed, no head] \arrow[d, dashed, no head] & \cdot \arrow[r, shift right] & (2,w_2-1) \arrow[l, shift right] \arrow[d, dashed, no head] \\
 & (k,1) \arrow[ul, shift right, "a_{k1}"'] \arrow[r, shift right, "b_{k2}"'] & (k,2) \arrow[l, shift right, "a_{k2}"'] \arrow[r, dashed, no head] & \cdot \arrow[r, shift right] & (k,w_k-1) \arrow[l, shift right]
\end{tikzcd}
$$

modulo the relations

$$\sum_i a_{i1} b_{i1} = \lambda_0 e_0, \quad a_{it+1} b_{it+1} - b_{it} a_{it} = \lambda_{it} e_{it}.$$

Translating the problem

Theorem

$\exists$ solution $M_1 + \cdots + M_k = 0$ with $M_i \in \bar{\mathcal{C}}_i$ $\Leftrightarrow$ *$\exists$ Π^λ-module of dimension vector d*

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$\exists$ *solution* $M_1 + \cdots + M_k = 0$ *with* $M_i \in \bar{\mathcal{C}}_i$	$\Leftrightarrow$	$\exists$ Π^λ*-module of dimension vector* d
$\exists$ *solution with* $M_i \in \mathcal{C}_i$	$\Leftrightarrow$	$\exists$ *strict* Π^λ*-module, dim* d

We call a Π^λ-module **strict** provided the linear maps A_{it} are all injective, and the B_{it} are all surjective.

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$\exists$ *irred. solution with* $M_i \in \mathcal{C}_i$	$\Leftrightarrow$	$\exists$ *simple* Π^λ*-module, dim* d

All simple Π^λ-modules N with $N_0 \neq 0$ are strict.

WCB. On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero, Duke Math. J. 118 (2003) 339–352.

Forgetful functor

There is a forgetful functor

$$\operatorname{mod} \Pi^{\lambda} \longrightarrow \operatorname{mod} \mathbb{C}Q.$$

Theorem

A $\mathbb{C}Q$-module N lies in the image if and only if $\lambda \cdot \underline{\dim}\, N' = 0$ for all indecomposable direct summands N' of N.

Here $\lambda \cdot f = \lambda_0 f_0 + \sum_{it} \lambda_{it} f_{it}$.

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Here $\lambda \cdot f = \lambda_0 f_0 + \sum_{it} \lambda_{it} f_{it}$.

Kac's Theorem says the dimension vectors of indecomposable $\mathbb{C}Q$-modules is precisely Φ^+.

Write $\Phi^+_\lambda := \{f \in \Phi^+ \mid \lambda \cdot f = 0\}$. Then there is a Π^λ-module of dimension vector f if and only if $f \in \mathbb{N}\Phi^+_\lambda$.

Reflection functors

For each vertex $v \in Q_0$ with $\lambda_v \neq 0$ there is a reflection functor

$$R_v\colon \operatorname{mod}\Pi^{\lambda} \longrightarrow \operatorname{mod}\Pi^{s_v^*(\lambda)}$$

acting as the usual reflection s_v on dimension vectors.

We have

$$s_v^*(\lambda)\cdot f = \lambda \cdot s_v(f).$$

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We have

$$s_v^*(\lambda) \cdot f = \lambda \cdot s_v(f).$$

Define Σ_λ to be those $f \in \Phi_\lambda^+$ such that, for every non-trivial decomposition

$$f = g_1 + \cdots + g_r, \qquad g_i \in \Phi_\lambda^+$$

we have

$$p(f) > p(g_1) + \cdots + p(g_r).$$

The reflection s_v sends Σ_λ to $\Sigma_{s_v^*(\lambda)}$.

Dimension vectors of simples

Theorem

If $f \in \Sigma_\lambda$, then the Π^λ-modules of dimension vector f form an irreducible affine variety, and the simple modules form a dense open subset.

WCB. Geometry of the moment map for representations of quivers, Compositio Math., 126 (2001) 257–293.

Reduction step

What about the converse?

Lemma

Suppose there exists a simple Π^λ-module of dimension vector f.
If v is a vertex with $\lambda_v = 0$, then either $f = e_v$ or $(f, e_v) \leq 0$.

Thus we can apply reflections and assume f is minimal (so either $f = e_v$, or f is in the fundamental region).

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Suppose there exists a simple Π^λ-module of dimension vector f in the fundamental region. If $f \notin \Sigma_\lambda$, then the lemma puts severe restrictions on the quiver and the parameter λ.

Non-existence of simples

We are reduced to showing that there are no simple modules in the following three cases.

(I) Q is extended Dynkin, with minimal positive root δ, $\lambda \cdot \delta = 0$ and $f = m\delta$ with $m \geq 2$.
Arm lengths are $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$.

(II) Q has arm lengths $(2,2,2,3)$, $(3,3,4)$, $(2,4,5)$ or $(2,3,7)$. This is an extended Dynkin subquiver Q' together with an extra vertex v.
We have $\lambda_v = 0$, $f_v = 1$, and $f|_{Q'} = m\delta$ with $m \geq 2$.

(III) f has two consecutive 1s on some arm. If we remove the connecting edge to get the quiver $Q' \amalg Q''$, then $\lambda \cdot f|_{Q'} = 0$.

The first and third are relatively straightforward. The second is more involved.

Additive DSP: Summary

Theorem

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Theorem

There is an irreducible solution with $M_i \in \mathcal{C}_i$ if and only if $d \in \Sigma_\lambda$.

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Given conjugacy classes $\mathcal{C}_i$, can we solve $M_1 \cdots M_k = 1$ with $M_i \in \mathcal{C}_i$? Are there irreducible solutions?

We can try and emulate the proof of the additive case.

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We can try and emulate the proof of the additive case.

Recall that we have the data ξ_{it} and d_{it}, and the quiver Q.

Define $q \in \mathbb{C}^{Q_0}$ via

$$q_0 = \prod_i \xi_{i1}, \quad q_{it} = \xi_{it}/\xi_{it+1}.$$

Compare with

$$\lambda_0 = \sum_i \xi_{i1}, \quad \lambda_{it} = \xi_{it} - \xi_{it+1}.$$

The multiplicative preprojective algebra

The **multiplicative preprojective algebra** Λ^q is the path algebra of

$$\begin{tikzcd}
 & (1,1) \arrow[dl, "a_{11}"'] \arrow[r, shift left, leftarrow, "a_{12}"] \arrow[r, shift right, "b_{12}"'] & (1,2) \arrow[r, dashed, no head] & {} \arrow[r, shift left, leftarrow] \arrow[r, shift right] & (1, w_1 - 1) \\
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\end{tikzcd}$$

modulo the relations $1 + a_{it} b_{it}$, $1 + b_{it} a_{it}$ invertible, and

$$(e_0 + a_{11} b_{11}) \cdots (e_0 + a_{k1} b_{k1}) = q_0 e_0$$

$$(e_{it} + a_{it+1} b_{it+1})(e_{it} + b_{it} a_{it})^{-1} = q_{it} e_{it}$$

c.f. $\quad a_{11} b_{11} + \cdots + a_{k1} b_{k1} = \lambda_0 e_0, \quad a_{it+1} b_{it+1} - b_{it} a_{it} = \lambda_{it} e_{it}.$

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All simple Λ^q-modules N with $N_0 \neq 0$ are strict

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No forgetful functor

Unfortunately, for the multiplicative preprojective algebra, there is no analogue of the forgetful functor

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and there is no description of the set of dimension vectors of Λ^q-modules.

We do have, though, that if there is a Λ^q-module of dimension vector f, then

$$1 = q^f := \prod_v q_v^{f_v}.$$

We therefore write $\Phi_q^+ := \{f \in \Phi^+ \mid q^f = 1\}$.

Reflection functors

Let v be a vertex with $q_v \neq 1$. Then there is a reflection functor

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Define Σ'_q to be those $f \in \Phi_q^+$ such that, for every non-trivial decomposition

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If $f \in \Sigma'_q$, then the Λ^q-modules of dim f, if non-empty, form an equidimensional variety, and the simple modules form a dense open subset.

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To show that the dimension vectors of the simple Λ_q-modules all lie in Σ'_q, we would again have to show that there are no simples in the same three cases as before.

Unclear how to proceed using multiplicative preprojective algebras.

A second approach

By introducing multiplicative preprojective algebras, WCB and Peter Shaw managed to emulate some, but not all, of the proof of the additive DSP.

We therefore need a complementary approach to the problem. This comes via the Riemann – Hilbert correspondence, relating the DSP to the category of sheaves on $\mathbb{P}^1$ equipped with a logarithmic connection.

Logarithmic connections

Fix distinct points $D = (a_1, \dots, a_k)$ in $\mathbb{P}^1$. Assume for convenience that $a_j \in \mathbb{C} = \mathbb{A}^1$.

Let E be a locally-free sheaf of rank n on $\mathbb{P}^1$. A **logarithmic connection** on E is a map

$$\nabla \colon E \to E \otimes \Omega^1(D) \cong E(k-2)$$

of sheaves of $\mathbb{C}$-modules satisfying the Leibniz-type rule

$$\nabla(fs) = f\nabla(s) + (x - a_1)\cdots(x - a_k)df\, s \quad f \in \mathbb{C}[x], \quad s \in E(\mathbb{A}^1).$$

Residues

For each point a_j there is a **residue map**

$$\mathrm{Res}_{a_j} \nabla \in \mathrm{End}(E_{a_j})$$

E_{a_j} is the fibre of E at a_j, so a $\mathbb{C}$-vector space.

Residues

For each point a_j there is a **residue map**

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The monodromy around a_j is conjugate to $\exp(-2\pi i \operatorname{Res}_{a_j} \nabla)$.

If we are to fix the conjugacy classes of the monodromies, then we need to fix the conjugacy classes of the residues.

We fix a transversal T to $\mathbb{Z}$ in $\mathbb{C}$, and write

$$\operatorname{conn}_{D,T} \mathbb{P}^1$$

for the category of sheaves equipped with a logarithmic connection, all of whose eigenvalues lie in T.

The Riemann – Hilbert Correspondence

We have the fundamental group

$$\pi_1 = \pi_1(\mathbb{P}^1 - D) = \langle g_1, \ldots, g_k \mid g_1 \cdots g_k = 1 \rangle$$

and the Riemann – Hilbert Correspondence says that taking monodromy yields an equivalence of categories

$$\mathrm{conn}_{D,T}\,\mathbb{P}^1 \cong \mathrm{mod}\,\pi_1.$$

In particular, $\mathrm{conn}_{D,T}\,\mathbb{P}^1$ is an abelian category.

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In particular, $\operatorname{conn}_{D,T} \mathbb{P}^1$ is an abelian category.

Take $\zeta_{jt} \in T$ such that $\xi_{jt} = \exp(-2\pi i \zeta_{jt})$. The data ζ_{jt}, d_{jt} for fixed j determines a conjugacy class $\mathcal{C}'_j$.

So we can reformulate the DSP as trying to construct logarithmic connections on locally-free sheaves such that $\operatorname{Res}_{a_j} \nabla \in \mathcal{C}'_j$.

Atiyah and Mihai

There is a functorial morphism (tubular mutation) $\phi_D \colon E(-k) \longrightarrow E$.

Theorem

There is a functorial exact commutative diagram in $\operatorname{coh}\mathbb{P}^1$

$$\begin{tikzcd}
0 \arrow[r] & E(-2) \arrow[r] \arrow[d, "\phi_D"] & A(E) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E(k-2) \arrow[r] & M_D(E) \arrow[r] & E \arrow[r] & 0
\end{tikzcd}$$

The sections of the top row are in bijection with connections on E.
The sections of the bottom row are in bijection with log. conn. on E.

The top row is due to Atiyah. The bottom row to Mihai.

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M.F. Atiyah. Complex analytic connections in fibre bundles,TAMS 85 (1957) 181–207.

A. Mihai. Sur le résidue et la monodromie d'une connexion méromorphe, C.R. Acad. Sc. Paris Sér. A 281 (1975) 435–438.

Residues again

The cokernel of ϕ_D can be identified with $\bigoplus_j E_{a_j}$.

If E has a logarithmic connection ∇, then the composite

$$E(-k) \xrightarrow{\phi_D} E \xrightarrow{\nabla} E(k-2)$$

factors through $\phi_D \colon E(-k) \to E(k-2)$, so we have an induced morphism between the cokernels. This map is the direct sum of the residues

$$\mathrm{Res}_{a_j} \nabla \in \mathrm{End}(E_{a_j}).$$

Parabolic bundles

We want to fix the conjugacy classes of the residues. One way to do this is to equip each fibre E_{a_i} with a flag

$$E_{a_i} = E_{i0} \supset E_{i1} \supset \cdots \supset E_{iw_i} = 0.$$

In other words, we have a **parabolic bundle** (E, E_{it}).

We set $\underline{\dim}(E, E_{it}) = (\operatorname{rank} E, \dim E_{it})$ in $\mathbb{Z}^{Q_0}$.

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A ζ-connection on the parabolic bundle (E, E_{it}) is a logarithmic connection ∇ on E such that

$$(\operatorname{Res}_{a_i} \nabla - \zeta_{it})(E_{it-1}) \subseteq E_{it}.$$

First translation

There is a forgetful functor

$$\mathsf{parbun}_{D,\zeta}\,\mathbb{P}^1 \to \mathsf{conn}_{D,T}\,\mathbb{P}^1$$

where $\mathsf{parbun}_{D,\zeta}\,\mathbb{P}^1$ is the category of parabolic bundles equipped with a ζ-connection.

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Its image is an abelian subcategory, and we have fully faithful left and a right adjoints from the image.

So we almost have a recollement,

but $\mathsf{parbun}_{D,\zeta}\,\mathbb{P}^1$ is only exact, not abelian.

Weighted projective lines

Following Lenzing, there is an equivalence between the category of parabolic bundles and the category of locally free sheaves on the weighted projective line $\mathbb{X}$ (of type D, w).

The category $\operatorname{coh}\mathbb{X}$ of all coherent sheaves on $\mathbb{X}$ is an hereditary abelian category, with finite dimensional homs and exts, and Serre functor

$$E \mapsto E(\omega).$$

Weighted projective lines

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The category $\operatorname{coh}\mathbb{X}$ of all coherent sheaves on $\mathbb{X}$ is an hereditary abelian category, with finite dimensional homs and exts, and Serre functor

$$E \mapsto E(\omega).$$

There is also a forgetful functor

$$\operatorname{coh}\mathbb{X} \longrightarrow \operatorname{coh}\mathbb{P}^1, \quad E \mapsto E_0$$

with fully faithful left and right adjoints (so a recollement).

Zeta connections

Theorem

There is a functorial short exact sequence in $\operatorname{coh} \mathbb{X}$

$$0 \longrightarrow E(\omega) \longrightarrow B_\zeta(E) \longrightarrow E \longrightarrow 0.$$

If E is locally free, then sections are in bijection with ζ-connections on the corresponding parabolic bundle (E_0, E_{it}).

We write $\operatorname{coh}_\zeta \mathbb{X}$ for the abelian category of pairs (E, σ) where σ is section for E.

WCB. Connections for weighted projective lines, J. Pure Appl. Alg. 215 (2011) 35–43.

Second translation

Theorem

$\exists$ *solution* $M_1 \cdots M_k = 1$ *with* $M_i \in \bar{\mathcal{C}}_i$ $\Leftrightarrow$ $\exists\ (E, \sigma) \in \operatorname{coh}_\zeta \mathbb{X}$, *locally free*, $\underline{\dim}\, E = d$

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$\exists$ *solution with* $M_i \in \mathcal{C}_i$	$\Leftrightarrow$	$\exists$ *strict* (E, σ), $\underline{\dim}\, E = d$

We call $(E, \sigma) \in \mathrm{coh}_\zeta \mathbb{X}$ **strict** provided it is locally free and the counit $L(E, \sigma) \to (E, \sigma)$ is an isomorphism.

Second translation

Theorem

$\exists$ *solution* $M_1 \cdots M_k = 1$ *with* $M_i \in \bar{\mathcal{C}}_i$	$\Leftrightarrow$	$\exists$ $(E,\sigma) \in \mathrm{coh}_\zeta \mathbb{X}$, *locally free,* $\underline{\dim} E = d$
$\exists$ *solution with* $M_i \in \mathcal{C}_i$	$\Leftrightarrow$	$\exists$ *strict* (E,σ), $\underline{\dim} E = d$
$\exists$ *irred. solution with* $M_i \in \mathcal{C}_i$	$\Leftrightarrow$	$\exists$ *simple* (E,σ), $\underline{\dim} E = d$

All simple (E,σ) of positive rank are strict.

Existence of sections

The Grothendieck group of $\mathbb{X}$ can be identified with $\mathbb{Z} \oplus \mathbb{Z}^{Q_0}$, where

$$[E] = (\deg E_0, \underline{\dim}\, E).$$

We can then use Serre Duality to obtain

Theorem

There exists a section for E if and only if $\deg E_0' + \zeta * \underline{\dim}\, E' = 0$ for all indecomposable direct summands E' of E.

Here

$$\zeta * f = \sum_{it} \zeta_{it}(f_{it-1} - f_{it}).$$

A simple computation

Recall that

$$\xi_{jt} = \exp(-2\pi i \zeta_{jt})$$

and

$$q_0 = \prod_j \xi_{j1}, \quad q_{jt} = \xi_{jt}/\xi_{jt+1}.$$

Thus

$$\zeta * f \in \mathbb{Z} \quad \Longleftrightarrow \quad q^f = 1.$$

Classes of indecomposable sheaves

The next result is an analogue of Kac's Theorem.

Theorem

There exists an indecomposable $E \in \operatorname{coh} \mathbb{X}$ with $[E] = (m, f)$ if and only if $f \in \Phi^+$ (or $f \in \Phi^-$, $f_0 = 0$, and $m > 0$).

WCB. Kac's Theorem for weighted projective lines, Journal E.M.S. 12 (2010), 1331–1345.

Existence of solutions

Combining these results gives

Theorem

There is a solution to the DSP if and only if $d \in \mathbb{N}\Phi_q^+$.

Existence of solutions

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Theorem

There is a solution to the DSP if and only if $d \in \mathbb{N}\Phi_q^+$.

We can also combine with the results using multiplicative preprojective algebras to get

Theorem

If $d \in \Sigma'_q$, *then there exists an irreducible solution to the DSP.*

Non-existence of simples

For the converse,

> if there is an irreducible solution, then $d \in \Sigma_q$,

we can use the reflection functors (as defined for the multiplicative preprojective algebra) and follow the proof of the additive DSP.

We are reduced to showing that there are no simple objects $(E, \sigma) \in \mathrm{coh}_\zeta \mathbb{X}$ with $\underline{\dim}\, E = f$ in the same three cases (I), (II), (III) described earlier.

Non-existence of simples

(I) Here Q is extended Dynkin, so $\operatorname{coh}\mathbb{X}$ is of tubular type. Let $h \geq 1$ be minimal such that $q^{h\delta} = 1$.

We need to show that if $(E, \sigma) \in \operatorname{coh}_\zeta \mathbb{X}$ has $\underline{\dim}\, E = mh\delta$ with $m \geq 2$, then (E, σ) is not simple.

We can reduce to the case when all indecomposable summands of E lie in a single tube. Passing to a suitable perpendicular category, the result then follows from the deformed preprojective algebra case.

Non-existence of simples

(II) Here $\operatorname{coh}\mathbb{X}$ is of extended tubular type. We can shorten one of the arms by one to obtain $\operatorname{coh}\mathbb{X}'$ of tubular type. These categories are related by a recollement.

This case is much more involved, and again relies on the corresponding deformed preprojective case.

(III) Here f has consecutive 1s on some arm, and is relatively easy.

WCB, A. Hubery. A new approach to simple modules for preprojective algebras, Alg. Rep. Th. 23 (2020) 1849–1860.

WCB, A. Hubery. The Deligne Simpson Problem, preprint.

DSP: Summary

Theorem

There is a solution to the DSP with $M_i \in \bar{\mathcal{C}}_i$ if and only if $d \in \mathbb{N}\Phi_q^+$.

Theorem

There is an irreducible solution with $M_i \in \mathcal{C}_i$ if and only if $d \in \Sigma_q'$.

Happy Birthday, Bill !