Conference in celebration of the work of Bill Crawley-Boevey,

Bielefeld and Manchester, Sept. 1, 2021, 12h-12h50, B. Keller

On Amiot's conjecture

Some memories 1985 – 1989

1985 Bill defends his Ph. D. thesis *Polycyclic-by-Finite Affine Group Schemes and Infinite Soluble Groups*

at Cambridge supervised by Steve Donkin

1986 Bill starts a postdoc with Sheila Brenner and Michael Butler

at Liverpool

(BK starts his Ph. D. thesis at Zurich supervised by P. Gabriel)

Stephen Donkin

At the ICRA in Beijing in 2000

M. Butler (1929-2012) and S. Brenner (1930-2002) at the ICRA in Cocoyoc in 1994

The Brenner-Butler theorem in tilting theory replaces the use of black magic with the use of adjoint functors.

More postdocs at Liverpool from the 80s/90s: Mike Prest, Alastair King, ...

Bill's research at Liverpool: Tame algebras and bocses, Functorial filtrations and the problem of an idempotent and a square-zero matrix ($\rightsquigarrow$ Gelfand's problem), ...

1988 Bill becomes a postdoc at Bielefeld in Ringel's group

Claus Michael Ringel
in 1988

1989 Peter Gabriel invites Bill for a seminar talk at Zurich

Gabriel's motto in interpersonal relations:

Viel Feind, viel Ehr! The more enemies, the more honour!

A quote from Gabriel during a master course (to Paul Schmutz):

Es macht mir nichts aus, Sie zu beleidigen. I do not mind offending you.

Peter Gabriel (1933-2015)

Gabriel did not want to apply the above motto to Bill!

But he was not happy with the subject of Bill's talk:

Regular modules for tame hereditary algebras

He suggested to Bill, as tactfully as he could, to turn to other problems.

One of the other problems Bill turned to:

The structure and representations of preprojective algebras.

This talk is about (two- and three-dimensional, deformed) preprojective algebras.

Amiot's conjecture predicts that all triang. cat. with certain properties can be constructed from deformed 3-dim. preprojective algebras = Ginzburg dg algebras.

Plan: 1. From preprojective algebras to Amiot's conjecture

2. (Constructive) criticism of the conjecture

3. Recent progress

1. From preprojective algebras to Amiot's conjecture

Q a finite connected acyclic quiver, e.g. $1 \xrightarrow{\beta} 2 \xrightarrow{\alpha} 3$.

$k = \mathbb{C}$ for simplicity

$\Pi(Q)$ the preprojective algebra (Gelfand-Ponomarev 1976) of Q over k, e.g.

$$1 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 2 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 3 \quad \text{with } \rho = \sum_{\gamma \in Q_1} [\gamma, \gamma^*] \quad \text{or with } -\beta^*\beta = 0,\ \beta\beta^* - \alpha\alpha^* = 0,\ \alpha\alpha^* = 0.$$

Facts

$\Pi(Q)$ as a right kQ-module

1) $$\Pi(Q)|_{kQ} \cong \bigoplus_{M \in \mathrm{pre}(kQ)} M$$

system of reprs. of the isoclasses of indec. preproj. kQ-modules

So $\dim \Pi(Q) < \infty \underset{\text{Gabriel}}{\iff} Q$ Dynkin isom. to its k-dual as a right module

2) $\dim \Pi(Q) < \infty \Rightarrow \Pi(Q)$ is Frobenius

$\Rightarrow$ $\mathrm{mod}\,\Pi(Q) = \{k\text{-fin.dim. right } kQ\text{-mod.}\}$ is Frobenius

$\xRightarrow{\text{Heller, Happel}}$ $\underline{\mathrm{mod}}\,\Pi(Q) = (\mathrm{mod}\,\Pi(Q))/(\text{proj.-inj.})$ is can. triang.

3) **ON THE EXCEPTIONAL FIBRES OF KLEINIAN SINGULARITIES**

By William Crawley-Boevey

American Journal of Mathematics 122 (2000), 1027–1037.

Lemma 1. *If M and N are finite dimensional $\Pi(Q)$-modules, then*

$$\dim \mathrm{Ext}^1(M,N) = \dim \mathrm{Hom}(M,N) + \dim \mathrm{Hom}(N,M) - (\underline{\dim}\, M, \underline{\dim}\, N).$$

symmetrized Euler form

Consequence: $\dim \mathrm{Ext}^1(M,N) = \dim \mathrm{Ext}^1(N,M)$.

Consequence of Bill's proof:

k-dual $\rightarrow$ $D\mathrm{Ext}^1(M,N) \simeq \mathrm{Ext}^1(N,M) \underset{\text{if } Q \text{ Dynkin}}{\simeq} \underline{\mathrm{Hom}}(N, \Sigma M)$

Reformulation for Q Dynkin: $\underline{\mathrm{mod}}\,\Pi(Q)$ is 2-Calabi-Yau as a triang. cat.

From now on we assume: Q is Dynkin with underlying graph Δ.

4) $\Pi(Q)$ is wild except if $\Delta \in \{A_2, A_3, A_4, D_4, A_5\}$.

5) However, $\Pi(Q)$ is always 2-representation-finite (in the sense of Iyama), i.e. $\underline{\mathrm{mod}}\, \Pi(Q)$ contains a (canonical) cluster-tilting object T.

equivalently: $\mathrm{mod}\, \Pi(Q)$

constructed by Geiss-Leclerc-Schröer (2006, 2007)

Def. (Iyama 2007): T is (2-) cluster-tilting if

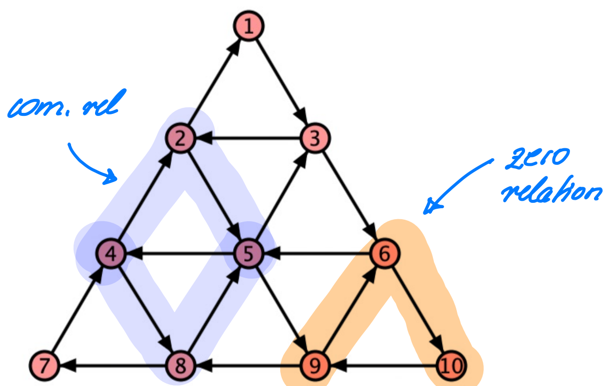
a) T is rigid, i.e. $\mathrm{Ext}^1(T,T) = 0$

b) T is a 2-step generator of $\underline{\mathrm{mod}}\, \Pi(Q)$, i.e. $\forall M \in \underline{\mathrm{mod}}\, \Pi(Q)$, there is a triangle

$$T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow \Sigma T_1$$

with $T_0, T_1 \in \mathrm{add}(T)$.

Example: $Q: 1 \to 2 \to 3 \to 4 \to 5$. Then the canonical cluster-tilting object T of $\underline{\mathrm{mod}}\, \Pi(kR)$ has endomorphism algebra $\underline{\mathrm{End}}(T)$ given by

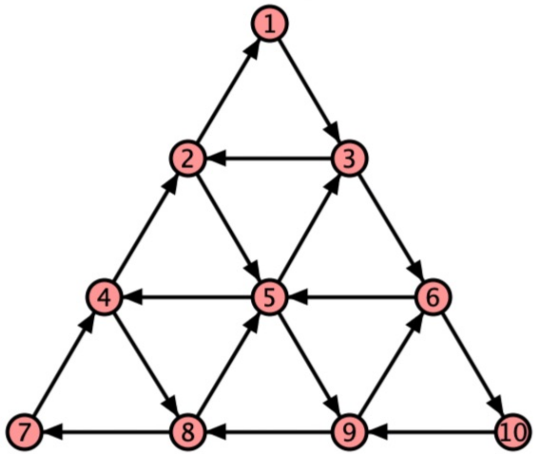


inner arrow $\rightsquigarrow$ commutativity rel.

boundary arrow $\rightsquigarrow$ zero relation

In other words, $\mathrm{End}(T)$ is the Jacobian algebra $J_{R,W}$ of the quiver with potential (R, W) given by

R :



, $W = \sum \triangle - \sum \nabla$

Whence a beautifully compact description of $J_{R,W}$. Such a description exists for $\underline{\mathrm{End}}(T)$, $T \in \underline{\mathrm{mod}}\,\Pi(Q)$, for an arbitrary Dynkin quiver Q.

canonical

Remark: If we know $\underline{\mathrm{End}}(T) \cong J_{R,W}$, we can describe the objects/isom.

of $\underline{\mathrm{mod}}\,\Pi(Q)$ because T is a rigid 2-step generator:

$$\{\text{objects of } \underline{\mathrm{mod}}\,\Pi(Q)\}/\text{isom.} \xleftarrow{\sim} \{\text{minimal morph. } T_1 \xrightarrow{f} T_0 \text{ of } \mathrm{add}(T)\}/\text{isom}$$

$$\mathrm{cone}(f) \longleftarrow\!\shortmid \; f$$

What about the morphisms ?

arbitrary quiver with potential

Important discovery (Victor Ginzburg 2006): Each Jacobian algebra $J_{R,W}$ has a canonical "enhancement" $\Gamma_{R,W}$ (now called the Ginzburg dg algebra).

Rk: $\Gamma_{R,W}$ is a dg (= differential graded) algebra which is connective ($H^p\Gamma_{R,W} = 0$, $\forall p>0$) and has $H^0\Gamma_{R,W} = J_{R,W}$. Moreover, it is (homologically) smooth (Kontsevich) and bimodule 3-CY (VdB 2009).

Thm (Amiot 2009): We have a canonical triangle equivalence

$\underline{\mathrm{mod}}\,\Pi(Q) \xrightarrow{\sim} \mathcal{C}_{R,W}$ where $\mathcal{C}_{R,W}$ is the

(generalized) cluster category $\mathcal{C}_{R,W} = \mathrm{per}(\Gamma_{R,W})/\mathrm{pvd}(\Gamma_{R,W})$.

perfect derived category

perfectly valued der. cat.: $\{M \in \mathcal{D}\Gamma \mid M|_k \in \mathrm{per}\,k\}$

M. Duflo, C. Amiot, B. Leclerc, I. Reiten
11-07-2008

Conjecture (Amiot 2010): Let $\mathcal{T}$ be a Hom-finite, Karoubian, triang. cat. s.th.

a) $\mathcal{T}$ is algebraic (i.e. $\mathcal{T} \xrightarrow[\Delta]{\sim} \underline{\mathcal{E}}$ for some Frob. cat. $\mathcal{E}$).

b) $\mathcal{T}$ is 2-Calabi-Yau as a triang. cat.

c) $\mathcal{T}$ contains a cluster-tilting object T.

Then $\mathcal{S} \xrightarrow[\Delta]{\sim} \mathcal{C}_{R,W}$, $T \longmapsto \Gamma_{R,W}$. In particular,

we have $\operatorname{End}(T) \simeq \operatorname{End}_{\mathcal{C}_{R,W}}(\Gamma_{R,W}) = \mathcal{J}_{Q,R}$.

Evidence accumulated so far:

1) K–Reiten (2008): Ok if $\operatorname{End}(T)$ is hereditary

2) Amiot (2009): $\mathcal{S} = \underline{\mathrm{mod}}\,\Pi(Q)$, Q Dynkin, T canonical

3) Buan-Iyama-Reiten-Smith (2011) combined with Amiot-Reiten-Todorov (2011) and Amiot-Iyama-Reiten-Todorov (2015): $\underline{\mathrm{sub}}(I)$, $\mathrm{sub}(I) \subseteq \mathrm{mod}\,\Pi(Q)$, Q acyclic

4) Amiot-Iyama-Reiten (2011) and Broomhead (2012): categories from dimer models

5) Amiot-Iyama-Reiten (2011) and Thanhoffer-VdB (2016): $\mathcal{S} = \underline{\mathrm{CM}}(R^G)$, $R = k[x,y,z]$, G cyclic

6) Kalck-Yang, Relative singularity categories III: cluster resolutions (preprint 06/2020)

7) Garcia Elsener: Monomial Gorenstein algebras and the stably CY-property (2021)

2. Criticism of the conjecture

Recall:

Conjecture (Amiot 2010): Let $\mathcal{C}$ be a Hom-finite triangulated category s.th.

a) $\mathcal{C}$ is algebraic ($\Leftrightarrow \mathcal{C} \xrightarrow[\Delta]{\sim} \underline{\mathcal{E}}$, $\mathcal{E}$ Frob.) $\Longleftrightarrow$ $\mathcal{C} \xrightarrow[\Delta]{\sim} H^0\mathcal{A}$, $\mathcal{A}$ pretriang. dg cat.

b) $\mathcal{C}$ is 2-Calabi-Yau as a triang. cat.

c) $\mathcal{C}$ contains a cluster-tilting object T.

Then $\exists\, (R,W)$ and $\mathcal{C} \xrightarrow[\Delta]{\sim} \mathcal{C}_{R,W}$, $T \longmapsto \Gamma_{R,W}$

In particular, we have $\mathrm{End}(T) \simeq \mathrm{End}_{\mathcal{C}_{R,W}}(\Gamma_{R,W}) = \mathcal{J}_{W,R}$.

Question: Which of the three assumptions is unnatural ? more precisely: unnaturally weak

Answer: Condition b) because it postulates structure on $H^0\mathcal{A}$ instead of $\mathcal{A}$!

Question: What is the correct lift to $\mathcal{A}$ of the notion of 2-CY-structure on $H^0\mathcal{A}$?

cyclic homology

Def. (Kontsevich): A (right) 2-CY-structure on $\mathcal{A}$ is a class $\eta \in DHC_{-2}(\mathcal{A})$

which is non degenerate, i.e. its image under

$$DHC_{-2}(\mathcal{A}) \longrightarrow DHH_{-2}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}\mathcal{A}^e}(\mathcal{A}, \Sigma^2 D\mathcal{A}^{op})$$

is an isomorphism in $\mathcal{D}\mathcal{A}^e$.

New condition: b') $\mathcal{A}$ (as in a)) carries a 2-CY structure in this sense.

Максим Л. Концевич
1964-

Sanity check :

$\dim J_{R,W} < \infty$

Thm : Suppose (R,W) is a Jacobi-finite quiver with potential. Then $\mathcal{S} = \mathcal{C}_{R,W}$ is Hom-finite and a), b'), c) hold for $\mathcal{A} = (\mathcal{C}_{R,W})_{dg}$ ← can. dg enhancement.

3, Recent progress

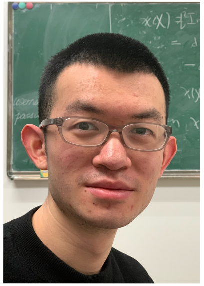
Thm : The modified conjecture is true.

Key ingredients : 1) Localization seq. in cyclic homology (1998).

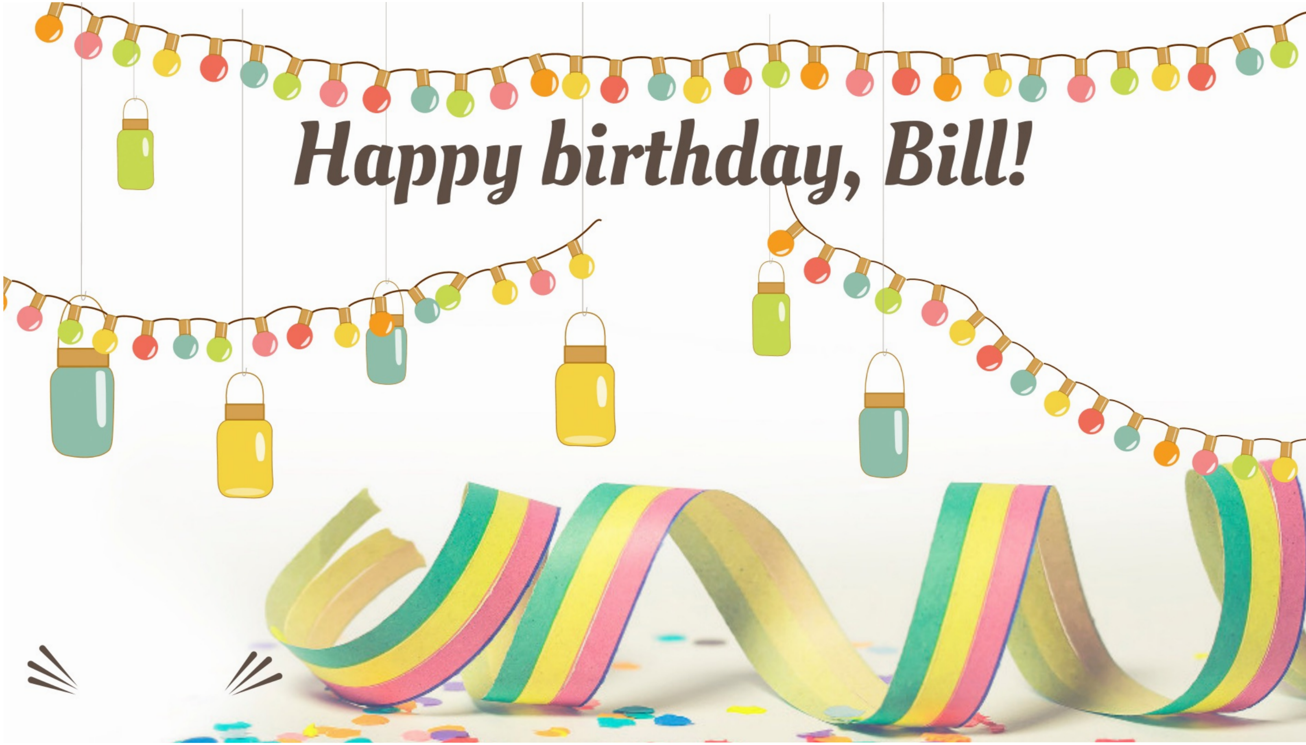
2) Van den Bergh's superpotential Thm (2015) : If A is a (pseudocompact) connective augmented dg algebra with a (left) 3-CY-structure, then A is weakly equivalent to $\Gamma_{R,W}$ for some (R,W).

Remains to be done: If $\mathcal{E}$ is a stably 2-CY cat. occurring in nature, then $\mathcal{A} = (\underline{\mathcal{E}})_{dg}$ carries a can. (right) 2-CY structure.

Rk: This should even hold for relative CY-structures (in the sense of Toën / Brav-Dyckerhoff). Ongoing work by Junyang Liu.



Junyang Liu



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