Categorification of perfect matchings

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[CKP] arXiv:2106.15924

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What is “categorification” here?

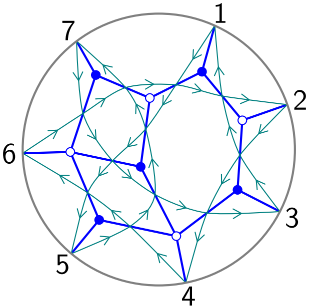
Enriching/explaining combinatorics by representation theory.

Behind the scenes: replacing cluster algebras by cluster categories.

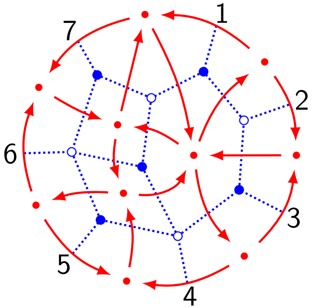
But not higher representation theory [Khovanov, Rouqier, ...]:

> replacing vector spaces by (abelian/triangulated) categories and linear maps by functors.

Context: consistent dimer models on a disc



dual
$\longleftrightarrow$

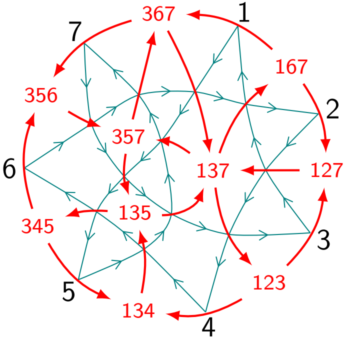


planar bipartite graph G
$+$ Postnikov conditions
on strands (zig-zag paths)

quiver with faces
$Q = (Q_0, Q_1, Q_2)$
$\circ$ = c-wise, $\bullet$ = ac-wise

Combinatorics: left source labelling of quiver vertices

[Postnikov, Scott] Label each quiver vertex $i \in Q_0$ by the *sources* of the strands for which i is on the *left* of the strand.



Ex: Labels J_i are in $\binom{[n]}{k} = \{k\text{-subsets of } \{1, \ldots, n\}\}$, for fixed k, the average increment (mod n) of the strand permutation π. Here $\pi = (2\,4\,6)(1\,5\,7\,3)$, with increments 2, 2, 3, 4, 2, 3, 5.

Hint: Consider the *necklace* $\mathcal{N}_\pi$ of boundary labels.

Geometry: Grassmannians and positroids

The Grassmannian $\mathrm{Gr}_{k,n}$ of (co)dimension k subspaces of $\mathbb{C}^n$ has homogeneous coordinate ring

$$\mathbb{C}[\mathrm{Gr}_{k,n}] = \mathbb{C}[\mathrm{Mat}_{k,n}]^{\mathrm{SL}_k},$$

i.e. SL_k-invariant functions of $k \times n$ matrices,

generated by Plücker coordinates (minors) $\Delta_J : J \in \binom{[n]}{k}$, satisfying quadratic Plücker relations.

The non-negative (real) Grassmannian

$$\mathrm{Gr}_{k,n}^{(\geqslant 0)} = \{w \in \mathrm{Gr}_{k,n}(\mathbb{R}) : \Delta_J(w) \geqslant 0, \ \forall J\} = \bigcup_{\mathcal{P}} \mathrm{Gr}_{\mathcal{P}}^{(>0)}$$

has a stratification indexed by *positroids* $\mathcal{P} \subseteq \binom{[n]}{k}$, where

$$\mathcal{P}(w) = \{J : \Delta_J(w) > 0\}, \text{ for } w \in \mathrm{Gr}_{k,n}^{(\geqslant 0)}.$$

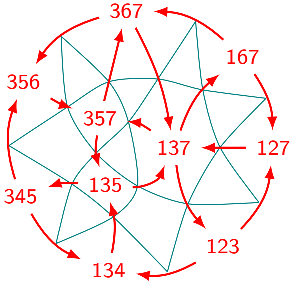
Clusters for positroid strata

A *cluster* $\mathcal{C} \subset \mathcal{P}$ is a subset such that

$$\mathrm{Gr}_{\mathcal{P}}^{(>0)} \to (\mathbb{R}_{>0})^{\mathcal{C}} : w \mapsto (\Delta_J(w))_{J \in \mathcal{C}}$$

is a bijection (i.e. a chart).

[Postnikov] Clusters $\mathcal{C}$ are given by left source labels $\{\mathrm{J}_i : i \in Q_0\}$ for some dimer model on a disc.

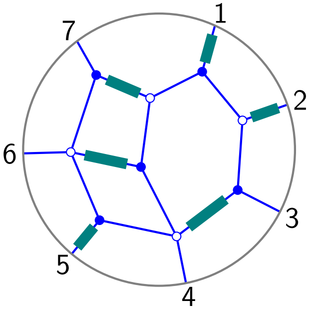


[Postnikov, Lam, Speyer,...] The positroid $\mathcal{P}$ is the set of *boundary values of perfect matchings* for any dimer model with same strand permutation π. The necklace $\mathcal{N}_\pi$ is 'minimal' in the positroid $\mathcal{P}_\pi$.

Uniform (Grassmannian) case: if $\pi(i) = i + k$, then $\mathcal{P}_\pi = \binom{[n]}{k}$.

Perfect matchings on a quiver with faces

A *perfect matching* μ is a collection of arrows in Q such that every face contains one arrow in μ.



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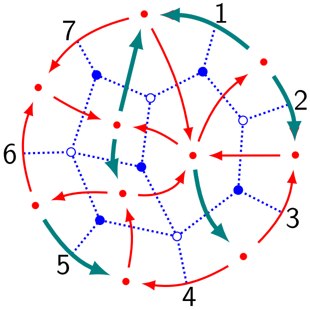
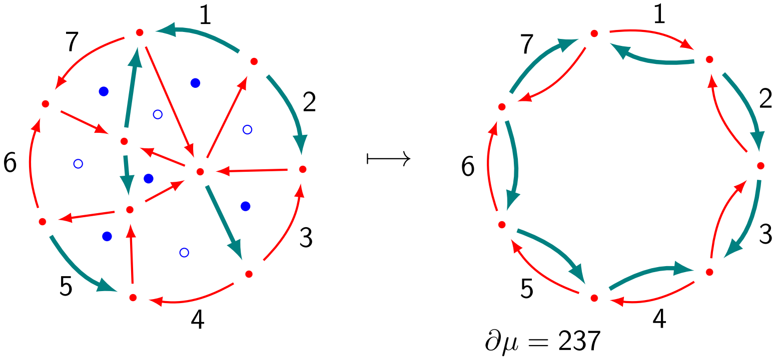


Boundary values of matchings

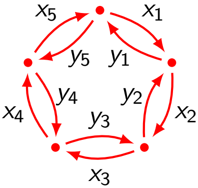
Boundary value $\partial\mu$ is the restriction of μ to the string of digons corresponding to boundary faces of Q. Identify with $J \in \binom{[n]}{k}$ giving matched c-wise arrows. [*Ex:* it's the same k.]

$$k = \#\{\text{c-w faces}\} - \#\{\text{ac-w faces}\} + \#\{\text{ac-w bdry arrows}\}$$



Categorification I: circle algebra [JKS]

Define the *circle algebra* $C = C_{k,n}$ as the (complete) path algebra of a circle of n digons, modulo relations $xy = yx$, $y^k = x^{n-k}$.



Centre $Z = \mathbb{C}[[t]]$ for $t = xy$ and C is a Z-order, in particular, C is free and fin. gen. over Z.

The category CM C of C-modules free and fin. gen. over Z is a Frobenius cluster category, in particular, stably 2-Calabi-Yau.

CM C contains 'rank 1' modules M_J, for all $J \in \binom{[n]}{k}$, with

- Z at each vertex,
- (x_j, y_j) acting by $(t, 1)$, for $j \in J$, or $(1, t)$, for $j \notin J$.

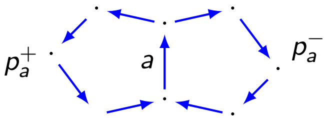
Thus M_J are 'rank 1 matrix factorisations of t' on each digon.

Cluster character $\Psi\colon$ CM $C \to \mathbb{C}[\mathrm{Gr}_{k,n}]$ with $\Psi(M_J) = \Delta_J$.

Categorification II: dimer algebra [BKM]

For a dimer model Q on a disc, the *dimer algebra* $A = A_Q$ is the complete path algebra $\widehat{\mathbb{C}Q}$ modulo relations

$$p_a^+ = p_a^-$$



for each internal arrow $a \in Q_1^{int}$. (Alt: it's a frozen Jacobi algebra).

A central element t acts, at any vertex, as the boundary path of any adjacent face (all equal by relations). Thus A is also a Z-order.

Prop [CKP] Consistency implies that $e_j A e_i = \langle \text{paths } i \text{ to } j \rangle \cong Z$, so projectives Ae_i are rank 1, for all idempotents e_i, $i \in Q_0$.

Categorification III: boundary algebra

Define *boundary algebra* $B = eAe$ from idempotent $e = \sum_{bdry\ i} e_i$.
Note: $B = C$ in uniform (Grassmannian) case.

Thm [CKP] B naturally has C as a subalgebra and the restriction $\mathrm{CM}\, B \to \mathrm{CM}\, C$ is a fully faithful embedding.

Thm [Pres] $T = eA = \bigoplus_{i \in Q_0} eAe_i$ is a cluster tilting object in $\mathrm{GP}\, B$ (Gorenstein projective B-modules), which is Frobenius cluster category contained in $\mathrm{CM}\, B$. Also $A = \mathrm{End}_B(T)^{\mathrm{op}}$.

Prop [CKP] M_J is in $\mathrm{CM}\, B$ if and only if $J \in \mathcal{P}$, the positroid.

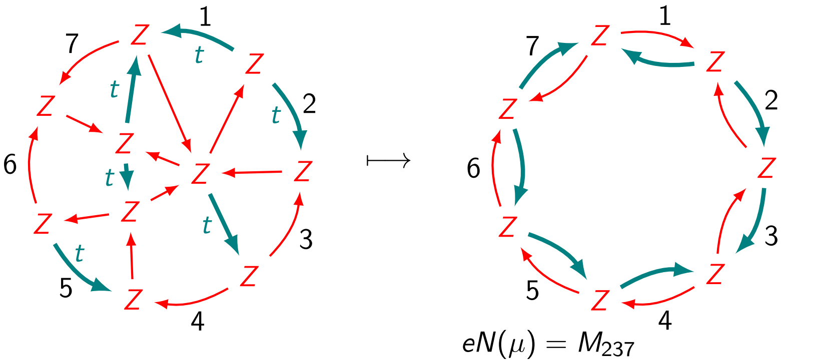
Proof by induction/restriction, i.e. $\mathrm{CM}\, A \to \mathrm{CM}\, B \colon N \mapsto eN$ and its adjoint $\mathrm{CM}\, B \to \mathrm{CM}\, A \colon M \mapsto \mathrm{Hom}_B(T, M)$, we see that

rank 1 $M \in \mathrm{CM}\, B$ are the restrictions of rank 1 $N \in \mathrm{CM}\, A$.

Categorification IV: matching modules [CKP]

Each matching μ gives a rank 1 module $N(\mu) \in \mathrm{CM}\, A$ with Z at each vertex and arrows a acting by t, if $a \in \mu$, or 1, if $a \notin \mu$.

Thm All rank 1 modules have this form. Note: $eN(\mu) \cong M_{\partial\mu}$.



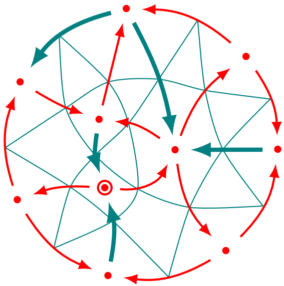
Thus $N(\mu)$ is a ‘rank 1 matrix factorisation of t’ on each face of Q.

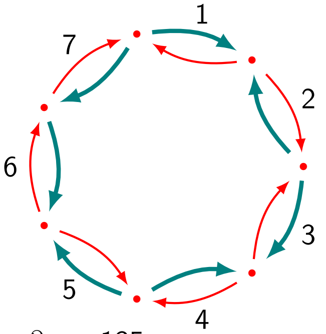
Projective matchings

Question: what are the boundary modules of the projectives, i.e. the summands $T_i = eAe_i$, for $i \in Q_0$?

Projectives Ae_i are rank 1, so $Ae_i \cong N(\mathtt{p}_i)$, for some matching $\mathtt{p}_i$.

$$a \in \mathtt{p}_i \iff \text{minimal path from } i \text{ to } ha \text{ does not go through } ta$$



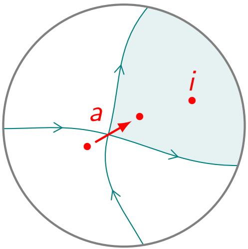


$\partial\mu = 135$

Muller-Speyer's downstream matchings

[Mu-Sp] defined canonical matchings $\mathtt{m}_i$, with $\partial\,\mathtt{m}_i = \mathtt{J}_i$, the left source label, by

$$a \in \mathtt{m}_i \iff i \in \text{d'stream wedge of } a.$$



Thm [CKP] $\mathtt{m}_i = \mathtt{p}_i$.

Cor 1 $T_i = eAe_i = M_{\mathtt{J}_i}$, so $T = \bigoplus_{i \in Q_0} M_{\mathtt{J}_i}$

Cor 2 $B = \bigoplus_{J \in \mathcal{N}_\pi} M_J$,

Punchline: The necklace $\mathcal{N}_\pi$ is the boundary algebra B, whose rank 1 modules are the positroid $\mathcal{P}_\pi$.

Proof via projective resolution

Thm [CKP] Each matching module $N(\mu)$ in CM A has a projective resolution

$$\bigoplus_{\substack{a\in\mu \\ int}} Ae_{ta} \to \bigoplus_{a\notin\mu} Ae_{ha} \to \bigoplus_{i\in Q_0} Ae_i \to N(\mu),$$

so we can compute the class $[N(\mu)] \in \mathrm{K}(\operatorname{Proj} A)$.

Thm [CKP] $[N(\mathtt{m}_i)] = [Ae_i]$ and thus (with work) $N(\mathtt{m}_i) \cong Ae_i$.
Note: [Mu-Sp] knows this as a combinatorial fact.

Question: is there a more direct combinatorial proof that $\mathtt{m}_i = \mathtt{p}_i$, and thus that the boundary values of the projective matchings are the left source labels?

Some references

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[Mu-Sp] G. Muller, D.E. Speyer, *The twist for positroid varieties*, Proc. L.M.S. **115** (2017) 1014–1071 (arXiv:1606.08383)

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