Categorification of perfect matchings

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What is “categorification” here?

Enriching/explaining combinatorics by representation theory.

Behind the scenes: replacing cluster algebras by cluster categories.

But not higher representation theory [Khovanov, Rouquier, ...]:

replacing vector spaces by (abelian/triangulated) categories
and linear maps by functors.
Context: consistent dimer models on a disc

planar bipartite graph $G$
+ Postnikov conditions on strands (zig-zag paths)

quiver with faces $Q = (Q_0, Q_1, Q_2)$
$\circ = c$-wise, $\bullet = ac$-wise
Combinatorics: left source labelling of quiver vertices

[Postnikov, Scott] Label each quiver vertex \( i \in Q_0 \) by the sources of the strands for which \( i \) is on the left of the strand.

**Ex:** Labels \( J_i \) are in \( {[n]\choose k} = \{ k\text{-subsets of } \{1, \ldots, n\}\} \), for fixed \( k \), the average increment (mod \( n \)) of the strand permutation \( \pi \). Here \( \pi = (246)(1573) \), with increments 2, 2, 3, 4, 2, 3, 5.

**Hint:** Consider the necklace \( \mathcal{N}_\pi \) of boundary labels.
The Grassmannian $\text{Gr}_{k,n}$ of (co)dimension $k$ subspaces of $\mathbb{C}^n$ has homogeneous coordinate ring

$$\mathbb{C}[\text{Gr}_{k,n}] = \mathbb{C}[[\text{Mat}_{k,n}]]_{\text{SL}_k},$$

i.e. $\text{SL}_k$-invariant functions of $k \times n$ matrices, generated by Plücker coordinates (minors) $\Delta_J : J \in \binom{[n]}{k}$, satisfying quadratic Plücker relations.

The non-negative (real) Grassmannian

$$\text{Gr}_{k,n}^{(\geq 0)} = \{ w \in \text{Gr}_{k,n}(\mathbb{R}) : \Delta_J(w) \geq 0, \ \forall J \} = \bigcup_{\mathcal{P}} \text{Gr}_{\mathcal{P}}^{(> 0)}$$

has a stratification indexed by $\textit{positroids} \ \mathcal{P} \subseteq \binom{[n]}{k}$, where

$$\mathcal{P}(w) = \{ J : \Delta_J(w) > 0 \}, \ \text{for} \ w \in \text{Gr}_{k,n}^{(\geq 0)}.$$
Clusters for positroid strata

A \textit{cluster} \( C \subset \mathcal{P} \) is a subset such that

\[
\text{Gr}^{(>0)}_{\mathcal{P}} \rightarrow (\mathbb{R}_{>0})^C : w \mapsto (\Delta J(w))_{J \in C}
\]

is a bijection (i.e. a chart).

[Postnikov] Clusters \( C \) are given by left source labels \( \{J_i : i \in Q_0\} \) for some dimer model on a disc.

[Postnikov, Lam, Speyer,...] The positroid \( \mathcal{P} \) is the set of \textit{boundary values of perfect matchings} for any dimer model with same strand permutation \( \pi \). The necklace \( \mathcal{N}_\pi \) is ‘minimal’ in the positroid \( \mathcal{P}_\pi \).

Uniform (Grassmannian) case: if \( \pi(i) = i + k \), then \( \mathcal{P}_\pi = \binom{[n]}{k} \).
Perfect matchings on a quiver with faces

A *perfect matching* $\mu$ is a collection of arrows in $Q$ such that every face contains one arrow in $\mu$. 
Boundary values of matchings

Boundary value $\partial \mu$ is the restriction of $\mu$ to the string of digons corresponding to boundary faces of $Q$. Identify with $J \in \binom{[n]}{k}$ giving matched c-wise arrows. [Ex: it’s the same $k$.]

$$k = \#\{\text{c-w faces}\} - \#\{\text{ac-w faces}\} + \#\{\text{ac-w bdry arrows}\}$$

![Diagram](image-url)
Define the circle algebra $C = C_{k,n}$ as the (complete) path algebra of a circle of $n$ digons, modulo relations $xy = yx$, $y^k = x^{n-k}$.

Centre $Z = \mathbb{C}[[t]]$ for $t = xy$ and $C$ is a $Z$-order, in particular, $C$ is free and fin. gen. over $Z$.

The category $\text{CM}_C$ of $C$-modules free and fin. gen. over $Z$ is a Frobenius cluster category, in particular, stably 2-Calabi-Yau.

$\text{CM}_C$ contains ‘rank 1’ modules $M_J$, for all $J \in \binom{[n]}{k}$, with
- $Z$ at each vertex,
- $(x_j, y_j)$ acting by $(t, 1)$, for $j \in J$, or $(1, t)$, for $j \not\in J$.

Thus $M_J$ are ‘rank 1 matrix factorisations of $t$’ on each digon.

Cluster character $\Psi : \text{CM}_C \to \mathbb{C}[\text{Gr}_{k,n}]$ with $\Psi(M_J) = \Delta_J$. 
For a dimer model $Q$ on a disc, the *dimer algebra* $A = A_Q$ is the complete path algebra $\widehat{\mathbb{C}Q}$ modulo relations

$$p_a^+ = p_a^-$$

for each internal arrow $a \in Q_1^{int}$. (Alt: it’s a frozen Jacobi algebra).

A central element $t$ acts, at any vertex, as the boundary path of any adjacent face (all equal by relations). Thus $A$ is also a $\mathbb{Z}$-order.

**Prop [CKP]** Consistency implies that $e_iAe_j = \langle \text{paths } i \text{ to } j \rangle \cong \mathbb{Z}$, so projectives $Ae_i$ are rank 1, for all idempotents $e_i$, $i \in Q_0$. 
Categorification III: boundary algebra

Define \textit{boundary algebra} $B = eAe$ from idempotent $e = \sum_{bdry \ i} e_i$.

Note: $B = C$ in uniform (Grassmannian) case.

\textbf{Thm \ ([CKP])} $B$ naturally has $C$ as a subalgebra and the restriction $\text{CM } B \to \text{CM } C$ is a fully faithful embedding.

\textbf{Thm \ ([Pres])} $T = eA = \bigoplus_{i \in Q_0} eAe_i$ is a cluster tilting object in $\text{GP } B$ (Gorenstein projective $B$-modules), which is Frobenius cluster category contained in $\text{CM } B$. Also $A = \text{End}_B(T)^\text{op}$.

\textbf{Prop \ ([CKP])} $M_J$ is in $\text{CM } B$ if and only if $J \in \mathcal{P}$, the positroid.

\textbf{Proof} by induction/restriction, i.e. $\text{CM } A \to \text{CM } B: N \mapsto eN$ and its adjoint $\text{CM } B \to \text{CM } A: M \mapsto \text{Hom}_B(T, M)$, we see that rank 1 $M \in \text{CM } B$ are the restrictions of rank 1 $N \in \text{CM } A$. 
Categorification IV: matching modules [CKP]

Each matching $\mu$ gives a rank 1 module $N(\mu) \in \text{CM} A$ with $Z$ at each vertex and arrows $a$ acting by $t$, if $a \in \mu$, or 1, if $a \notin \mu$.

**Thm** All rank 1 modules have this form. Note: $eN(\mu) \cong M_{\partial \mu}$.

Thus $N(\mu)$ is a ‘rank 1 matrix factorisation of $t$’ on each face of $Q$. 
Projective matchings

**Question:** what are the boundary modules of the projectives, i.e. the summands \( T_i = eAe_i \), for \( i \in Q_0 \)?

Projectives \( Ae_i \) are rank 1, so \( Ae_i \cong N(p_i) \), for some matching \( p_i \).

\[ a \in p_i \iff \text{minimal path from } i \text{ to } ha \text{ does not go through } ta \]
Muller-Speyer’s downstream matchings

[[Mu-Sp]] defined canonical matchings $m_i$, with $\partial m_i = J_i$, the left source label, by

$$a \in m_i \iff i \in \text{d’stream wedge of } a.$$ 

**Thm** [[CKP]] $m_i = p_i$.

**Cor 1** $T_i = eAe_i = M_{J_i}$, so $T = \bigoplus_{i \in Q_0} M_{J_i}$

**Cor 2** $B = \bigoplus_{J \in N_{\pi}} M_{J}$.

**Punchline:** The necklace $N_{\pi}$ is the boundary algebra $B$, whose rank 1 modules are the positroid $P_{\pi}$. 
Proof via projective resolution

Thm [CKP] Each matching module $N(\mu)$ in CM $A$ has a projective resolution

\[ \bigoplus_{a \in \mu \text{ int}} A_{e_{ta}} \rightarrow \bigoplus_{a \notin \mu} A_{e_{ha}} \rightarrow \bigoplus_{i \in Q_0} A_{e_i} \rightarrow N(\mu), \]

so we can compute the class $[N(\mu)] \in K(\text{Proj } A)$.

Thm [CKP] $[N(m_i)] = [A_{e_i}]$ and thus (with work) $N(m_i) \cong A_{e_i}$. Note: [Mu-Sp] knows this as a combinatorial fact.

Question: is there a more direct combinatorial proof that $m_i = p_i$, and thus that the boundary values of the projective matchings are the left source labels?
Some references


