Uniqueness of exact Borel subalgebras and bocses

(joint work with Vanessa Miemietz)

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Bill in July 2009

Quivers and relations

Theorem (Morita '58)

For every finite dimensional algebra Λ *there is a basic algebra* Λ^b *such that* $\operatorname{mod}\Lambda \cong \operatorname{mod}\Lambda^b$*. If* Λ *is Morita equivalent to* Γ*, then* $\Lambda^b \cong \Gamma^b$*.*

Theorem (Gabriel's structure theorem '73)

Over $\Bbbk = \overline{\Bbbk}$*, every basic algebra is isomorphic to* $\Bbbk Q/I$ *for* Q *a finite quiver and* I *an admissible ideal.*

Lemma (Govorov '73, Butler, cf. Bongartz '83)

$$\mathbb{D}\operatorname{Ext}^1(L(\mathtt{i}), L(\mathtt{j})) \cong e_{\mathtt{j}} Q_+/Q_+^2 e_{\mathtt{i}} \qquad \textit{arrows}$$

$$\mathbb{D}\operatorname{Ext}^2(L(\mathtt{i}), L(\mathtt{j})) \cong e_{\mathtt{j}} I/(IQ_+ + Q_+ I) e_{\mathtt{i}} \qquad \textit{relations}$$

Koszul algebras

How to reconstruct the relations from Ext^2 between simples?

Definition

Let $A = \Bbbk Q/I$ be a graded algebra, $\deg \alpha = 1$ for $\alpha \in Q_1$. Then A is **Koszul** if $\mathrm{Ext}_A^{*,\bullet}(\mathbb{L}, \mathbb{L})$ is generated by degree $(1, 1)$ as an algebra.

Theorem (Beilinson–Ginzburg–Soergel ’96)

If A is a basic Koszul algebra, then A is isomorphic to the quadratic dual of $\mathrm{Ext}_A^{*}(\mathbb{L}, \mathbb{L})$*. In fact, this uses only* Ext^1 *and* Ext^2*.*

What about the general case? Most algebras are not Koszul!

A_∞-algebras

Definition (Stasheff '63)

An A_∞**-algebra** is a graded vector space $\mathscr{E}$ together with graded linear maps $m_n \colon \mathscr{E}^{\otimes n} \to \mathscr{E}$ of degree $2-n$ such that for all $n \geq 1$:

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$

- $n=1$: $m_1 m_1 = 0$, cochain complex;
- $n=2$: $m_1 m_2 - m_2(\mathrm{id} \otimes m_1 + m_1 \otimes \mathrm{id}) = 0$, graded Leibniz rule;
- $n=3$: $m_1 m_3 + m_3 m_1^{\otimes} + m_2(m_2 \otimes \mathrm{id} - \mathrm{id} \otimes m_2) = 0$, associative up to homotopy, etc.

A_∞-morphisms

Definition

Given A_∞-algebras $\mathscr{E}$ and $\mathscr{E}'$, an A_∞**-morphism** $\mathscr{E} \to \mathscr{E}'$ is given by graded linear maps $f_n \colon \mathscr{E}^{\otimes n} \to \mathscr{E}'$ of degree $1-n$ such that for all $n \geq 1$:

$$\begin{aligned} &\sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) \\ = &\sum_{\sum j_\ell = n} (-1)^{\Sigma(\ell-i)(j_i-1)} m_k(f_{j_1} \otimes \cdots \otimes f_{j_k}) \end{aligned}$$

Lemma

f is an isomorphism iff f_1 is an isomorphism.

Definition

f is an A_∞**-quasi-isomorphism** if f_1 is a quasi-isomorphism.

Relation to dg algebras

There are two important cases in which an A_∞-algebra happens to be associative. We will only deal with such cases.

Definition

An A_∞-algebra is called a **dg algebra** if $m_n = 0$ for all $n \geq 3$. An A_∞-algebra is called **minimal** if $m_1 = 0$.

Lemma

If $\mathscr{E}$ and $\mathscr{E}'$ are minimal A_∞-algebras, then every A_∞-quasi-isomorphism between $\mathscr{E}$ and $\mathscr{E}'$ is an isomorphisms.

Kadeishvili's theorem

Theorem (Kadeishvili '83)

Let $\mathscr{D}$ be a dg algebra. Then there exists an A_∞-structure on $H^*(\mathscr{D})$ together with an A_∞-quasi-isomorphism $H^*(\mathscr{D}) \to \mathscr{D}$. This structure is unique up to (non-unique) isomorphism.

We apply this to

$$\mathscr{D} = \mathrm{Hom}_A^*(P_M^\bullet, P_M^\bullet)$$

and its homology

$$\mathscr{E} = \mathrm{Ext}^*(M, M).$$

Call an A_∞-structure arising this way **canonical.**

Construction of quiver and relations

Theorem (Keller '99)

Let $A = \Bbbk Q/I$, then A is quasi-isomorphic to the dual bar construction, i.e. A_∞-Koszul dual, of $\mathrm{Ext}^*(\mathbb{L},\mathbb{L})$.

Theorem (Keller '01)

Given a presentation $A = \Bbbk Q/I$, then there is a canonical A_∞-structure on $\mathrm{Ext}_A^*(\mathbb{L},\mathbb{L})$ *and a splitting* $I/(IQ_+ + Q_+I) \to I \hookrightarrow \Bbbk Q$ *which can be identified with the dual of the map*

$$(m_n)_{n\in\mathbb{N}}\colon \bigoplus_n \mathrm{Ext}_A^1(\mathbb{L},\mathbb{L})^{\otimes n} \to \mathrm{Ext}_A^2(\mathbb{L},\mathbb{L}).$$

Uniqueness of the basic algebra follows from uniqueness in Kadeishvili's theorem.

Example

Let $A = \Bbbk(\ 1 \xrightarrow{\gamma} 2 \xrightarrow{\beta} 3 \xrightarrow{\alpha} 4\ )/(\alpha\beta\gamma)$.

Then
$$\mathrm{Ext}^*_A(\mathbb{L},\mathbb{L}) \cong \Bbbk\left(\begin{tikzcd} 1 \arrow[r, "c"] \arrow[rrr, dotted, bend right, "r"'] & 2 \arrow[r, "b"] & 3 \arrow[r, "a"] & 4 \end{tikzcd}\right)/(cb, ba)$$

with $m_3(a,b,c) = r$.

Exceptional collections and quasi-hereditary algebras

Definition (Beilinson '78, Bondal '89)

$\Delta(\mathtt{1}), \ldots, \Delta(\mathtt{n})$ form an **exceptional collection** if

- $\mathrm{End}(\Delta(\mathtt{i})) \cong \Bbbk$, and $\mathrm{Ext}^k(\Delta(\mathtt{i}), \Delta(\mathtt{i})) = 0$ for $k \neq 0$;
- $\mathrm{Ext}^k(\Delta(\mathtt{i}), \Delta(\mathtt{j})) = 0$ for $\mathtt{i} > \mathtt{j}$ and all k.

Bill investigated these for path algebras of quivers in '93.

Definition (Cline–Parshall–Scott '88)

An algebra Λ is **quasi-hereditary** if there is an exceptional collection of modules such that Λ has a filtration with subquotients isomorphic to Δ.

Example

Blocks of BGG category $\mathscr{O}$, Schur algebras, $\mathrm{gldim} \leq 2$.

Short summary of the rest of the talk

- Everything that could be done before for $\mathrm{Ext}^*_A(\mathbb{L},\mathbb{L})$ can be done for quasi-hereditary algebras for $\mathrm{Ext}^*_\Lambda(\Delta,\Delta)$ as well.
- The only difference is that everything gets much more technical because $\mathrm{Hom}_\Lambda(\Delta,\Delta) \neq 0$.
- At the moment, we don't know of a different route to prove uniqueness.

Regular exact Borel subalgebras

Definition (Koenig ’95, Kleiner–Roiter ’75)

Let R be a quasi-hereditary algebra. A subalgebra $B \subseteq R$ is called an **exact Borel subalgebra** if

- B has the same indexing set of simple modules;
- B is quasi-hereditary with simple standard modules;
- $R \otimes_B -$ is exact;
- $R \otimes_B L_B(\mathtt{i}) \cong \Delta_R(\mathtt{i})$.

It is called **regular** (resp. **basic**) if in addition

- $\mathrm{Ext}_B^{\geq 1}(L(\mathtt{i}), L(\mathtt{j})) \to \mathrm{Ext}_R^{\geq 1}(\Delta(\mathtt{i}), \Delta(\mathtt{j}))$ is an isomorphism for all $\mathtt{i}, \mathtt{j}$ (and B is basic).

Analogue of Gabriel's structure theorem

Theorem (Koenig–K–Ovsienko '14)

For every quasi-hereditary algebra Λ, there is a Morita equivalent algebra R which has a regular basic exact Borel subalgebra.

Theorem (K–Miemietz '21, cf. Conde '20)

This algebra extension is unique up to isomorphism, i.e. if R and S are Morita equivalent quasi-hereditary algebras with regular basic exact Borel subalgebras A and B, respectively, then there exists an algebra isomorphism from R to S sending A to B.

Conde has an inductive combinatorial formula computing R from Λ.

Analogue of the Govorov, Butler–Bongartz

It is convenient to work with $V = \mathrm{Hom}_{B^{\mathrm{op}}}(R, B)$.

Lemma (Ovsienko (unpublished), cf. K–Miemietz '21)

Let R be a quasi-hereditary algebra with regular basic exact Borel subalgebra $B \cong \Bbbk Q/I$, then

$$\mathbb{D}\,\mathrm{Ext}^1_R(\Delta(\mathtt{i}), \Delta(\mathtt{j})) \cong e_{\mathtt{j}} Q_+/Q_+^2 e_{\mathtt{i}}$$
$$\mathbb{D}\,\mathrm{Ext}^2_R(\Delta(\mathtt{i}), \Delta(\mathtt{j})) \cong e_{\mathtt{j}} I/(IQ_+ + Q_+ I) e_{\mathtt{i}}$$
$$\mathbb{D}\,\mathrm{Hom}_R(\Delta(\mathtt{i}), \Delta(\mathtt{j})) \cong e_{\mathtt{j}} V/(VJ_B + J_B V) e_{\mathtt{i}}$$

Uniqueness of regular exact Borel subalgebras

Uniqueness of B up to isomorphism follows from

Theorem (K–Miemietz '21)

Let R be a quasi-hereditary algebra with basic regular exact Borel subalgebra B, then

$$\operatorname{Ext}_B^{\geq 1}(\mathbb{L},\mathbb{L}) \cong \operatorname{Ext}_R^{\geq 1}(\Delta,\Delta)$$

as (non-unital) A_∞-algebras (for a certain compatible choice of canonical A_∞-structures).

Construction via A_∞-Koszul duality

By results of Sweedler, Kleiner, Burt–Butler, Roiter, Brzeziński–Koenig–K, and K–Miemietz it suffices to construct a dg algebra since

regular exact Borel subalgebras
$\leftrightarrow$ regular directed corings (B, V)
$\leftrightarrow$ semifree directed dg algebras $(T_B(\overline{V}), \partial)$

Theorem (Koenig–K–Ovsienko '14)

The semifree dg algebra needed is the quotient of the dual bar construction of $\mathrm{Ext}^*_\Lambda(\Delta, \Delta)$ *by the differential ideal generated by the negative degree part.*

An inverse construction

Theorem (K–Miemietz '21)

Let $B \subseteq R$ *be a regular basic exact Borel subalgebra and let* $V = \mathrm{Hom}_{B^{\mathrm{op}}}(R, B)$ *and* $\overline{V}$ *be the kernel of the counit. Then given a presentation* $B = \Bbbk Q/I$, *there is a canonical* A_∞*-structure and a splitting* $I/(IQ_+ + Q_+I) \to \Bbbk Q$ *which can be identified with the dual of the map*

$$(m_n)_{n\in\mathbb{N}} \colon \bigoplus_n \mathrm{Ext}^1_R(\Delta,\Delta)^{\otimes n} \to \mathrm{Ext}^2_R(\Delta,\Delta)$$

...

contd.

and a splitting $A \to \Bbbk Q$ such that

$$(m_n)_{n\in\mathbb{N}}\colon \bigoplus_{i+j=n-1} (\mathrm{Ext}^1_R)^{\otimes i} \otimes \mathrm{rad} \otimes (\mathrm{Ext}^1_R)^{\otimes j} \to \mathrm{Ext}^1$$

can be identified with the dual of

$$Q_+/Q_+^2 \to B \xrightarrow{\partial_0} \overline{V} \cong B \otimes \Phi \otimes B \to \Bbbk Q \otimes \Phi \otimes \Bbbk Q, \text{ and}$$

$$\bigoplus_{i+j+k=n-2} (\mathrm{Ext}^1_R)^{\otimes i} \otimes \mathrm{rad} \otimes (\mathrm{Ext}^1_R)^{\otimes j} \otimes \mathrm{rad} \otimes (\mathrm{Ext}^R_1)^{\otimes k} \xrightarrow{(m_n)_{n\in\mathbb{N}}} \mathrm{rad}$$

can be identified with the dual of

$$\Phi \to \overline{V} \xrightarrow{\partial_1} \overline{V} \otimes_B \overline{V} \cong B \otimes \Phi \otimes B \otimes \Phi \otimes B \to \Bbbk Q \otimes \Phi \otimes \Bbbk Q \otimes \Phi \otimes \Bbbk Q$$

Dual extension algebra [Xi '95, Thuresson '20]

$$\Lambda =$$

$$\begin{tikzcd} \Bbbk(& 1 \arrow[r, shift left, "\gamma"] & 2 \arrow[l, shift left, "\gamma'"] \arrow[r, shift left, "\beta"] & 3 \arrow[l, shift left, "\beta'"] \arrow[r, shift left, "\alpha"] & 4 \arrow[l, shift left, "\alpha'"] &)/(\alpha\beta\gamma, \gamma'\beta'\alpha', \gamma\gamma', \beta\beta', \alpha\alpha') \end{tikzcd}$$

Then

$$\begin{tikzcd} \mathrm{Ext}^*_\Lambda(\Delta, \Delta) \cong \Bbbk(& 1 \arrow[r, shift left, "a"] \arrow[r, shift right, dashed, "\varphi"'] \arrow[rrr, bend left, dotted, "r"] & 2 \arrow[r, shift left, "a"] \arrow[r, shift right, dashed, "\varphi"'] & 3 \arrow[r, shift left, "a"] \arrow[r, shift right, dashed, "\varphi"'] & 4 &)/(a^2, \varphi^3, \varphi a). \end{tikzcd}$$

The unique regular basic exact Borel subalgebra is

$$\begin{tikzcd} B = \Bbbk(& 1 \arrow[r, "a"'] \arrow[rr, bend left] \arrow[rrr, bend right] & 2 \arrow[r, "a"'] \arrow[rr, bend left] & 3 \arrow[r, "a"'] & 4 &)/(a^3) \text{ as a subalgebra of} \end{tikzcd}$$

$$R \cong \mathrm{End}_\Lambda(P(1) \oplus P(2) \oplus 2P(3) \oplus 4P(4))^{\mathrm{op}}.$$