Quiver Heisenberg algebras and rad$^n$-approximations
(a joint work with M. Herschend)

at a conference in celebration of the work of Bill Crawley-Boevey

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1. Introduction

Setup.

- \( k \): an algebraically closed field of arbitrary characteristic.
- \( Q \): a finite acyclic quiver.

1.1. The preprojective algebras \( \Pi(Q) \) of \( Q \).

We denote by \( \overline{Q} \) the double of \( Q \).

![Diagram](image)

Recall that for a vertex \( i \in Q_0 \), the mesh relation \( \rho_i \) is defined by

\[
\rho_i := \sum_{\alpha \in Q_1 : t(\alpha) = i} \alpha \alpha^* - \sum_{\alpha \in Q_1 : h(\alpha) = i} \alpha^* \alpha.
\]

The following is also called the mesh relation:

\[
\rho := \sum_{i \in Q_0} \rho_i.
\]

The preprojective algebra is defined to be the path of the double quiver \( \overline{Q} \) with the mesh relations:

\[
\Pi(Q) = \frac{k\overline{Q}}{(\rho)} = \frac{k\overline{Q}}{(\rho_i \mid i \in Q_0)}.
\]

We equip \( \overline{Q} \) with the grading \((\ast\text{-}grading)\)

\[
\deg^* \alpha := 0, \quad \deg^* \alpha^* := 1 \text{ for } \alpha \in Q_1.
\]

Then \( \deg^* \rho_i = 1 \) and \( \Pi(Q) \) is a \( \ast \)-graded algebra.

We have \( \Pi(Q)_0 = kQ \).
1.2. Quiver Heisenberg Algebras $^v\Lambda(Q)$.

The quiver Heisenberg algebra $^v\Lambda(Q)$ has a parameter $v \in k^\times Q_0$, i.e., a collection $v = (v_i)_{i \in Q_0}$ of non-zero element of $k$ indexed by $Q_0$.

**Definition.** Let $v \in k^\times Q_0$.

1. For $i \in Q_0$, we set
   
   $v_i := v_i^{-1} \rho_i, \quad v := \sum_{i \in Q_0} v_i = \sum_{i \in Q_0} v_i^{-1} \rho_i$.

2. For $a \in \overline{Q_1}$, the quiver Heisenberg relation $^v\eta_a$ is defined to be
   
   $v \eta_a := [a, v \varrho] = a^v \varrho - v \varrho a = v_{h(a)}^{-1} a \rho_{h(a)} - v_{t(a)}^{-1} \rho_{t(a)} a$.

3. We define the quiver Heisenberg algebra $^v\Lambda(Q)$ to be
   
   the path algebra of $Q$ with the quiver Heisenberg relations:
   
   $^v\Lambda(Q) := \frac{kQ}{(v\eta_a | a \in \overline{Q_1})}$.

- The QH-relations $^v\eta_a$ are homogeneous w.r.t $*$-grading.
- Hence $^v\Lambda(Q)$ is a $*$-graded algebra. We have $^v\Lambda(Q)_0 = kQ$.
- The element $^v\varrho$ is central in $^v\Lambda(Q)$ and $\Pi(Q) = ^v\Lambda(Q)/(^v\varrho)$.

Hence there is an exact seq of $*$-graded $^v\Lambda$-bimodules:

$$^v\Lambda(-1) \xrightarrow{v\varrho} ^v\Lambda \xrightarrow{v\pi} \Pi \longrightarrow 0$$

where $(-1)$ denote the shift of $*$-degree by $-1$.

**Remark.** Originally, I and Martin studied the case $v = (1, 1, \cdots, 1)$.

In that case, the quiver Heisenberg relation $^v\eta_a = [a, \rho]$ can be looked as a quiver version of the Heisenberg relations $[x, [x, y]], [y, [x, y]]$.

Hence the name quiver Heisenberg algebras.
1.3. Related algebras and preceding results.

We point out the following isomorphism of algebras:

\[ ^v\Lambda(Q) \cong \frac{k[z][Q]}{(\rho_i - (v_i z)e_i \mid i \in Q_0)}, \]

(where \( e_i \) is the idempotent element corresponding to \( i \in Q_0 \))

from which we see that \( ^v\Lambda(Q) \) is a special case of

- The central extension of the preprojective algebras
  by Etingof-Rains (2006)

\[ \Pi(Q)_{\lambda,\mu} := \frac{k[z][Q]}{(\rho_i - (\lambda_i z + \mu_i)e_i \mid i \in Q_0)} \]

where \( \lambda_i, \mu_i \in k \) for each \( i \in Q_0 \).

This algebra is a special case of the following algebra.

- The \( N = 1 \)-quiver algebra by Cachazo-Katz-Vafa (2001)

\[ \Pi(Q)_P := \frac{k[z][Q]}{(\rho_i - P_i(z)e_i \mid i \in Q_0)} \]

where \( P_i(z) \in k[z] \) for each \( i \in Q_0 \).

This algebra is obtained as a pull-back of

- The deformation family of the preprojective algebras
  by Crawley-Boevey-Holland (1998)

\[ \Pi(Q)_\bullet := \frac{k[z_1, \ldots, z_r][Q]}{(\rho_i - z_ie_i \mid i \in Q_0)} \]

where \( r = \#Q_0 \).
**Theorem** (Etingof-Rains). Assume $\text{char } k = 0$.

If $Q$ is a Dynkin quiver with the Coxeter number $h$ and $r := \#Q_0$, then for generic $v \in k^\times Q$,

$$\dim v^\Lambda(Q) = \sum_{M \in \text{ind } Q} (\dim M)^2 = \frac{rh^2(h + 1)}{12}.$$

**Theorem** (Herschend-M). Assume $\text{char } k = 0$ and $v = (1, 1, \cdots, 1)$.

Then as $kQ$-modules,

$$v^\Lambda(Q)e_i \cong \bigoplus_{M \in \text{ind } P(Q)} M^{\oplus \dim e_i M} \quad (\spadesuit)$$

where $P(Q)$ denotes the set of the preprojective modules of $kQ$.

**Corollary**. Assume $\text{char } k = 0$ and $v = (1, 1, \cdots, 1)$.

1. $v^\Lambda(Q) \cong \bigoplus_{M \in \text{ind } P(Q)} M^{\oplus \dim M}.$

2. If $Q$ is a Dynkin quiver, then

$$v^\Lambda(Q) \cong \bigoplus_{M \in \text{ind } Q} M^{\oplus \dim M}.$$

The aims of this talk are

1. to remove the assumptions $\text{char } k = 0$ and $v = (1, 1, \cdots, 1)$ and
2. to give an understanding of the above theorem.
1.4. **Example: $A_3$-quiver.** Let $Q$ be a directed $A_3$-quiver.

$$Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \quad \overline{Q} : 1 \xleftarrow{\alpha^*} 2 \xrightarrow{\beta^*} 3.$$  

The mesh relations are

$$\rho_1 = \alpha \alpha^*, \quad \rho_2 = -\alpha^* \alpha + \beta \beta^*, \quad \rho_3 = -\beta^* \beta.$$  

Let $v = (v_1, v_2, v_3) \in k^X Q_0$.

The quiver Heisenberg relations are

$$v \eta_{\alpha} = [\alpha, v \varrho] = v_2^{-1} \alpha \rho_2 - v_1^{-1} \rho \alpha = v_2^{-1} \alpha \beta \beta^* - (v_2^{-1} + v_1^{-1}) \alpha \alpha^* \alpha$$  

$$v \eta_{\beta} = -(v_2^{-1} + v_3^{-1}) \beta \beta^* \beta + v_2^{-1} \alpha^* \alpha \beta,$$  

$$v \eta_{\alpha^*} = (v_1^{-1} + v_2^{-1}) \alpha^* \alpha \alpha^* - v_2^{-1} \beta \beta^* \alpha^*,$$  

$$v \eta_{\beta^*} = (v_2^{-1} + v_3^{-1}) \beta^* \beta \beta^* - v_2^{-1} \beta^* \alpha \alpha^* \alpha.$$  

Assume $v = (1, 1, 1)$, char $k = 0$.

Then the isomorphism $(\clubsuit)$ tells, for example, that

$$v \Lambda e_2 = \begin{array}{c}
2 \\
\oplus 2 \\
1 \\
3
\end{array} \quad \text{for} \quad \begin{array}{c}
1 \\
2 \\
3
\end{array} \quad \text{with} \quad \begin{array}{c}
2 \\
3
\end{array}.$$
Then, looking homogeneous part w.r.t \(*\)-grading,

\[ v\Lambda_0 e_2 = 1^2, \quad v\Lambda_1 e_2 = 1^{2\oplus 3}, \quad v\Lambda_2 e_2 = 2^3. \]

Since \(v\Lambda_0 = kQ\), we see

\[ v\Lambda_0 e_2 = kQe_2 =: P_2. \]

Observe that

\[ v\Lambda_1 e_2 = 1^{2\oplus 3} \]

is the middle term of Auslander-Reiten sequence stating from \(P_2\).

(not a coincidence, but a consequence of "universal AR-sequence").

**What about \(v\Lambda_2 e_2\)?**

### Rough statement of the main theorem

Let \(M \in \text{ind } Q\).

Assume \(n \in \mathbb{N}\) and \(v\) satisfy some conditions.

Then, the module \(v\Lambda_n \otimes_{kQ} M\) provides a minimal left \(\text{rad}^n\)-approximation of \(M\).

Thus, in the above case \(v\Lambda_2 e_2 = v\Lambda_2 \otimes_{kQ} P_2\) provides a minimal left \(\text{rad}^2\)-approximation of \(P_2\).

As a consequence, we can deduce

\[ v\Lambda_2 e_2 \cong I_2 \]

where \(I_2 := D(e_2 kQ)\).
2. \textbf{rad}^n-\textbf{APPROXIMATIONS}

We discuss \textbf{rad}^n-approximations in $\mathbf{D}^b(\textit{R mod})$
where $\textit{R}$ is a finite dimensional algebra of finite global dimension.

2.0.1. \textit{The radical rad and n-th power rad}^n.

Let $M, N \in \mathbf{D}^b(\textit{R mod})$.

Recall that the radical $\text{rad}(M, N)$ is defined to be
a subspace of $\text{Hom}_{\mathbf{D}^b(\textit{R mod})}(M, N)$ consisting of
such morphisms $f : M \to N$ that satisfy the following property:
for any $L \in \text{ind} \mathbf{D}^b(\textit{R mod})$ and any morphisms
\[ s : L \to M, \quad t : N \to L, \]
the composition $tfs : L \to L$ is not an isomorphism.

\[
\begin{array}{c}
M \xrightarrow{f} N \\
\downarrow s \quad \downarrow t \\
L \quad L
\end{array}
\]

The radicals $\{\text{rad}(M, N)\}_{M,N}$ form an ideal $\text{rad}$ of $\mathbf{D}^b(\textit{R mod})$
( an $k$-linear additive sub-bi-functor of $\text{Hom}_{\mathbf{D}^b(\textit{R mod})}$).

For $n \geq 2$, we denote the $n$-th power of $\text{rad}$ by $\text{rad}^n$.

In other words, $\text{rad}^n(M, N)$ is a subspace of $\text{Hom}_{\mathbf{D}^b(\textit{R mod})}(M, N)$
consisting of those morphisms $f : M \to N$ that are obtained as
$n$-times compositions of morphisms in $\text{rad}$.

\[ f : M \xrightarrow{g_1} L_1 \xrightarrow{g_2} L_2 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} L_{n-1} \xrightarrow{g_n} N \]

where $g_1, \cdots, g_n \in \text{rad}$. 
2.0.2. \(\text{rad}^n\)-approximations.

**Definition.** Let \(n \geq 1\).

(1) A morphism \(f : M \to N\) is called a left approximation of \(M\) with respect to \(\text{rad}^n\) (or, left \(\text{rad}^n\)-approximation of \(M\)) if

\[
\begin{align*}
\text{(i)} & \quad f \in \text{rad}^n(M, N) \\
\text{(ii)} & \quad \text{any morphism } g : M \to L \text{ belonging to } \text{rad}^n(M, L) \text{ factors through } f, \text{i.e., there exists } h : N \to L \text{ such that } g = hf.
\end{align*}
\]

(2) A morphism \(f : M \to N\) is called a left minimal if \(h : N \to N\) satisfies \(hf = f\), then it is an isomorphism.

(3) A morphism \(f : M \to N\) is called a minimal left \(\text{rad}^n\)-approximation if it is both left minimal and a left \(\text{rad}^n\)-approximation.

**Remark.** (1) We define a (minimal) right \(\text{rad}^n\)-approximation of \(M\), which is a morphism \(N \to M\), in a dual way.

(2) More generally, we can define (minimal) left or right approximations with respect to an ideal.
Lemma. Let $M \in \text{ind D}^b(R\text{mod})$.

Then a morphism $f : M \to N$ is minimal left almost split if and only if it is a minimal left $\text{rad}$-approximation.

A point here is that if $M$ is a domain of a left almost split morphism $f : M \to N$, it must be indecomposable. But the notion of minimal left $\text{rad}$-approximation makes sense for a non-indecomposable object.

Since any $M \in \text{ind D}^b(R\text{mod})$ admits a minimal left almost split morphism $f : M \to N$,

Corollary. For $M \in \text{D}^b(R\text{mod})$ and $n \geq 1$,

a (minimal) left $\text{rad}^n$-approximation $M \to N$ of $M$ exists.

A minimal left $\text{rad}^n$-approximation of $M$ is unique up to isomorphism under $M$.

2.0.3. A description of $\text{rad}^n$-approximations.

We give $\text{rad}^n$-versions of well-know description of $\text{rad}$-approximations.

Theorem. Let $n \geq 1$ and $M \in \text{ind D}^b(R\text{mod})$.

Let $\lambda_n : M \to L_n$ be a minimal left $\text{rad}^n$-approximation of $M$.

Then

$$L_n \cong \bigoplus_{K \in \text{ind D}^b(R\text{mod})} K^{\oplus d^n_K}$$

where

$$d^n_K := \dim \frac{\text{rad}^n(M, K)}{\text{rad}^{n+1}(M, K)}.$$
3. \( \text{rad}^n \)-approximations in \( \mathcal{D}^b(kQ \text{ mod}) \)

Let \( M \in \text{ind} \mathcal{D}^b(kQ \text{ mod}) \) and
\[
\lambda_n : M \rightarrow L_n \text{ a minimal left } \text{rad}^n \text{-approximation of } M \text{ for } n \geq 0.
\]

- If \( Q \) is non-Dynkin, then, \( L_n \neq 0 \) for \( n \geq 0 \).

**Theorem.** Let \( Q \) be a Dynkin quiver with the Coxeter number \( h \).
Then,

1. \( L_n \neq 0 \) if and only if \( 0 \leq n \leq h - 2 \).
2. \( L_{h-2} = S(M) \) where \( S \) denotes a Serre functor of \( \mathcal{D}^b(kQ \text{ mod}) \).

**Example.** In the case \( M = P_i = kQe_i \), we have
\[
L_{h-2} = S(P_i) = I_i = \mathcal{D}(e_i kQ)
\]
and a minimal left \( \text{rad}^{h-2} \)-approximation is a non-zero morphism

\[ f : P_i \rightarrow I_i. \]

3.1. **Description of \( \bigoplus_n L_n \).**

We exclude the case where \( Q \) is wild and \( M \) is a shift of a regular module.

Let \( \mathcal{C}_M \) be the connected component of AR-quiver that contains \( M \).
Then for each \( K \in \text{ind} \mathcal{C}_M \), the radical filtration terminates

\[
\text{Hom}_{kQ}(M, K) \supset \text{rad}(M, K) \supset \cdots \supset \text{rad}^n(M, K) \supset \cdots.
\]

Consequently,

**Theorem.** In the above setting,

\[
\bigoplus_{n \geq 0} L_n = \bigoplus_{K \in \text{ind} \mathcal{C}_M} K^{\oplus \dim \text{Hom}(M,K)}.
\]
4. The derived quiver Heisenberg algebras

4.1. For a moment assume char \( k \neq 2 \).

**Lemma.** The \( QHA \) \( v\Lambda(Q) \) is the Jacobi algebra:

\[
v\Lambda(Q) = P \left( \overline{Q}, -\frac{1}{2} v\varrho \right).
\]

**Definition.** The derived quiver Heisenberg algebra \( v\tilde{\Lambda}(Q) \) is defined to be the Ginzburg dg-algebra

\[
v\tilde{\Lambda}(Q) = G \left( \overline{Q}, -\frac{1}{2} v\varrho \right).
\]

Explicitly,

<table>
<thead>
<tr>
<th></th>
<th>( e_i )</th>
<th>( \alpha )</th>
<th>( \alpha^* )</th>
<th>( \alpha^\circ )</th>
<th>( \alpha^\triangleright )</th>
<th>( t_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ch deg</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>

The values of \( d \) are defined as:

\[
d(\alpha) := 0, \quad d(\alpha^*) := 0,
\]

\[
d(\alpha^\circ) := -v\eta_{\alpha^*}, \quad d(\alpha^\triangleright) := v\eta_{\alpha},
\]

\[
d(t_i) := \sum_{\alpha:t(\alpha)=i} \alpha\alpha^\circ - \sum_{\alpha:h(\alpha)=i} \alpha^\circ\alpha + \sum_{\alpha:h(\alpha)=i} \alpha^*\alpha^\triangleright - \sum_{\alpha:t(\alpha)=i} \alpha^\triangleright\alpha^*.
\]
The point here is that

Although the potential \(-\frac{1}{2}v_\varphi\) contains the fraction \(\frac{1}{2}\),
but the differential of \(G((Q, -\frac{1}{2}v_\varphi)\) does not.

Therefore, the explicit definition of \(v\tilde{\Lambda}(Q)\) even works
for the case \(\text{char } k = 2\).

4.2. From now \(\text{char } k\) is arbitrary, again.

**Definition.** We define the derived quiver Heisenberg algebra \(v\tilde{\Lambda}(Q)\)
as a DG-algebra given by the above cohomological graded quiver and
the differentials.

We may equip \(v\tilde{\Lambda}(Q)\) with the \(*\)-grading as below:

<table>
<thead>
<tr>
<th>(e_i)</th>
<th>(\alpha)</th>
<th>(\alpha^*)</th>
<th>(\alpha^\circ)</th>
<th>(t_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{ch deg})</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(\text{deg}^*)</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Lemma.**

\(H^0(v\tilde{\Lambda}(Q)) \cong v\Lambda(Q)\).

- Recall that \(v_\varphi \in v\Lambda\) is central and there is an exact seq of \(*\)-graded
\(v\Lambda\)-bimdoules:

\[v\Lambda(-1) \xrightarrow{v_\varphi} v\Lambda \xrightarrow{v\pi} \Pi \rightarrow 0\]

**Lemma.** (1) The element \(v_\varphi \in v\tilde{\Lambda}(Q)\) is “homotopical central”.
(2) The right multiplication

\[\cdot v_\varphi : v\tilde{\Lambda}(Q) \rightarrow v\tilde{\Lambda}(Q)\]

can be regraded as a morphism of \(*\)-graded DG-\(v\tilde{\Lambda}\)-bimdoules.
Let $\tilde{\Pi} = \tilde{\Pi}(Q)$ be the derived preprojective algebra:

$$\tilde{\Pi}(Q) := T_{kQ} \Theta[1],$$

$$\Theta := R\text{Hom}_{kQ}(D(kQ), kQ).$$

We call the tensor degree of $\tilde{\Pi}(Q)$, the $*$-grading. Thus, in particular

$$\tilde{\Pi}_1 = \Theta[1].$$

Note we have $H^0(\tilde{\Pi}(Q)) = \Pi(Q)$.

**Theorem.** There exists a $*$-graded DG-algebra homomorphism

$$v\tilde{\pi} : v\tilde{\Lambda}(Q) \to \tilde{\Pi}(Q)$$

such that $H^0(v\tilde{\pi}) = v\pi$.

The above morphisms constitute an exact triangle

$$v\tilde{\Lambda}(-1) \overset{v\varrho}{\longrightarrow} v\tilde{\Lambda} \overset{v\tilde{\pi}}{\longrightarrow} \tilde{\Pi} \to$$

in the derived category of $*$-graded DG-$v\tilde{\Lambda}$-bimodules.

We denote the $*$-degree 1-part of the exact triangle by $vA^R$, which is an exact triangle of DG-$kQ$-bimodules:

$$vA^R : kQ \overset{v\varrho}{\longrightarrow} v\tilde{\Lambda}_1 \overset{v\tilde{\pi}_1}{\longrightarrow} \tilde{\Pi}_1 \overset{-v\tilde{\theta}[1]}{\longrightarrow}$$

where we set the co-connecting morphism of $vA^R$ by

$$v\tilde{\theta} : \Theta = \tilde{\Pi}_1[-1] \longrightarrow kQ.$$
5. Universal Auslander-Reiten triangle

5.1. Weighted trace.

Let \( U \in \mathcal{D}^b(k \text{ mod}) \). The trace of \( \phi : U \to U \) is defined to be

\[
\text{Tr}_k(\phi) := \sum_{n \in \mathbb{Z}} (-1)^n \text{Tr}_k[H^n(\phi) : H^n(U) \to H^n(U)].
\]

**Definition.** Let \( v \in k^\times Q_0 \).

For \( M \in \mathcal{D}^b(kQ \text{ mod}) \) and \( f : M \to M \), we define

\[
v \text{Tr}(f) := \sum_{i \in Q_0} v_i \text{Tr}(e_i f).
\]

where \( e_i f : e_i M \to e_i M \in \mathcal{D}^b(k \text{ mod}) \).

**Example.** If \( M \in kQ \text{ mod} \), then

\[
v \text{Tr}(\text{id}_M) = \sum_{i \in Q_0} v_i \dim(e_i M) = v \cdot \dim(M)
\]

5.2. Trace formula.

The endofunctor \( S^{-1} := \Theta \otimes_{kQ}^L - \) of \( \mathcal{D}^b(kQ \text{ mod}) \) is the inverse of a Serre functor \( S \). So, \( v \tilde{\theta}_M := v \tilde{\theta} \otimes^L M \) is

\[
v \tilde{\theta}_M : S^{-1}M \to M.
\]

**Theorem.** Let \( M \in \mathcal{D}^b(kQ \text{ mod}) \) and \( f \in \text{Hom}_{kQ}(M, M) \).

Then,

\[
\langle f, v \tilde{\theta}_M \rangle_{S^{-1}} = v \text{Tr}(f)
\]

where \( \langle -, + \rangle_{S^{-1}} \) denotes the paring of Serre duality

\[
\langle -, + \rangle_{S^{-1}} : \text{Hom}_{kQ}(M, M) \otimes_k \text{Hom}_{kQ}(S^{-1}M, M) \to k.
\]
5.3. Universal Auslander-Reiten triangle.

**Definition.** An element $v \in k^\times Q_0$ is said to have the *property (I)* if for all $M \in \text{ind } Q$ we have

$$0 \neq v \cdot \text{dim}(M) = v \text{ Tr}(\text{id}_M) = \langle \text{id}_M, v \tilde{\theta}_M \rangle_{S-1} \text{ in } k.$$  

**Example.** (1) Assume that $\text{char } k = 0$.

If $v_1, v_2, \cdots, v_r > 0$, then $v \in k^\times Q$ has the property (I).

(2) If $v_1, v_2, \cdots, v_r \in k$ are linearly independent over the prime field $\mathbb{P}$ of $k$, then $v \in k^\times Q_0$ has the property (I).

Using Happel’s criterion for AR-triangle, we deduce

**Theorem** (Universal Auslander-Reiten triangle).

Assume that $v \in k^\times Q_0$ has the property (I).

Then for any $M \in D^b(kQ\text{ mod})$

the exact triangle $v\text{AR}_M := v\text{AR} \otimes^L M$ is a direct sum of AR-triangles starting from indec. summand of $M$

$$v\text{AR}_M : M \xrightarrow{v\tilde{\varrho}_M} \tilde{\Lambda}_1 \otimes^L_{kQ} M \xrightarrow{v\tilde{\pi}_1, M} \tilde{\Pi}_1 \otimes^L_{kQ} M \xrightarrow{-v\tilde{\theta}_M[1]}.$$  

In other words, the morphism $v\tilde{\varrho}_M : M \to v\tilde{\Lambda}_1 \otimes^L_{kQ} M$ is a minimal left rad-approximation of $M$ and

the morphism $v\tilde{\pi}_1, M : v\tilde{\Lambda}_1 \otimes^L_{kQ} M \to \tilde{\Pi}_1 \otimes^L_{kQ} M$ is a minimal right rad-approximation of $\tilde{\Pi}_1 \otimes^L_{kQ} M$.

**Remark.** If we fix $M$ first, then we can weakened the assumption on $v$ to “$v \cdot \text{dim} N \neq 0$ for each indec. summand $N$ of $M$.”
6. $\text{rad}^n$-approximations and $v\tilde{\Lambda}(Q)$

Recall there is a $\ast$-graded DG-algebra morphism

$$v\tilde{\pi} : v\tilde{\Lambda} \to \tilde{\Pi}$$

Let $v\tilde{\pi}_n : v\tilde{\Lambda}_n \to \tilde{\Pi}_n$ be the $\ast$-degree $n$-part and $v\tilde{\pi}_{n,M} = v\tilde{\pi}_n \otimes_{kQ} M$.

**Theorem.** Assume that $v \in k^\times Q_0$ has the property (I).

Let $M \in \mathcal{D}^b(kQ \text{ mod})$.

Then the morphism

$$v\tilde{\pi}_{n,M} : v\tilde{\Lambda}_n \otimes_{kQ} M \to \tilde{\Pi}_n \otimes_{kQ} M$$

is a minimal right $\text{rad}^n$-approximation of $\tilde{\Pi}_n \otimes_{kQ} M$ for $n$ belonging to 

$$\begin{aligned}
0 \leq n &\leq h - 2 & \text{(Q is Dynkin)}, \\
0 \leq n &\quad & \text{(Q is non-Dynkin)}.
\end{aligned}$$

Taking $\mathcal{R}\text{Hom}_{kQ^{op}}(-, \tilde{\Pi}_n)$ of the right modules version of this theorem, inductively, we can deduce

**Theorem.** Assume that $v \in k^\times Q_0$ has the property (I).

Let $M \in \mathcal{D}^b(kQ \text{ mod})$.

Then $v\tilde{\Lambda}_n \otimes_{kQ} M$ provides a minimal left $\text{rad}^n$-approximation of $M$ for $n$ belonging to (♠).

I.e., there exists a minimal left $\text{rad}^n$-approximation morphism

$$v\beta_M^{(n)} : M \to v\tilde{\Lambda}_n \otimes_{kQ} M$$

for $n$ belonging to (♠).
Theorem. Assume that $v \in k^\times Q_0$ has the property (I).

Let $M \in \text{ind } D^b(kQ \text{ mod})$.

Except the case where $Q$ is wild and $M$ is a shift of a regular module, we have the following isomorphism

$$
\bigoplus_{n} v\widetilde{\Lambda}_n \otimes_{kQ} M \cong \bigoplus_{K \in \text{ind } \mathcal{C}_M} K^{\oplus \dim \text{Hom}(M,K)}.
$$

where in LHS, $n$ runs through $(\spadesuit)$.

Corollary. Assume that $v \in k^\times Q_0$ has the property (I).

Then as $kQ$-modules

$$
v\Lambda \cong \bigoplus_{K \in \text{ind } \mathcal{P}(Q)} K^{\oplus \dim K}
$$

7. WHAT IS A MINIMAL LEFT $\text{rad}^n$-APPROXIMATION MORPHISM

We have shown that there exists a minimal left $\text{rad}^n$-approximation morphism

$$
v\beta^{(n)}_M : M \to v\widetilde{\Lambda}_n \otimes_{kQ} M.
$$

A natural candidate is the multiplication of the $n$-th power $v\varrho^n$ of $v\varrho$

**Question:** Is

$$
v\varrho^n_M : M \to v\widetilde{\Lambda}_n \otimes_{kQ} M
$$

a minimal left $\text{rad}^n$-approximation of $M$?
Example. Let $Q$ be a directed $A_3$-quiver.

$$Q: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3, \quad \overline{Q}: 1 \xrightarrow{\alpha^*} 2 \xrightarrow{\beta^*} 3.$$ 

The property (I) precisely says that $\upsilon$ satisfies the followings

$$\upsilon_1 \neq 0, \upsilon_2 \neq 0, \upsilon_3 \neq 0,$$

$$\upsilon_1 + \upsilon_2 \neq 0, \upsilon_2 + \upsilon_3 \neq 0, \upsilon_1 + \upsilon_2 + \upsilon_3 \neq 0.$$

Proposition. Assume that $\upsilon \in \mathbb{k}^\times Q_0$ has the property (I).

Then, the morphism

$$\upsilon \varrho^2_{P_2} : P_2 \rightarrow \upsilon \tilde{\Lambda}_2 \otimes_{kQ}^L P_2$$

is a minimal left $\text{rad}^2$-approximation if and only if

$$\upsilon_1 + 2\upsilon_2 + \upsilon_3 \neq 0.$$ 

We note

$$\upsilon_1 + 2\upsilon_2 + \upsilon_3 = \upsilon \cdot \dim(\upsilon \tilde{\Lambda}_1 \otimes_{kQ}^L P_2).$$

7.1. Minimal left $\text{rad}^2$-approximation.

Theorem. Assume that $\upsilon \in \mathbb{k}^\times Q_0$ has the property (I).

Let $M \in \text{ind } \textbf{D}^b(\mathbb{k}Q \text{ mod})$. Assume that

$$\upsilon \cdot \dim(\upsilon \tilde{\Lambda}_1 \otimes_{kQ}^L M) \neq 0.$$ 

Then the morphism

$$\upsilon \varrho^2_M : M \rightarrow \upsilon \tilde{\Lambda}_2 \otimes_{kQ}^L M$$

is a minimal left $\text{rad}^2$-approximation of $M$. 
7.2. Minimal left rad$^n$-approx (char $k = 0$).

**Theorem.** Assume char $k = 0$.

Let $M \in \text{ind } \mathcal{D}^b(kQ \mod)$ and $n$ belongs to (♠).

Then for a generic parameter $v \in k^\times Q_0$, the morphism

$$v \varrho_n^M : M \to v\tilde{\Lambda}_n \otimes_{kQ} M$$

is a minimal left rad$^n$-approximation of $M$.

**Corollary.** Assume char $k = 0$.

Let $Q$ be a Dynkin quiver with the Coxeter number $h$.

Then for a generic parameter $v \in k^\times Q_0$ the morphism

$$v \varrho_n^M : M \to v\tilde{\Lambda}_n \otimes_{kQ} M$$

is a minimal left rad$^n$-approximation of $M$ for all $M \in \mathcal{D}^b(kQ \mod)$ and $n = 1, 2, \cdots, h - 2$.

7.3. Minimal left rad$^n$-approx $(Q = A_N$-quiver$)$.

**Theorem.** Let $N \geq 1$ and $Q$ an $A_N$-quiver (note $h = N + 1$).

Assume that $k$ has a primitive $h$-th root of unity.

Then for a generic $v \in k^\times Q_0$, the morphism

$$v \varrho_n^M : M \to \tilde{\Lambda}_n \otimes_{kQ} M$$

is a minimal left rad$^n$-approximation of $M$ for all $M \in \mathcal{D}^b(kQ \mod)$ and $n = 1, 2, \cdots, h - 2$. 
Thank you