Quiver Heisenberg algebras and rad^n-approximations

(a joint work with M. Herschend)

at a conference in celebration of the work of Bill Crawley-Boevey

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Setup .

- $\mathbf{k}$: *an algebraically closed field of arbitrary characteristic.*
- $\boldsymbol{Q}$: *a finite acyclic quiver.*

1.1. The preprojective algebras $\boldsymbol{\Pi(Q)}$ of $\boldsymbol{Q}$.

We denote by $\overline{\boldsymbol{Q}}$ the double of $\boldsymbol{Q}$.

$$\boxed{\boldsymbol{Q}} \qquad \boldsymbol{i} \xrightarrow{\ \alpha\ } \boldsymbol{j} \qquad \wr \qquad \begin{tikzcd} \boldsymbol{i} \arrow[r, bend left, "\alpha"] & \boldsymbol{j} \arrow[l, bend left, "\alpha^*"] \end{tikzcd} \qquad \boxed{\overline{\boldsymbol{Q}}}$$

Recall that for a vertex $\boldsymbol{i} \in \boldsymbol{Q_0}$, the mesh relation $\boldsymbol{\rho_i}$ is defined by

$$\boldsymbol{\rho_i := \sum_{\alpha \in Q_1 : t(\alpha) = i} \alpha\alpha^* - \sum_{\alpha \in Q_1 : h(\alpha) = i} \alpha^*\alpha.}$$

The following is also called the mesh relation:

$$\boldsymbol{\rho := \sum_{i \in Q_0} \rho_i.}$$

The preprojective algebra is defined to be

the path of the double quiver $\overline{\boldsymbol{Q}}$ with the mesh relations:

$$\boldsymbol{\Pi(Q) = \frac{\mathbf{k}\overline{Q}}{(\rho)} = \frac{\mathbf{k}\overline{Q}}{(\rho_i \mid i \in Q_0)}.}$$

We equip $\overline{\boldsymbol{Q}}$ with the grading ($*$-grading)

$$\boldsymbol{\deg^* \alpha := 0,\ \deg^* \alpha^* := 1} \text{ for } \boldsymbol{\alpha \in Q_1.}$$

Then $\boldsymbol{\deg^* \rho_i = 1}$ and $\boldsymbol{\Pi(Q)}$ is a $*$-graded algebra.

We have $\boldsymbol{\Pi(Q)_0 = \mathbf{k}Q}$.

1.2. **Quiver Heisenberg Algebras ${}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})$.**

The quiver Heisenberg algebra ${}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})$ has a parameter $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_{\mathbf{0}}$, i.e., a collection $\boldsymbol{v} = (\boldsymbol{v}_i)_{i \in Q_0}$ of non-zero element of $\mathbf{k}$ indexed by $\boldsymbol{Q}_{\mathbf{0}}$.

Definition . *Let* $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_{\mathbf{0}}$.

(1) For $\boldsymbol{i} \in \boldsymbol{Q}_{\mathbf{0}}$*, we set*

$$ {}^{v}\boldsymbol{\varrho}_i := \boldsymbol{v}_i^{-1}\boldsymbol{\rho}_i, \quad {}^{v}\boldsymbol{\varrho} := \sum_{i \in Q_0} {}^{v}\boldsymbol{\varrho}_i = \sum_{i \in Q_0} \boldsymbol{v}_i^{-1}\boldsymbol{\rho}_i. $$

(2) For $\boldsymbol{a} \in \overline{\boldsymbol{Q}}_{\mathbf{1}}$*, the* quiver Heisenberg relation ${}^{v}\boldsymbol{\eta}_{\boldsymbol{a}}$ *is defined to be*

$$ {}^{v}\boldsymbol{\eta}_{\boldsymbol{a}} := [\boldsymbol{a}, {}^{v}\boldsymbol{\varrho}] = \boldsymbol{a}\,{}^{v}\boldsymbol{\varrho} - {}^{v}\boldsymbol{\varrho}\boldsymbol{a} = \boldsymbol{v}_{h(a)}^{-1}\boldsymbol{a}\boldsymbol{\rho}_{h(a)} - \boldsymbol{v}_{t(a)}^{-1}\boldsymbol{\rho}_{t(a)}\boldsymbol{a}. $$

(3) We define the quiver Heisenberg algebra ${}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})$ *to be*

the path algebra of $\overline{\boldsymbol{Q}}$ *with the quiver Heisenberg relations:*

$$ {}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q}) := \frac{\mathbf{k}\overline{\boldsymbol{Q}}}{({}^{v}\boldsymbol{\eta}_{a} \mid \boldsymbol{a} \in \overline{\boldsymbol{Q}}_{1})}. $$

• The QH-relations ${}^{v}\boldsymbol{\eta}_{\boldsymbol{a}}$ are homogeneous w.r.t $*$-grading.

Hence ${}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})$ is a $*$-graded algebra. We have ${}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})_{\mathbf{0}} = \mathbf{k}\boldsymbol{Q}$.

• The element ${}^{v}\boldsymbol{\varrho}$ is central in ${}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})$ and $\boldsymbol{\Pi}(\boldsymbol{Q}) = {}^{v}\boldsymbol{\Lambda}(\boldsymbol{Q})/({}^{v}\boldsymbol{\varrho})$.

Hence there is an exact seq of $*$-graded ${}^{v}\boldsymbol{\Lambda}$-bimdoules:

$$ {}^{v}\boldsymbol{\Lambda}(-\mathbf{1}) \xrightarrow{{}^{v}\varrho} {}^{v}\boldsymbol{\Lambda} \xrightarrow{{}^{v}\pi} \boldsymbol{\Pi} \longrightarrow \mathbf{0} $$

where $(-\mathbf{1})$ denote the shift of $*$-degree by $-\mathbf{1}$.

Remark . *Originally, I and Martin studied the case* $\boldsymbol{v} = (\mathbf{1}, \mathbf{1}, \cdots, \mathbf{1})$.

In that case, the quiver Heisenberg relation ${}^{v}\boldsymbol{\eta}_{a} = [\boldsymbol{a}, \boldsymbol{\rho}]$ *can be looked as a quiver version of the Heisenberg relations* $[\boldsymbol{x}, [\boldsymbol{x}, \boldsymbol{y}]], [\boldsymbol{y}, [\boldsymbol{x}, \boldsymbol{y}]]$.

Hence the name quiver Heisenberg algebras.

1.3. **Related algebras and preceding results.**

We point out the following isomorphism of algebras:

$$ {}^{v}\Lambda(Q) \cong \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - (v_i z)e_i \mid i \in Q_0)}, $$

(where e_i is the idempotent element corresponding to $i \in Q_0$)

from which we see that ${}^{v}\Lambda(Q)$ is a special case of

- The central extension of the preprojective algebras by Etingof-Rains (2006)

$$ \Pi(Q)_{\lambda,\mu} := \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - (\lambda_i z + \mu_i)e_i \mid i \in Q_0)} $$

where $\lambda_i, \mu_i \in \mathbf{k}$ for each $i \in Q_0$.

This algebra is a special case of the following algebra.

- The $N = 1$-quiver algebra by Cachazo-Katz-Vafa (2001)

$$ \Pi(Q)_P := \frac{\mathbf{k}[z]\overline{Q}}{(\rho_i - P_i(z)e_i \mid i \in Q_0)} $$

where $P_i(z) \in \mathbf{k}[z]$ for each $i \in Q_0$.

This algebra is obtained as a pull-back of

- The deformation family of the preprojective algebras by Crawley-Boevey-Holland (1998)

$$ \Pi(Q)_{\bullet} := \frac{\mathbf{k}[z_1, \cdots, z_r]\overline{Q}}{(\rho_i - z_i e_i \mid i \in Q_0)} $$

where $r = \#Q_0$.

Theorem (Etingof-Rains). *Assume* $\mathbf{char\,k = 0}$.

If $\boldsymbol{Q}$ *is a Dynkin quiver with the Coxeter number* $\boldsymbol{h}$ *and* $\boldsymbol{r := \#Q_0}$*, then for generic* $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}$*,*

$$\dim {}^{v}\Lambda(Q) = \sum_{M \in \operatorname{ind} Q} (\dim M)^2 = \frac{rh^2(h+1)}{12}.$$

Theorem (Herschend-M). *Assume* $\mathbf{char\,k = 0}$ *and* $\boldsymbol{v = (1, 1, \cdots, 1)}$.

Then as $\mathbf{k}\boldsymbol{Q}$*-modules,*

$${}^{v}\Lambda(Q)e_i \cong \bigoplus_{M \in \operatorname{ind} \mathscr{P}(Q)} M^{\oplus \dim e_i M} \qquad (\clubsuit)$$

where $\mathscr{P}(\boldsymbol{Q})$ *denotes the set of the preprojective modules of* $\mathbf{k}\boldsymbol{Q}$.

Corollary . *Assume* $\mathbf{char\,k = 0}$ *and* $\boldsymbol{v = (1, 1, \cdots, 1)}$.

(1)

$${}^{v}\Lambda(Q) \cong \bigoplus_{M \in \operatorname{ind} \mathscr{P}(Q)} M^{\oplus \dim M}.$$

(2) If $\boldsymbol{Q}$ *is a Dynkin quiver, then*

$${}^{v}\Lambda(Q) \cong \bigoplus_{M \in \operatorname{ind} Q} M^{\oplus \dim M}.$$

The aims of this talk are

(1) to remove the assumptions $\mathbf{char\,k = 0}$ **and** $\boldsymbol{v = (1, 1, \cdots, 1)}$ **and**

(2) to give an understanding of the above theorem.

1.4. **Example: A_3-quiver.** Let Q be a directed A_3-quiver.

$$Q:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3\,, \quad \overline{Q}:\ 1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3\,.$$

The mesh relations are

$$\rho_1 = \alpha\alpha^*,\ \rho_2 = -\alpha^*\alpha + \beta\beta^*,\ \rho_3 = -\beta^*\beta.$$

Let $v = (v_1, v_2, v_3) \in \mathbf{k}^{\times Q_0}$.

The quiver Heisenberg relations are

$$\begin{aligned}
{}^v\eta_\alpha &= [\alpha, {}^v\varrho] = v_2^{-1}\alpha\rho_2 - v_1^{-1}\rho\alpha = v_2^{-1}\alpha\beta\beta^* - (v_2^{-1} + v_1^{-1})\alpha\alpha^*\alpha \\
{}^v\eta_\beta &= -(v_2^{-1} + v_3^{-1})\beta\beta^*\beta + v_2^{-1}\alpha^*\alpha\beta, \\
{}^v\eta_{\alpha^*} &= (v_1^{-1} + v_2^{-1})\alpha^*\alpha\alpha^* - v_2^{-1}\beta\beta^*\alpha^*, \\
{}^v\eta_{\beta^*} &= (v_2^{-1} + v_3^{-1})\beta^*\beta\beta^* - v_2^{-1}\beta^*\alpha^*\alpha.
\end{aligned}$$

Assume $v = (1, 1, 1),\ \operatorname{char} \mathbf{k} = 0$.

Then the isomorphism $(\clubsuit)$ tells, for example, that

$${}^v\Lambda e_2 = \begin{array}{ccc} & 2 & \\ 1 & & 3 \\ & 2^{\oplus 2} & \\ 1 & & 3 \\ & 2 & \end{array}$$

$$\begin{tikzcd}[column sep=small]
 & & \begin{smallmatrix}3\\2\\1\end{smallmatrix} \arrow[dr] & & \\
 & \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] \arrow[dr] & & \begin{smallmatrix}3\\2\end{smallmatrix} \arrow[dr] & \\
1 \arrow[ur] & & 2 \arrow[ur] & & 3
\end{tikzcd}$$

Then, looking homogeneous part w.r.t $*$-grading,

$$ {}^{v}\boldsymbol{\Lambda}_0 \boldsymbol{e}_2 = \begin{smallmatrix} & \mathbf{2} \\ \mathbf{1} & \end{smallmatrix}, \quad {}^{v}\boldsymbol{\Lambda}_1 \boldsymbol{e}_2 = \begin{smallmatrix} & & \mathbf{3} \\ & \mathbf{2}^{\oplus 2} & \\ \mathbf{1} & & \end{smallmatrix}, \quad {}^{v}\boldsymbol{\Lambda}_2 \boldsymbol{e}_2 = \begin{smallmatrix} & \mathbf{3} \\ \mathbf{2} & \end{smallmatrix}. $$

Since ${}^{v}\boldsymbol{\Lambda}_0 = \mathbf{k}\boldsymbol{Q}$, we see

$$ {}^{v}\boldsymbol{\Lambda}_0 \boldsymbol{e}_2 = \mathbf{k}\boldsymbol{Q}\boldsymbol{e}_2 =: \boldsymbol{P}_2. $$

Observe that

$$ {}^{v}\boldsymbol{\Lambda}_1 \boldsymbol{e}_2 = \begin{smallmatrix} & & \mathbf{3} \\ & \mathbf{2}^{\oplus 2} & \\ \mathbf{1} & & \end{smallmatrix} $$

is the middle term of Auslander-Reiten sequence stating from $\boldsymbol{P}_2$.

(not a coincidence, but a consequence of "universal AR-sequence".)

What about ${}^{v}\boldsymbol{\Lambda}_2 \boldsymbol{e}_2$?

<u>Rough statement of the main theorem</u>

Let $\boldsymbol{M} \in \mathbf{ind}\, \boldsymbol{Q}$.

Assume $\boldsymbol{n} \in \mathbb{N}$ and $\boldsymbol{v}$ satisfy some conditions.

Then, the module ${}^{v}\boldsymbol{\Lambda}_{\boldsymbol{n}} \otimes_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M}$ provides a minimal left $\mathbf{rad}^{\boldsymbol{n}}$-approximation of $\boldsymbol{M}$.

Thus, in the above case ${}^{v}\boldsymbol{\Lambda}_2 \boldsymbol{e}_2 = {}^{v}\boldsymbol{\Lambda}_2 \otimes_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{P}_2$ provides a minimal left $\mathbf{rad}^{\mathbf{2}}$-approximation of $\boldsymbol{P}_2$.

As a consequence, we can deduce

$$ {}^{v}\boldsymbol{\Lambda}_2 \boldsymbol{e}_2 \cong \boldsymbol{I}_2 $$

where $\boldsymbol{I}_2 := \mathrm{D}(\boldsymbol{e}_2 \mathbf{k}\boldsymbol{Q})$.

2. $\mathbf{rad}^n$-APPROXIMATIONS

We discuss $\mathbf{rad}^n$-approximations in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})$ where $\boldsymbol{R}$ is a finite dimensional algebra of finite global dimension.

2.0.1. *The radical* $\mathbf{rad}$ *and* $\boldsymbol{n}$*-th power* $\mathbf{rad}^n$*.*

Let $\boldsymbol{M}, \boldsymbol{N} \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})$.

Recall that the radical $\mathbf{rad}(\boldsymbol{M}, \boldsymbol{N})$ is defined to be a subspace of $\mathbf{Hom}_{\mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})}(\boldsymbol{M}, \boldsymbol{N})$ consisting of such morphisms $\boldsymbol{f} : \boldsymbol{M} \to \boldsymbol{N}$ that satisfy the following property:

for any $\boldsymbol{L} \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})$ and any morphisms

$$\boldsymbol{s} : \boldsymbol{L} \to \boldsymbol{M}, \quad \boldsymbol{t} : \boldsymbol{N} \to \boldsymbol{L},$$

the composition $\boldsymbol{tfs} : \boldsymbol{L} \to \boldsymbol{L}$ is not an isomorphism.

$$\begin{tikzcd} \boldsymbol{M} \arrow[rr, "\boldsymbol{f}"] & & \boldsymbol{N} \arrow[dl, "\boldsymbol{t}"] \\ & \boldsymbol{L} \arrow[ul, "\boldsymbol{s}"] & \end{tikzcd} .$$

The radicals $\{\mathbf{rad}(\boldsymbol{M}, \boldsymbol{N})\}_{\boldsymbol{M},\boldsymbol{N}}$ form an ideal $\mathbf{rad}$ of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})$ (an $\mathbf{k}$-linear additive sub-bi-functor of $\mathbf{Hom}_{\mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})}$).

For $\boldsymbol{n} \geq \mathbf{2}$, we denote the $\boldsymbol{n}$-th power of $\mathbf{rad}$ by $\mathbf{rad}^n$.

In other words, $\mathbf{rad}^n(\boldsymbol{M}, \boldsymbol{N})$ is a subspace of $\mathbf{Hom}_{\mathsf{D}^{\mathrm{b}}(\boldsymbol{R}\,\mathbf{mod})}(\boldsymbol{M}, \boldsymbol{N})$ consisting of those morphisms $\boldsymbol{f} : \boldsymbol{M} \to \boldsymbol{N}$ that are obtained as $\boldsymbol{n}$-times compositions of morphisms in $\mathbf{rad}$.

$$\boldsymbol{f} : \boldsymbol{M} \xrightarrow{g_1} \boldsymbol{L}_1 \xrightarrow{g_2} \boldsymbol{L}_2 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-1}} \boldsymbol{L}_{n-1} \xrightarrow{g_n} \boldsymbol{N}$$

where $\boldsymbol{g}_1, \cdots, \boldsymbol{g}_n \in \mathbf{rad}$.

2.0.2. $\mathbf{rad}^n$*-approximations.*

Definition . *Let* $n \geq 1$.

(1) A morphism $f : M \to N$ *is called a left approximation of* M *with respect to* $\mathbf{rad}^n$ *(or, left* $\mathbf{rad}^n$*-approximation of* M*)*

if (i) f *belongs to* $\mathbf{rad}^n(M, N)$ *and*

(ii) any morphism $g : M \to L$ *belonging to* $\mathbf{rad}^n(M, L)$

factors through f*, i.e.,*

there exists $h : N \to L$ *such that* $g = hf$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g"'] & N \arrow[d, dashed, "h"] \\ & L \end{tikzcd}$$

(2) A morphism $f : M \to N$ *is called a left minimal*

if $h : N \to N$ *satisfies* $hf = f$*, then it is an isomorphism.*

$$\begin{tikzcd} & N \arrow[d, "h"] \\ M \arrow[ur, "f"] \arrow[r, "f"'] & N \end{tikzcd} \quad \Longrightarrow \quad h \text{ is an isom.}$$

(3) A morphism $f : M \to N$ *is called a minimal left* $\mathbf{rad}^n$*-approximation if it is both left minimal and a left* $\mathbf{rad}^n$*-approximation.*

Remark . *(1) We define a (minimal) right* $\mathbf{rad}^n$*-approximation of* M*, which is a morphism* $N \to M$*, in a dual way.*

(2) More generally, we can define (minimal) left or right approximations with respect to an ideal.

Lemma . *Let* $M \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(R\,\mathbf{mod})$.

Then a morphism $f : M \to N$ *is minimal left almost split if and only if it is a minimal left* **rad**-*approximation.*

A point here is that if M is a domain of a left almost split morphism $f : M \to N$, it must be indecomposable. But the notion of minimal left **rad**-approximation makes sense for a non-indecomposable object.

Since any $M \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(R\,\mathbf{mod})$ admits a minimal left almost split morphism $f : M \to N$,

Corollary . *For* $M \in \mathsf{D}^{\mathrm{b}}(R\,\mathbf{mod})$ *and* $n \geq 1$,

a (minimal) left $\mathbf{rad}^n$-*approximation* $M \to N$ *of* M *exists.*

A minimal left $\mathbf{rad}^n$-*approximation of* M *is unique up to isomorphism under* M.

2.0.3. *A description of* $\mathbf{rad}^n$-*approximations.*

We give $\mathbf{rad}^n$-versions of well-know description of **rad**-approximations.

Theorem . *Let* $n \geq 1$ *and* $M \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(R\,\mathbf{mod})$.

Let $\lambda_n : M \to L_n$ *be a minimal left* $\mathbf{rad}^n$-*approximation of* M. *Then*

$$L_n \cong \bigoplus_{K \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(R\,\mathbf{mod})} K^{\oplus d_K^n}$$

where

$$d_K^n := \dim \frac{\mathbf{rad}^n(M, K)}{\mathbf{rad}^{n+1}(M, K)}.$$

3. $\mathbf{rad}^n$-APPROXIMATIONS IN $\mathsf{D}^{\mathrm{b}}(\mathrm{k}Q\,\mathbf{mod})$

Let $M \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(\mathrm{k}Q\,\mathbf{mod})$ and

$\lambda_n : M \to L_n$ a minimal left $\mathbf{rad}^n$-approximation of M for $n \geq 0$.

- If Q is non-Dynkin, then, $L_n \neq 0$ for $n \geq 0$.

Theorem . *Let Q be a Dynkin quiver with the Coxeter number h. Then,*

(1) $L_n \neq 0$ if and only if $0 \leq n \leq h-2$.

(2) $L_{h-2} = \mathsf{S}(M)$ where S denotes a Serre functor of $\mathsf{D}^{\mathrm{b}}(\mathrm{k}Q\,\mathbf{mod})$.

Example . *In the case $M = P_i = \mathrm{k}Qe_i$, we have*

$$L_{h-2} = \mathsf{S}(P_i) = I_i = \mathrm{D}(e_i\mathrm{k}Q)$$

and a minimal left $\mathbf{rad}^{h-2}$-approximation is a non-zero morphism

$$f : P_i \longrightarrow I_i.$$

3.1. Description of $\bigoplus_n L_n$.

We exclude the case where Q is wild and M is a shift of a regular module.

Let $\mathscr{C}_M$ be the connected component of AR-quiver that contains M.

Then for each $K \in \mathbf{ind}\,\mathscr{C}_M$, the radical filtration terminates

$$\mathbf{Hom}_{\mathrm{k}Q}(M,K) \supset \mathbf{rad}(M,K) \supset \cdots \supset \mathbf{rad}^n(M,K) \supset \cdots .$$

Consequently,

Theorem . *In the above setting,*

$$\bigoplus_{n\geq 0} L_n = \bigoplus_{K\in \mathbf{ind}\,\mathscr{C}_M} K^{\oplus \dim \mathbf{Hom}(M,K)}.$$

4. The derived quiver Heisenberg algebras

4.1. For a moment assume $\mathrm{char}\, \mathrm{k} \neq 2$.

Lemma . *The QHA ${}^v\boldsymbol{\Lambda}(\boldsymbol{Q})$ is the Jacobi algebra:*

$$ {}^v\boldsymbol{\Lambda}(\boldsymbol{Q}) = \mathscr{P}\left(\overline{\boldsymbol{Q}}, -\frac{1}{2}{}^v\boldsymbol{\varrho\rho}\right). $$

Definition . *The derived quiver Heisenberg algebra ${}^v\widetilde{\boldsymbol{\Lambda}}(\boldsymbol{Q})$ is defined to be the Ginzburg dg-algebra*

$$ {}^v\widetilde{\boldsymbol{\Lambda}}(\boldsymbol{Q}) = \mathscr{G}\left(\overline{\boldsymbol{Q}}, -\frac{1}{2}{}^v\boldsymbol{\varrho\rho}\right). $$

Explicitly,

$$ \boxed{\boldsymbol{Q}} \qquad \boldsymbol{i} \xrightarrow{\ \alpha\ } \boldsymbol{j} $$

$$ \begin{tikzcd} \boldsymbol{i} \arrow[loop left, "t_i"] \arrow[rr, bend left=20, "\boldsymbol{\alpha}"] \arrow[rr, bend right=50, "\boldsymbol{\alpha}^{\circledast}"'] & & \boldsymbol{j} \arrow[loop right, "t_j"] \arrow[ll, bend right=50, "\boldsymbol{\alpha}^{\circ}"'] \arrow[ll, bend left=20, "\boldsymbol{\alpha}^{*}"] \end{tikzcd} $$

	$\boldsymbol{e_i}$	$\boldsymbol{\alpha}$	$\boldsymbol{\alpha^*}$	$\boldsymbol{\alpha^{\circledast}}$	$\boldsymbol{\alpha^{\circ}}$	$\boldsymbol{t_i}$
ch deg	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-2}$

The values of $\boldsymbol{d}$ are defined as:

$$ \boldsymbol{d(\alpha) := 0, \quad d(\alpha^*) := 0,} $$

$$ \boldsymbol{d(\alpha^{\circ}) := -{}^v\eta_{\alpha^*}, \quad d(\alpha^{\circledast}) := {}^v\eta_{\alpha},} $$

$$ \boldsymbol{d(t_i) := \sum_{\alpha: t(\alpha)=i} \alpha\alpha^{\circ} - \sum_{\alpha: h(\alpha)=i} \alpha^{\circ}\alpha + \sum_{\alpha: h(\alpha)=i} \alpha^*\alpha^{\circledast} - \sum_{\alpha: t(\alpha)=i} \alpha^{\circledast}\alpha^*.} $$

The point here is that

> Although the potential $-\frac{1}{2}{}^v\varrho\rho$ contains the fraction $\frac{1}{2}$, but the differential of $\mathfrak{G}\left(\overline{Q}, -\frac{1}{2}{}^v\varrho\rho\right)$ does not.
>
> Therefore, the explicit definition of ${}^v\widetilde{\Lambda}(Q)$ even works for the case $\mathbf{char\,k = 2}$.

4.2. **From now char k is arbitrary, again.**

Definition . *We define the derived quiver Heisenberg algebra* ${}^v\widetilde{\Lambda}(Q)$ *as a DG-algebra given by the above cohomological graded quiver and the differentials.*

We may equip ${}^v\widetilde{\Lambda}(Q)$ *with the* $*$*-grading as below:*

	e_i	α	α^*	$\alpha^{\circledast}$	α°	t_i
ch deg	0	0	0	−1	−1	−2
deg*	0	0	1	1	2	2

Lemma .

$$\mathrm{H}^0({}^v\widetilde{\Lambda}(Q)) \cong {}^v\Lambda(Q).$$

- Recall that ${}^v\varrho \in {}^v\Lambda$ is central and there is an exact seq of $*$-graded ${}^v\Lambda$-bimdoules:

$${}^v\Lambda(-1) \xrightarrow{{}^v\varrho} {}^v\Lambda \xrightarrow{{}^v\pi} \Pi \longrightarrow 0$$

Lemma . *(1) The element* ${}^v\varrho \in {}^v\widetilde{\Lambda}(Q)$ *is "homotopical central".*
(2) The right multiplication

$$\cdot{}^v\varrho : {}^v\widetilde{\Lambda}(Q) \to {}^v\widetilde{\Lambda}(Q)$$

can be regraded as a morphism of $*$*-graded DG-*${}^v\widetilde{\Lambda}$*-bimdoules.*

Let $\widetilde{\mathbf{\Pi}} = \widetilde{\mathbf{\Pi}}(\boldsymbol{Q})$ be the derived preprojective algebra:

$$\widetilde{\mathbf{\Pi}}(\boldsymbol{Q}) := \mathsf{T}_{\mathbf{k}\boldsymbol{Q}}\, \mathbf{\Theta}[\mathbf{1}],$$

$$\mathbf{\Theta} := \mathbb{R}\mathbf{Hom}_{\mathbf{k}\boldsymbol{Q}}(\mathrm{D}(\mathbf{k}\boldsymbol{Q}), \mathbf{k}\boldsymbol{Q}).$$

We call the tensor degree of $\widetilde{\mathbf{\Pi}}(\boldsymbol{Q})$, the $*$-grading. Thus, in particular

$$\widetilde{\mathbf{\Pi}}_1 = \mathbf{\Theta}[\mathbf{1}].$$

Note we have $\mathrm{H}^{\mathbf{0}}(\widetilde{\mathbf{\Pi}}(\boldsymbol{Q})) = \mathbf{\Pi}(\boldsymbol{Q})$.

Theorem . *There exists a $*$-graded DG-algebra homomorphism*

$${}^{v}\tilde{\boldsymbol{\pi}} : {}^{v}\widetilde{\mathbf{\Lambda}}(\boldsymbol{Q}) \to \widetilde{\mathbf{\Pi}}(\boldsymbol{Q})$$

such that $\mathrm{H}^{\mathbf{0}}({}^{v}\tilde{\boldsymbol{\pi}}) = {}^{v}\boldsymbol{\pi}$.

The above morphisms constitute an exact triangle

$${}^{v}\widetilde{\mathbf{\Lambda}}(-\mathbf{1}) \xrightarrow{.^{v}\boldsymbol{\varrho}} {}^{v}\widetilde{\mathbf{\Lambda}} \xrightarrow{{}^{v}\tilde{\boldsymbol{\pi}}} \widetilde{\mathbf{\Pi}} \to$$

in the derived category of $*$-graded DG-${}^{v}\widetilde{\mathbf{\Lambda}}$-bimdoules.

We denote the $*$-degree $\mathbf{1}$-part of the exact triangle by ${}^{v}\mathsf{AR}$, which is an exact triangle of DG-$\mathbf{k}\boldsymbol{Q}$-bimodules:

$${}^{v}\mathsf{AR} : \mathbf{k}\boldsymbol{Q} \xrightarrow{.^{v}\boldsymbol{\varrho}} {}^{v}\widetilde{\mathbf{\Lambda}}_1 \xrightarrow{{}^{v}\tilde{\boldsymbol{\pi}}_1} \widetilde{\mathbf{\Pi}}_1 \xrightarrow{-{}^{v}\tilde{\boldsymbol{\theta}}[1]}$$

where we set the co-connecting morphism of ${}^{v}\mathsf{AR}$ by

$${}^{v}\tilde{\boldsymbol{\theta}} : \mathbf{\Theta} = \widetilde{\mathbf{\Pi}}_1[-\mathbf{1}] \longrightarrow \mathbf{k}\boldsymbol{Q}.$$

5. Universal Auslander-Reiten triangle

5.1. Weighted trace.

Let $\boldsymbol{U} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\,\mathbf{mod})$. The trace of $\boldsymbol{\phi} : \boldsymbol{U} \to \boldsymbol{U}$ is defined to be

$$\mathbf{Tr}_{\mathrm{k}}(\phi) := \sum_{n\in\mathbb{Z}} (-1)^n \,\mathbf{Tr}_{\mathrm{k}}[\mathrm{H}^n(\phi) : \mathrm{H}^n(\boldsymbol{U}) \to \mathrm{H}^n(\boldsymbol{U})].$$

Definition . *Let* $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q_0}$.

For $\boldsymbol{M} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\boldsymbol{Q}\,\mathbf{mod})$ *and* $\boldsymbol{f} : \boldsymbol{M} \to \boldsymbol{M}$, *we define*

$$^{v}\,\mathbf{Tr}(f) := \sum_{i\in Q_0} v_i \,\mathbf{Tr}_{\mathrm{k}}(e_i f).$$

where $\boldsymbol{e_i f} : \boldsymbol{e_i M} \to \boldsymbol{e_i M} \;\in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\,\mathbf{mod})$.

Example . *If* $\boldsymbol{M} \in \mathbf{k}\boldsymbol{Q}\,\mathbf{mod}$, *then*

$$^{v}\,\mathbf{Tr}(\mathbf{id}_M) = \sum_{i\in Q_0} v_i \,\mathbf{dim}(e_i M) = v\cdot \underline{\mathbf{dim}}(M)$$

5.2. Trace formula.

The endofunctor $\mathsf{S}^{-1} := \boldsymbol{\Theta} \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} -$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}\boldsymbol{Q}\,\mathbf{mod})$ is the inverse of a Serre functor S. So, ${}^{v}\tilde{\boldsymbol{\theta}}_M := {}^{v}\tilde{\boldsymbol{\theta}} \otimes^{\mathbb{L}} \boldsymbol{M}$ is

$${}^{v}\tilde{\theta}_M : \mathsf{S}^{-1}M \longrightarrow M.$$

Theorem . *Let* $\boldsymbol{M} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\boldsymbol{Q}\,\mathbf{mod})$ *and* $\boldsymbol{f} \in \mathbf{Hom}_{\mathbf{k}Q}(\boldsymbol{M},\boldsymbol{M})$.

Then,

$$\langle f, {}^{v}\tilde{\theta}_M\rangle_{\mathsf{S}^{-1}} = {}^{v}\,\mathbf{Tr}(f)$$

where $\langle -, +\rangle_{\mathsf{S}^{-1}}$ *denotes the paring of Serre duality*

$$\langle -, +\rangle_{\mathsf{S}^{-1}} : \mathbf{Hom}_{\mathrm{k}Q}(M,M) \otimes_{\mathrm{k}} \mathbf{Hom}_{\mathrm{k}Q}(\mathsf{S}^{-1}M, M) \longrightarrow \mathbf{k}.$$

5.3. **Universal Auslander-Reiten triangle.**

Definition . *An element* $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_0$ *is said to have the property (I) if for all* $\boldsymbol{M} \in \mathbf{ind}\,\boldsymbol{Q}$ *we have*

$$\mathbf{0} \neq \boldsymbol{v} \cdot \underline{\mathbf{dim}}(\boldsymbol{M}) = {}^{\boldsymbol{v}}\,\mathbf{Tr}(\mathbf{id}_{\boldsymbol{M}}) = \langle \mathbf{id}_{\boldsymbol{M}}, {}^{\boldsymbol{v}}\tilde{\boldsymbol{\theta}}_{\boldsymbol{M}} \rangle_{\mathsf{S}^{-1}} \text{ in } \mathbf{k}.$$

Example . *(1) Assume that* $\mathbf{char}\,\mathbf{k} = \mathbf{0}$*.*

If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r > \mathbf{0}$*, then* $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}$ *has the property (I).*

(2) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_r \in \mathbf{k}$ *are linearly independent over the prime field* $\mathbb{P}$ *of* $\mathbf{k}$*, then* $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_0$ *has the property (I).*

Using Happel's criterion for AR-triangle, we deduce

Theorem (Universal Auslander-Reiten triangle)**.**

Assume that $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_0$ *has the property (I).*

Then for any $\boldsymbol{M} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\boldsymbol{Q}\,\mathbf{mod})$ *the exact triangle* ${}^{\boldsymbol{v}}\mathsf{AR}_{\boldsymbol{M}} := {}^{\boldsymbol{v}}\mathsf{AR} \otimes^{\mathbb{L}} \boldsymbol{M}$ *is a direct sum of AR-triangles starting from indec. summand of* $\boldsymbol{M}$

$${}^{\boldsymbol{v}}\mathsf{AR}_{\boldsymbol{M}} : \boldsymbol{M} \xrightarrow{{}^{\boldsymbol{v}}\tilde{\boldsymbol{\varrho}}_{\boldsymbol{M}}} \widetilde{\boldsymbol{\Lambda}}_1 \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M} \xrightarrow{{}^{\boldsymbol{v}}\tilde{\boldsymbol{\pi}}_{1,\boldsymbol{M}}} \widetilde{\boldsymbol{\Pi}}_1 \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M} \xrightarrow{-{}^{\boldsymbol{v}}\tilde{\boldsymbol{\theta}}_{\boldsymbol{M}}[\mathbf{1}]} .$$

In other words, the morphism ${}^{\boldsymbol{v}}\tilde{\boldsymbol{\varrho}}_{\boldsymbol{M}} : \boldsymbol{M} \to {}^{\boldsymbol{v}}\widetilde{\boldsymbol{\Lambda}}_1 \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M}$ *is a minimal left* $\mathbf{rad}$*-approximation of* $\boldsymbol{M}$ *and*

the morphism ${}^{\boldsymbol{v}}\tilde{\boldsymbol{\pi}}_{1,\boldsymbol{M}} : {}^{\boldsymbol{v}}\widetilde{\boldsymbol{\Lambda}}_1 \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M} \to \widetilde{\boldsymbol{\Pi}}_1 \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M}$ *is a minimal right* $\mathbf{rad}$*-approximation of* $\widetilde{\boldsymbol{\Pi}}_1 \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M}$*.*

Remark . *If we fix* $\boldsymbol{M}$ *first, then we can weakened the assumption on* $\boldsymbol{v}$ *to "*$\boldsymbol{v} \cdot \underline{\mathbf{dim}}\boldsymbol{N} \neq \mathbf{0}$ *for each indec. summand* $\boldsymbol{N}$ *of* $\boldsymbol{M}$*".*

6. $\mathbf{rad}^n$-APPROXIMATIONS AND ${}^v\widetilde{\mathbf{\Lambda}}(Q)$

Recall there is a $*$-graded DG-algebra morphism

$$ {}^v\tilde{\boldsymbol{\pi}} : {}^v\widetilde{\mathbf{\Lambda}} \to \widetilde{\mathbf{\Pi}} $$

Let ${}^v\tilde{\boldsymbol{\pi}}_n : {}^v\widetilde{\mathbf{\Lambda}}_n \to \widetilde{\mathbf{\Pi}}_n$ be the $*$-degree $\boldsymbol{n}$-part and ${}^v\tilde{\boldsymbol{\pi}}_{n,M} = {}^v\tilde{\boldsymbol{\pi}}_n \otimes^{\mathbb{L}} \boldsymbol{M}$.

Theorem . *Assume that $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_0$ has the property (I).*

Let $\boldsymbol{M} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\boldsymbol{Q}\,\mathbf{mod})$.

Then the morphism

$$ {}^v\tilde{\boldsymbol{\pi}}_{n,M} : {}^v\widetilde{\mathbf{\Lambda}}_n \otimes^{\mathbb{L}}_{\mathbf{k}Q} \boldsymbol{M} \to \widetilde{\mathbf{\Pi}}_n \otimes^{\mathbb{L}}_{\mathbf{k}Q} \boldsymbol{M} $$

is a minimal right $\mathbf{rad}^n$-approximation of $\widetilde{\mathbf{\Pi}}_n \otimes^{\mathbb{L}}_{\mathbf{k}Q} \boldsymbol{M}$ for

$$ \boldsymbol{n} \text{ belonging to } \begin{cases} \mathbf{0} \le \boldsymbol{n} \le \boldsymbol{h} - \mathbf{2} & (\boldsymbol{Q} \text{ is Dynkin }), \\ \mathbf{0} \le \boldsymbol{n} & (\boldsymbol{Q} \text{ is non-Dynkin }). \end{cases} \qquad (\spadesuit) $$

Taking $\mathbb{R}\mathbf{Hom}_{\mathbf{k}Q^{\mathrm{op}}}(-, \widetilde{\mathbf{\Pi}}_n)$ of the right modules version of this theorem, inductively, we can deduce

Theorem . *Assume that $\boldsymbol{v} \in \mathbf{k}^{\times}\boldsymbol{Q}_0$ has the property (I).*

Let $\boldsymbol{M} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}\boldsymbol{Q}\,\mathbf{mod})$.

Then ${}^v\widetilde{\mathbf{\Lambda}}_n \otimes^{\mathbb{L}}_{\mathbf{k}Q} \boldsymbol{M}$ provides a minimal left $\mathbf{rad}^n$-approximation of $\boldsymbol{M}$ for $\boldsymbol{n}$ belonging to $(\spadesuit)$.

I.e., there exists a minimal left $\mathbf{rad}^n$-approximation morphism

$$ {}^v\boldsymbol{\beta}^{(n)}_M : \boldsymbol{M} \to {}^v\widetilde{\mathbf{\Lambda}}_n \otimes^{\mathbb{L}}_{\mathbf{k}Q} \boldsymbol{M} $$

for $\boldsymbol{n}$ belonging to $(\spadesuit)$.

Theorem . *Assume that* $\boldsymbol{v} \in \mathrm{k}^{\times}\boldsymbol{Q}_0$ *has the property (I).*

Let $\boldsymbol{M} \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(\mathrm{k}\boldsymbol{Q}\,\mathbf{mod})$*.*

Except the case where $\boldsymbol{Q}$ *is wild and* $\boldsymbol{M}$ *is a shift of a regular module,*

we have the following isomorphism

$$\bigoplus_{\boldsymbol{n}} {}^{\boldsymbol{v}}\widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{n}} \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M} \cong \bigoplus_{\boldsymbol{K} \in \mathbf{ind}\,\mathscr{C}_{\boldsymbol{M}}} \boldsymbol{K}^{\oplus\,\mathbf{dim}\,\mathbf{Hom}(\boldsymbol{M},\boldsymbol{K})}.$$

where in LHS, $\boldsymbol{n}$ *runs through* $(\spadesuit)$*.*

Corollary . *Assume that* $\boldsymbol{v} \in \mathrm{k}^{\times}\boldsymbol{Q}_0$ *has the property (I).*

Then as $\mathbf{k}\boldsymbol{Q}$*-modules*

$${}^{\boldsymbol{v}}\boldsymbol{\Lambda} \cong \bigoplus_{\boldsymbol{K} \in \mathbf{ind}\,\mathscr{P}(\boldsymbol{Q})} \boldsymbol{K}^{\oplus\,\mathbf{dim}\,\boldsymbol{K}}$$

7. What is a minimal left $\mathbf{rad}^{\boldsymbol{n}}$-approximation morphism

We have shown that

there exists a minimal left $\mathbf{rad}^{\boldsymbol{n}}$-approximation morphism

$${}^{\boldsymbol{v}}\boldsymbol{\beta}^{(\boldsymbol{n})}_{\boldsymbol{M}} : \boldsymbol{M} \to {}^{\boldsymbol{v}}\widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{n}} \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M}.$$

A natural candidate is the multiplication of the $\boldsymbol{n}$-th power ${}^{\boldsymbol{v}}\boldsymbol{\varrho}^{\boldsymbol{n}}$ of ${}^{\boldsymbol{v}}\boldsymbol{\varrho}$

Question: Is

$${}^{\boldsymbol{v}}\boldsymbol{\varrho}^{\boldsymbol{n}}_{\boldsymbol{M}} : \boldsymbol{M} \to {}^{\boldsymbol{v}}\widetilde{\boldsymbol{\Lambda}}_{\boldsymbol{n}} \otimes^{\mathbb{L}}_{\mathbf{k}\boldsymbol{Q}} \boldsymbol{M}$$

a minimal left rad$^{\boldsymbol{n}}$-approximation of $\boldsymbol{M}$?

Example . *Let $\boldsymbol{Q}$ be a directed $\boldsymbol{A_3}$-quiver.*

$$\boldsymbol{Q}:\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3\,,\quad \overline{\boldsymbol{Q}}:\ 1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3\,.$$

The property (I) precisely says that $\boldsymbol{v}$ satisfies the followings

$$\boldsymbol{v_1 \neq 0,\ v_2 \neq 0,\ v_3 \neq 0,}$$

$$\boldsymbol{v_1 + v_2 \neq 0,\ v_2 + v_3 \neq 0,\ v_1 + v_2 + v_3 \neq 0.}$$

Proposition . *Assume that $\boldsymbol{v} \in \mathrm{k}^{\times}\boldsymbol{Q_0}$ has the property (I). Then, the morphism*

$$^{v}\boldsymbol{\varrho}^{2}_{\boldsymbol{P_2}} : \boldsymbol{P_2} \to {}^{v}\widetilde{\boldsymbol{\Lambda}}_2 \otimes^{\mathbb{L}}_{\mathrm{k}\boldsymbol{Q}} \boldsymbol{P_2}$$

is a minimal left $\mathbf{rad^2}$*-approximation if and only if*

$$\boldsymbol{v_1 + 2v_2 + v_3 \neq 0.}$$

We note

$$\boldsymbol{v_1 + 2v_2 + v_3 = v \cdot \underline{\mathbf{dim}}({}^{v}\widetilde{\boldsymbol{\Lambda}}_1 \otimes^{\mathbb{L}}_{\mathrm{k}\boldsymbol{Q}} \boldsymbol{P_2}).}$$

7.1. Minimal left $\mathbf{rad^2}$-approximation.

Theorem . *Assume that $\boldsymbol{v} \in \mathrm{k}^{\times}\boldsymbol{Q_0}$ has the property (I). Let $\boldsymbol{M} \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(\mathrm{k}\boldsymbol{Q}\,\mathbf{mod})$. Assume that*

$$\boldsymbol{v} \cdot \underline{\mathbf{dim}}({}^{v}\widetilde{\boldsymbol{\Lambda}}_1 \otimes^{\mathbb{L}}_{\mathrm{k}\boldsymbol{Q}} \boldsymbol{M}) \neq \mathbf{0}.$$

Then the morphism

$$^{v}\boldsymbol{\varrho}^{2}_{\boldsymbol{M}} : \boldsymbol{M} \longrightarrow {}^{v}\widetilde{\boldsymbol{\Lambda}}_2 \otimes^{\mathbb{L}}_{\mathrm{k}\boldsymbol{Q}} \boldsymbol{M}$$

is a minimal left $\mathbf{rad^2}$*-approximation of $\boldsymbol{M}$.*

7.2. Minimal left $\mathbf{rad}^n$-approx ($\mathbf{char\,k = 0}$).

Theorem . *Assume* $\mathbf{char\,k = 0}$.

Let $M \in \mathbf{ind}\,\mathsf{D}^{\mathrm{b}}(\mathrm{k}Q\,\mathbf{mod})$ *and* n *belongs to* $(\spadesuit)$.

Then for a generic parameter $v \in \mathrm{k}^{\times}Q_0$, *the morphism*

$$ {}^{v}\varrho^{n}_{M} : M \to {}^{v}\widetilde{\Lambda}_n \otimes^{\mathbb{L}}_{\mathrm{k}Q} M $$

is a minimal left $\mathbf{rad}^n$*-approximation of* M.

Corollary . *Assume* $\mathbf{char\,k = 0}$.

Let Q *be a Dynkin quiver with the Coxeter number* h.

Then for a generic parameter $v \in \mathrm{k}^{\times}Q_0$ *the morphism*

$$ {}^{v}\varrho^{n}_{M} : M \to {}^{v}\widetilde{\Lambda}_n \otimes^{\mathbb{L}}_{\mathrm{k}Q} M $$

is a minimal left $\mathbf{rad}^n$*-approximation of* M

for all $M \in \mathsf{D}^{\mathrm{b}}(\mathrm{k}Q\,\mathbf{mod})$ *and* $n = 1, 2, \cdots, h-2$,

7.3. Minimal left $\mathbf{rad}^n$-approx ($Q = A_N$-quiver).

Theorem . *Let* $N \geq 1$ *and* Q *an* A_N*-quiver (note* $h = N + 1$*).*

Assume that $\mathbf{k}$ *has a primitive* h*-th root of unity.*

Then for a generic $v \in \mathrm{k}^{\times}Q_0$, *the morphism*

$$ {}^{v}\varrho^{n}_{M} : M \to \widetilde{\Lambda}_n \otimes^{\mathbb{L}}_{\mathrm{k}Q} M $$

is a minimal left $\mathbf{rad}^n$*-approximation of* M

for all $M \in \mathsf{D}^{\mathrm{b}}(\mathrm{k}Q\,\mathbf{mod})$ *and* $n = 1, 2, \cdots, h-2$.

Thank you