Finite approximations as a tool for studying triangulated categories

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Overview

1. A bunch of definitions
2. Two of the main theorems
3. Where we're headed, followed by background
4. The main theorems, sources of examples
5. First applications
6. More general theory and the next applications

A bunch of definitions

Reminder

Following a 1974 article of Lawvere, a metric on a category is a function that assigns a positive real number (length) to every morphism, satisfying:

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$$\text{Length}(\mathrm{id}) = 0 ,$$

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1. For any identity map $\mathrm{id} : X \longrightarrow X$ we have

$$\mathsf{Length}(\mathrm{id}) \quad = \quad 0 \; ,$$

2. and if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then

$$\mathsf{Length}(gf) \quad \leq \quad \mathsf{Length}(f) + \mathsf{Length}(g) \; .$$

Definition (Equivalence of metrics)

We'd like to view two metrics on a category $\mathcal{C}$ as equivalent if the identity functor $\mathrm{id} : \mathcal{C} \longrightarrow \mathcal{C}$ is uniformly continuous in both directions.

More formally:

Let $\mathcal{C}$ be a category. Two metrics

$$\mathsf{Length}_1 \qquad \text{and} \qquad \mathsf{Length}_2$$

are declared equivalent if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\{\mathsf{Length}_1(f) \ < \ \delta\} \qquad \Longrightarrow \qquad \{\mathsf{Length}_2(f) \ < \ \varepsilon\}$$

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Definition (Cauchy sequences)

Let $\mathcal{C}$ be a category with a metric. A Cauchy sequence in $\mathcal{C}$ is a sequence $E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow \cdots$ of composable morphisms such that, for any $\varepsilon > 0$, there exists an $M > 0$ such that the morphisms $E_i \longrightarrow E_j$ satisfy

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whenever $i, j > M$.

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We will assume the category $\mathcal{C}$ is $\mathbb{Z}$–linear. This means that $\mathrm{Hom}(a, b)$ is an abelian group for every pair of objects $a, b \in \mathcal{C}$, and that composition is bilinear.

Definition (The categories $\mathfrak{L}(\mathcal{C})$, $\mathfrak{C}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$)

Let $\mathcal{C}$ be a $\mathbb{Z}$–linear category with a metric. Let $Y : \mathcal{C} \longrightarrow \mathrm{Mod}\text{–}\mathcal{C}$ be the Yoneda map, that is the map sending an object $c \in \mathcal{C}$ to the functor $Y(c) = \mathrm{Hom}(-, c)$, viewed as an additive functor $\mathcal{C}^{\mathrm{op}} \longrightarrow Ab$.

1. Let $\mathfrak{L}(\mathcal{C})$ be the *completion* of $\mathcal{C}$, meaning full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are the colimits in $\mathrm{Mod}\text{–}\mathcal{C}$ of Cauchy sequences in $\mathcal{C}$.
2. Let $\mathfrak{C}(\mathcal{C})$ be the full subcategory of $\mathrm{Mod}\text{–}\mathcal{C}$ whose objects are *compactly supported*. By this we mean that $F : \mathcal{C}^{\mathrm{op}} \longrightarrow Ab$ belongs to $\mathfrak{C}(\mathcal{C})$ if there exists an $\varepsilon > 0$ so that

$$\{\mathrm{Length}(a \to b) < \varepsilon\} \quad \Longrightarrow \quad \{F(b) \longrightarrow F(a) \text{ is an isomorphism}\}.$$

3. Finally let $\mathfrak{S}(\mathcal{C}) = \mathfrak{C}(\mathcal{C}) \cap \mathfrak{L}(\mathcal{C})$.

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Equivalent metrics lead to identical $\mathfrak{L}(\mathcal{C})$, $\mathfrak{C}(\mathcal{C})$ and $\mathfrak{S}(\mathcal{C})$.

Heuristic

We want to specialize the above to a situation in which we can actually prove something.

Let $\mathcal{S}$ be a triangulated category with a Lawvere metric. We will only consider "translation invariant" metrics, meaning for any homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ c & \xrightarrow{g} & d \end{array}$$

we must have

$$\text{Length}(f) \quad = \quad \text{Length}(g)$$

Heuristic, continued

Given any $f : a \longrightarrow b$ we may form the homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{g} & x \end{array}$$

and our assumption tells us that

$$\mathrm{Length}(f) \quad = \quad \mathrm{Length}(g)$$

Hence it suffices to know the lengths of the morphisms $0 \longrightarrow x$.

Heuristic, continued

We will soon be assuming that the metric is non-archimedean. Replacing the metric by an equivalent (if necessary), we may also assume our metric takes values in the set of rational numbers of the form $\{0, \infty\} \cup \{2^n \mid n \in \mathbb{Z}\}$. To know everything about the metric it therefore suffices to specify the balls

$$B_n \quad = \quad \left\{ x \in \mathcal{S} \;\middle|\; \text{the morphism } 0 \longrightarrow x \text{ has length } \leq \frac{1}{2^n} \right\}$$

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If $f : x \longrightarrow y$ is any morphism, to compute its length you complete to a triangle $x \xrightarrow{f} y \longrightarrow z$, and then

$$\text{Length}(f) \quad = \quad \inf \left\{ \frac{1}{2^n} \;\middle|\; z \in B_n \right\}$$

Definition (good metric)

Let $\mathcal{S}$ be a triangulated category. A good metric on $\mathcal{S}$ is a sequence of full subcategories $\{B_n,\ n \in \mathbb{Z}\}$, containing 0 and satisfying

1. We want: if $x \xrightarrow{f} y \xrightarrow{g} z$ are composable morphisms, then $\text{Length}(gf) \leq \max\big(\text{Length}(f), \text{Length}(g)\big)$.

   This translates to $B_n * B_n = B_n$, which means that if there exists a triangle $b \longrightarrow x \longrightarrow b'$ with $b, b' \in B_n$, then $x \in B_n$.

2. $B_{n+1}[-1] \cup B_{n+1} \cup B_{n+1}[1] \subset B_n$.

Example

Suppose $\mathcal{S}$ has a t-structure. The $B_n = \mathcal{S}^{\leq -n}$ works.

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Example

Suppose $\mathcal{S}$ has a t-structure. The $B_n = \mathcal{S}^{\leq -n}$ works.

Theorem (1)

Let $\mathcal{S}$ be a triangulated category with a good metric. Some slides ago we defined a category

$$\mathfrak{S}(\mathcal{S}) = \mathfrak{L}(\mathcal{S}) \cap \mathfrak{C}(\mathcal{S}) \ .$$

Now define the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

With this definition of distinguished triangles, the category $\mathfrak{S}(\mathcal{S})$ is triangulated.

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Example (the six triangulated categories to keep in mind)

Let R be an associative ring.

1. $\mathrm{D}(R)$ will be our shorthand for $\mathrm{D}(R\text{–}\mathrm{Mod})$; the objects are all cochain complexes of R-modules, no conditions.
2. $\mathrm{D}^b(R\text{–}\mathrm{proj})$ is the derived category of bounded complexes of finitely generated, projective R–modules.
3. Suppose the ring R is coherent. Then $\mathrm{D}^b(R\text{–}\mathrm{mod})$ is the bounded derived category of finitely presented R–modules.

Example (the six triangulated categories to keep in mind, continued)

Let X be a quasicompact, quasiseparated scheme.

4. $\mathrm{D}_{\mathrm{qc}}(X)$ will be our shorthand for $\mathrm{D}_{\mathrm{qc}}(\mathcal{O}_X\text{–}\mathrm{Mod})$. The objects are the complexes of $\mathcal{O}_X$–modules, and the only condition is that the cohomology must be quasicoherent.

5. The objects of $\mathrm{D}^{\mathrm{perf}}(X) \subset \mathrm{D}_{\mathrm{qc}}(X)$ are the perfect complexes. A complex $F \in \mathrm{D}_{\mathrm{qc}}(X)$ is *perfect* if there exists an open cover $X = \cup_i U_i$ such that, for each U_i, the restriction map $u_i^* : \mathrm{D}_{\mathrm{qc}}(X) \longrightarrow \mathrm{D}_{\mathrm{qc}}(U_i)$ takes F to an object $u_i^*(F)$ isomorphic in $\mathrm{D}_{\mathrm{qc}}(U_i)$ to a bounded complex of vector bundles.

6. Assume X is noetherian. The objects of $\mathrm{D}^b_{\mathrm{coh}}(X) \subset \mathrm{D}_{\mathrm{qc}}(X)$ are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

Theorem (1, continued)

Now let R be an associative ring. Then the category $\mathsf{D}^b(R\text{–proj})$ admits an intrinsic metric [up to equivalence], so that

$$\mathfrak{S}\left[\mathsf{D}^b(R\text{–proj})\right] = \mathsf{D}^b(R\text{–mod}).$$

If we further assume that R is coherent then there is on $\left[\mathsf{D}^b(R\text{–mod})\right]^{\mathrm{op}}$ an intrinsic metric [again up to equivalence], such that

$$\mathfrak{S}\left(\left[\mathsf{D}^b(R\text{–mod})\right]^{\mathrm{op}}\right) = \left[\mathsf{D}^b(R\text{–proj})\right]^{\mathrm{op}} .$$

Theorem (1, continued)

Let X be a quasicompact, separated scheme. There is an intrinsic equivalence class of metrics on $\mathsf{D}^{\mathrm{perf}}(X)$ for which

$$\mathfrak{S}\left[\mathsf{D}^{\mathrm{perf}}(X)\right] = \mathsf{D}^{b}_{\mathsf{coh}}(X) \ .$$

Now assume that X is a noetherian, separated scheme. Then the category $\left[\mathsf{D}^{b}_{\mathsf{coh}}(X)\right]^{\mathrm{op}}$ can be given intrinsic metrics [up to equivalence], so that

$$\mathfrak{S}\left(\left[\mathsf{D}^{b}_{\mathsf{coh}}(X)\right]^{\mathrm{op}}\right) = \left[\mathsf{D}^{\mathrm{perf}}(X)\right]^{\mathrm{op}} \ .$$

Where we're headed: the big theorem that has much of what has preceded as corollaries

Theorem (the really central result)

The triangulated categories $\mathrm{D}(R)$ *and* $\mathrm{D}_{\mathrm{qc}}(X)$ *are approximable.*

Where we're headed: formal definition of approximability

Let $\mathcal{T}$ be a triangulated category with coproducts. It is approximable if:

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ so that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.
- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}_A^{[-A,A]}$.

Analogy to keep in mind: Fourier series

Triangulated category $\mathcal{T}$	Space of functions $f : \mathbb{S}^1 \longrightarrow \mathbb{C}$
Compact generator $G \in \mathcal{T}$	Choice of function, e.g. $g(x) = e^{2\pi ix}$
t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$	Banach norm, e.g. L^p–norm
$[1] : \mathcal{T} \longrightarrow \mathcal{T}$	The automorphism sending f to $\frac{f}{2}$
$\overline{\langle G \rangle}_A^{[-A,A]}$	The vector space spanned by $\left\{e^{2\pi inx} \mid -A \leq n \leq A\right\}$

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Background: compact generation, t–structures and the subcategories $\overline{\langle G\rangle}_A^{[-A,A]}$

Assume $\mathcal{T}$ is a triangulated category with coproducts.

An object $G \in \mathcal{T}$ is compact if $\mathrm{Hom}(G,-)$ commutes with coproducts.

The compact object $G \in \mathcal{T}$ generates $\mathcal{T}$ if every nonzero object $X \in \mathcal{T}$ admits a nonzero map $G[i] \longrightarrow X$, for some $i \in \mathbb{Z}$.

Example (the standard *t*–structure on D(R))

We define two full subcategories of D(R):

- $$\mathrm{D}(R)^{\leq 0} \quad = \quad \{A\in \mathrm{D}(R)\mid H^i(A)=0 \text{ for all } i>0\}$$
- $$\mathrm{D}(R)^{\geq 0} \quad = \quad \{A\in \mathrm{D}(R)\mid H^i(A)=0 \text{ for all } i<0\}$$

Definition

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- $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$ and $\mathcal{T}^{\geq 0} \subset \mathcal{T}^{\geq 0}[1]$
- $\mathrm{Hom}\Big(\mathcal{T}^{\leq 0}[1]\, ,\, \mathcal{T}^{\geq 0}\Big) = 0$
- Every object $B \in \mathcal{T}$ admits a triangle $A \longrightarrow B \longrightarrow C \longrightarrow$ with $A \in \mathcal{T}^{\leq 0}[1]$ and $C \in \mathcal{T}^{\geq 0}$.

Example (the standard *t*–structure on $\mathrm{D}(R)$)

We define two full subcategories of $\mathrm{D}(R)$:

- $$\mathrm{D}(R)^{\leq 0} = \{A \in \mathrm{D}(R) \mid H^i(A) = 0 \text{ for all } i > 0\}$$
- $$\mathrm{D}(R)^{\geq 0} = \{A \in \mathrm{D}(R) \mid H^i(A) = 0 \text{ for all } i < 0\}$$

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Notation

Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ and an integer $n \in \mathbb{Z}$ we define

$$\mathcal{T}^{\leq -n} = \mathcal{T}^{\leq 0}[n] \qquad \textit{and} \qquad \mathcal{T}^{\geq -n} = \mathcal{T}^{\geq 0}[n]$$

Reminder

We can define a good metric by setting

$$B_n \quad = \quad \mathcal{T}^{\leq -n} \; .$$

The black box construction of $\overline{\langle G\rangle}_A^{[-A,A]}$, of $\overline{\langle G\rangle}^{(-\infty,A]}$ and of $\langle G\rangle_A$

Let $\mathcal{T}$ be a triangulated category, and let $A > 0$ be an integer. I ask the audience to accept, as a black box, that there are sensible constructions of the following three full subcategories of $\mathcal{T}$:

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2.
3.

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1. $\langle G\rangle_A$. This is classical, it consists of the objects of $\mathcal{T}$ obtainable from G using no more than A extensions.
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3. Also assumes $\mathcal{T}$ has coproducts: $\overline{\langle G\rangle}_A^{[-A,A]}$. This is new, both the allowed suspensions and the number of extensions allowed are bounded.

Definition (formal definition of approximability)

Let $\mathcal{T}$ be a triangulated category with coproducts. It is approximable if:

There exists a compact generator $G \in \mathcal{T}$, a t–structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$, and an integer $A > 0$ so that

- $G^{\perp}$ contains $\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}$.

  This means: $\mathrm{Hom}\big(G\,,\,\mathcal{T}^{\leq -A} \cup \mathcal{T}^{\geq A}\big) = 0$.

- For every object $F \in \mathcal{T}^{\leq 0}$ there exists a triangle $E \longrightarrow F \longrightarrow D$, with $D \in \mathcal{T}^{\leq -1}$ and $E \in \overline{\langle G \rangle}_A^{[-A,A]}$.

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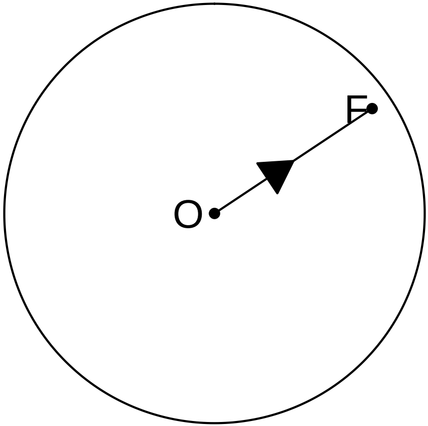
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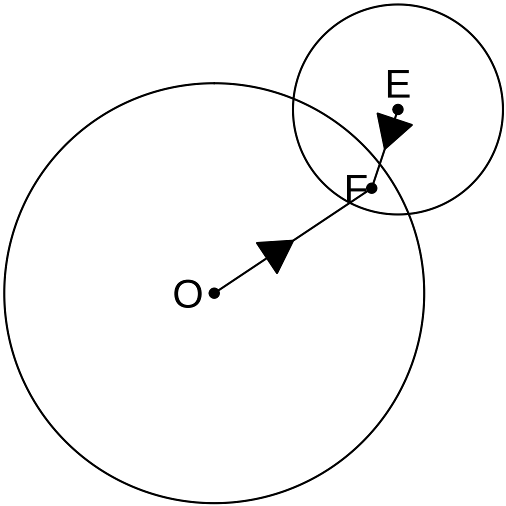
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The main theorems—sources of examples

1. If $\mathcal{T}$ has a compact generator G such that $\mathrm{Hom}\big(G, G[i]\big) = 0$ for all $i \geq 1$, then $\mathcal{T}$ is approximable.

2. Let X be a quasicompact, separated scheme. Then the category $\mathrm{D}_{\mathrm{qc}}(X)$ is approximable.

3. [Joint with Jesse Burke and Bregje Pauwels]: Suppose we are given a recollement of triangulated categories

$$\mathcal{R} \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \mathcal{S} \; \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \; \mathcal{T}$$

with $\mathcal{R}$ and $\mathcal{T}$ approximable. Assume further that the category $\mathcal{S}$ is compactly generated, and any compact object $H \in \mathcal{S}$ has the property that $\mathrm{Hom}\big(H, H[i]\big) = 0$ for $i \gg 0$. Then the category $\mathcal{S}$ is also approximable.

References for the fact(s) that the nontrivial examples of approximable triangulated categories really are examples

Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, https://arxiv.org/abs/1806.05342.

Amnon Neeman, *Strong generators in* $\mathrm{D}^{\mathrm{perf}}(X)$ *and* $\mathrm{D}^b_{\mathrm{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

It's time to come to applications. Before stating the first two we remind the audience what the terms used in the theorems mean.

An old definition

Let $\mathcal{S}$ be a triangulated category, and let $G \in \mathcal{S}$ be an object.

G is a strong generator if there exists an integer $\ell > 0$ with $\mathcal{S} = \langle G \rangle_\ell$.

The category $\mathcal{S}$ is strongly generated or regular if there exists a strong generator $G \in \mathcal{S}$.

The main theorems—first applications

1. Let X be a quasicompact, separated scheme. The category $\mathrm{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X has an open cover by affine schemes $\mathrm{Spec}(R_i)$, with each R_i of finite global dimension.

   Remark: if X is noetherian and separated, this simplifies to saying that $\mathrm{D}^{\mathrm{perf}}(X)$ is strongly generated if and only if X is regular and finite dimensional.

2. Let X be a finite-dimensional, separated, noetherian, quasiexcellent scheme. Then the category $\mathrm{D}^b_{\mathrm{coh}}(X)$ is strongly generated.

Ko Aoki, *Quasiexcellence implies strong generation*, J. Reine Angew. Math. (published online 14 August 2021, 6 pages), see also https://arxiv.org/abs/2009.02013.

Amnon Neeman, *Strong generators in* $\mathbf{D}^{\mathrm{perf}}(X)$ *and* $\mathbf{D}^b_{\mathrm{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.

Moving on to further theory and the next applications

Amnon Neeman, *Metrics on triangulated categories*, J. Pure Appl. Algebra **224** (2020), no. 4, 106206, 13.

Amnon Neeman, *Approximable triangulated categories*, Representations of Algebras, Geometry and Physics, Contemp. Math., vol. 769, Amer. Math. Soc., Providence, RI, 2021, pp. 111–155.

Moving on to further theory and the next applications

- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, https://arxiv.org/abs/1804.02240.
- Amnon Neeman, *The category $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ as functors on $\mathcal{T}_c^b$*, https://arxiv.org/abs/1806.05777.
- Amnon Neeman, *The categories $\mathcal{T}^c$ and $\mathcal{T}_c^b$ determine each other*, https://arxiv.org/abs/1806.06471.

‘

Let us begin in a generality which does not assume the full power of approximability.

‘

Definition (equivalent *t*–structures)

Let $\mathcal{T}$ be any triangulated category, and let $\big(\mathcal{T}_1^{\leq 0},\mathcal{T}_1^{\geq 0}\big)$ and $\big(\mathcal{T}_2^{\leq 0},\mathcal{T}_2^{\geq 0}\big)$ be two *t*–structures on $\mathcal{T}$. We declare them equivalent if the metrics they induce are equivalent.

To spell it out: the two *t*–structures are equivalent if there exists an integer $A>0$ with

$$\mathcal{T}_1^{\leq -A}\subset\mathcal{T}_2^{\leq 0}\subset\mathcal{T}_1^{\leq A}.$$

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Preferred t–structures

Let $\mathcal{T}$ be a triangulated category with coproducts, and let $G \in \mathcal{T}$ be a compact object. A 2003 theorem of Alonso, Jeremías and Souto teaches us that $\mathcal{T}$ has a unique t–structure $\left(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\right)$ *generated by* G.

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More precisely the following formula delivers a t–structure:

$$\mathcal{T}_G^{\leq 0} = \overline{\langle G \rangle}^{(-\infty,0]}\ , \qquad \mathcal{T}_G^{\geq 0} = \left(\left[\mathcal{T}_G^{\leq 0}\right]^{\perp}\right)[1]\ .$$

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If G and H are two compact generators for $\mathcal{T}$, then the t–structures $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ and $\big(\mathcal{T}_H^{\leq 0}, \mathcal{T}_H^{\geq 0}\big)$ are equivalent.

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We say that a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ is in the preferred equivalence class if it is equivalent to $\big(\mathcal{T}_G^{\leq 0}, \mathcal{T}_G^{\geq 0}\big)$ for some compact generator G, hence for every compact generator.

‘

Given a t–structure $\left(\mathcal{T}^{\leq 0},\mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^{-}=\bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^{+}=\bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^{b}=\mathcal{T}^{-}\cap\mathcal{T}^{+}$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^{-}$, $\mathcal{T}^{+}$ and $\mathcal{T}^{b}$.

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Given a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ it is customary to define the categories

$$\mathcal{T}^- = \bigcup_n \mathcal{T}^{\leq n}, \qquad \mathcal{T}^+ = \bigcup_n \mathcal{T}^{\geq -n}, \qquad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+$$

It's obvious that equivalent t–structures yield identical $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Now assume that $\mathcal{T}$ has coproducts and there exists a single compact generator G. Then there is a preferred equivalence class of t–structures, and a correponding preferred $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$. These are intrinsic, they're independent of any choice. In the remainder of the slides we only consider the "preferred" $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$.

Definition (the subtler categories $\mathcal{T}_c^b \subset \mathcal{T}_c^-$)

Let $\mathcal{T}$ be a triangulated category with coproducts, and assume it has a compact generator G. Choose a t–structure $\big(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\big)$ in the preferred equivalence class.

Heuristic: the full subcategory $\mathcal{T}_c^-$ should be thought of as the closure of $\mathcal{T}^c$ with respect to the metric—every object of $\mathcal{T}_c^-$ admits arbitrarily good approximations by compacts.

To spell it out more formally:

$$\mathcal{T}_c^- \quad = \quad \left\{ F \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{For every } \varepsilon > 0 \text{ there exists a morphism} \\ f : E \longrightarrow F \\ \text{with } E \text{ compact and } \mathrm{Length}(f) < \varepsilon \end{array} \right\}$$

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We furthermore define $\mathcal{T}_c^b = \mathcal{T}^b \cap \mathcal{T}_c^-$.

It's obvious that the category $\mathcal{T}_c^-$ is intrinsic. As $\mathcal{T}_c^-$ and $\mathcal{T}^b$ are both intrinsic, so is their intersection $\mathcal{T}_c^b$.

We have defined all this intrinsic structure, assuming only that $\mathcal{T}$ is a triangulated category with coproducts and with a single compact generator. In this generality we know that the subcategories $\mathcal{T}^-$, $\mathcal{T}^+$ and $\mathcal{T}^b$ are thick.

If we furthermore assume that $\mathcal{T}$ is approximable, then the subcategories $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ are also thick.

It can be proved that:

Example (The special case $\mathcal{T} = \mathbf{D}(R)$, with R a coherent ring)

$$\begin{array}{rclrclrcl}
\mathcal{T}^+ &=& \mathbf{D}^+(R), & \mathcal{T}^- &=& \mathbf{D}^-(R), & \mathcal{T}^c &=& \mathbf{D}^b(R\text{–proj}), \\
\mathcal{T}^b &=& \mathbf{D}^b(R), & \mathcal{T}^-_c &=& \mathbf{D}^-(R\text{–proj}), & \mathcal{T}^b_c &=& \mathbf{D}^b(R\text{–mod})
\end{array}$$

Example (The special case $\mathcal{T} = \mathbf{D}_{\mathrm{qc}}(X)$, with X a noetherian, separated scheme)

$$\begin{array}{rclrclrcl}
\mathcal{T}^+ &=& \mathbf{D}^+_{\mathrm{qc}}(X), & \mathcal{T}^- &=& \mathbf{D}^-_{\mathrm{qc}}(X), & \mathcal{T}^c &=& \mathbf{D}^{\mathrm{perf}}(X), \\
\mathcal{T}^b &=& \mathbf{D}^b_{\mathrm{qc}}(R), & \mathcal{T}^-_c &=& \mathbf{D}^-_{\mathrm{coh}}(X), & \mathcal{T}^b_c &=& \mathbf{D}^b_{\mathrm{coh}}(X)
\end{array}$$

Analogue to keep in mind, for what's coming

Consider the space S of Lebesgue measurable real-valued functions on $\mathbb{R}$. The pairing taking $f, g \in S$ to

$$\langle f, g \rangle = \int fg \, d\mu$$

is a map

$$S \times S \xrightarrow{\langle -,- \rangle} \mathbb{R} \cup \{\infty\}.$$

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If $f \in L^p$ and $g \in L^q$, with $\frac{1}{p} + \frac{1}{q} = 1$, then $\langle f, g \rangle \in \mathbb{R}$ and we deduce two maps

$$L^p \longrightarrow \mathrm{Hom}(L^q, \mathbb{R}), \qquad L^q \longrightarrow \mathrm{Hom}(L^p, \mathbb{R})$$

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maps, which turn out to be isometries

$$L^p \longrightarrow \mathrm{Hom}(L^q, \mathbb{R}), \qquad L^q \longrightarrow \mathrm{Hom}(L^p, \mathbb{R})$$

Let R be a commutative ring, and assume $\mathcal{T}$ is an R-linear category. The pairing sending $A, B \in \mathcal{T}$ to $\mathrm{Hom}(A, B)$ gives a map

$$\mathcal{T}^{\mathrm{op}} \times \mathcal{T} \longrightarrow R\text{–}\mathrm{Mod}$$

and we deduce two ordinary Yoneda maps

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big) \\ \mathcal{T}^{\mathrm{op}} &\longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}\ ,\ R\text{–}\mathrm{Mod}\Big) \end{aligned}$$

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$$\mathcal{T} \longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$
$$\mathcal{T}^{\mathrm{op}} \longrightarrow \mathrm{Hom}_R\Big(\mathcal{T}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

If $\mathcal{T}$ is also an approximable triangulated category, we can restrict to obtain restricted Yoneda maps

1. $$\mathcal{T}_c^- \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\Big(\big[\mathcal{T}^c\big]^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

2. $$\big[\mathcal{T}_c^-\big]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\Big(\mathcal{T}_c^b\ ,\ R\text{–}\mathrm{Mod}\Big)$$

Theorem (first general theorem about approximable categories)

Let R be a noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}(G, G[n])$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}^b_c \xhookrightarrow{\quad i \quad} \mathcal{T}^-_c \xrightarrow{\quad \mathcal{Y} \quad} \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}, R\text{–}\mathrm{Mod}\big)$$

$$[\mathcal{T}^c]^{\mathrm{op}} \xhookrightarrow{\quad \widetilde{\imath} \quad} [\mathcal{T}^-_c]^{\mathrm{op}} \xrightarrow{\quad \widetilde{\mathcal{Y}} \quad} \mathrm{Hom}_R\big(\mathcal{T}^b_c, R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ a and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

   A homological functor $H : \mathcal{T}^-_c \longrightarrow R\text{–}\mathrm{Mod}$ is locally finite if, for every object C, the R–module $H^i(C)$ is finite for every $i \in \mathbb{Z}$ and vanishes if $i \gg 0$.

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Theorem (first general theorem about approximable categories)

Let R be a noetherian ring, and let $\mathcal{T}$ be an R–linear, approximable triangulated category. Suppose there exists in $\mathcal{T}$ a compact generator G so that $\mathrm{Hom}(G, G[n])$ is a finite R–module for all $n \in \mathbb{Z}$. Consider the functors

$$\mathcal{T}_c^b \overset{i}{\hookrightarrow} \mathcal{T}_c^- \xrightarrow{\ \mathcal{Y}\ } \mathrm{Hom}_R\big([\mathcal{T}^c]^{\mathrm{op}}, R\text{–}\mathrm{Mod}\big)$$

$$[\mathcal{T}^c]^{\mathrm{op}} \overset{\widetilde{\imath}}{\hookrightarrow} [\mathcal{T}_c^-]^{\mathrm{op}} \xrightarrow{\ \widetilde{\mathcal{Y}}\ } \mathrm{Hom}_R\big(\mathcal{T}_c^b, R\text{–}\mathrm{Mod}\big)$$

where i and $\widetilde{\imath}$ are the obvious inclusions. Then

1. *The functor $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ are both full, and the essential images are the locally finite homological functors.*
2. *The composites $\mathcal{Y} \circ i$ a and $\widetilde{\mathcal{Y}} \circ \widetilde{\imath}$ are both fully faithful, and the essential images are the finite homological functors.*

A homological functor $H : \mathcal{T}_c^- \longrightarrow R\text{–}\mathrm{Mod}$ is locally finite if, for every object C, the R–module $H^n(C)$ is finite for every $n \in \mathbb{Z}$ and vanishes if $n \gg 0$ or $n \ll 0$

Let X be a scheme proper over a noetherian ring R. Then $\mathcal{T} = \mathsf{D}_{\mathsf{qc}}(X)$ satisfies the hypotheses of the theorem.

Corollary

The functor

$$\mathsf{D}^b_{\mathsf{coh}}(X) \xrightarrow{\quad \mathcal{Y}\circ i \quad} \mathrm{Hom}_R\Big(\big[\mathsf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ R\text{–}\mathrm{Mod}\Big)$$

gives an equivalence of $\mathsf{D}^b_{\mathsf{coh}}(X)$ *with the category of finite homological functors* $\big[\mathsf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}} \longrightarrow R\text{–}\mathrm{Mod}$.

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The above delivers a map taking $B \in \mathrm{D}^b_{\mathrm{coh}}(X^{\mathrm{an}})$ to a finite homological functor $\big[\mathrm{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}} \longrightarrow \mathbb{C}\text{–mod}$.

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$$\begin{array}{ccc}
\mathrm{D}^b_{\mathsf{coh}}(X^{\mathrm{an}}) & \searrow & \\
\Big\downarrow \mathcal{R} & & \mathrm{Hom}_R\Big(\big[\mathrm{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}}\ ,\ \mathbb{C}\text{–Mod}\Big) \\
\mathrm{D}^b_{\mathsf{coh}}(X) & \xrightarrow{\ \mathcal{Y}\circ i\ } \nearrow &
\end{array}$$

Representablity produced for us a functor $\mathcal{R} : \mathrm{D}^b_{\mathrm{coh}}(X^{\mathrm{an}}) \longrightarrow \mathrm{D}^b_{\mathrm{coh}}(X)$, which is easily seen to be right adjoint to $\mathcal{L}$.

To prove Serre's GAGA theorem it suffices to show that, in the adjunction $\mathcal{L} \dashv \mathcal{R}$, the unit and counit of adjuction are isomorphisms. And for this it suffices to produce a set of objects $P \subset \mathrm{D}^{\mathrm{perf}}(X)$, with $P[1] = P$ and such that

1. $P^\perp = \{0\}$.
2. $\mathcal{L}(P)^\perp = \{0\}$.
3. For every object $p \in P$ and every object $x \in \mathrm{D}^b_{\mathrm{coh}}(X)$, the natural map

$$\mathrm{Hom}(p, x) \longrightarrow \mathrm{Hom}\big(\mathcal{L}(p)\,,\mathcal{L}(x)\big)$$

   is an isomorphism.

But this is easy: we let P be the collection of perfect complexes supported at closed points.

Theorem (reminder: first theorem of the talk)

Let $\mathcal{S}$ be a triangulated category with a good metric. Many slides ago we defined a category

$$\mathfrak{S}(\mathcal{S}) = \mathfrak{L}(\mathcal{S}) \cap \mathfrak{C}(\mathcal{S}) \ .$$

We also defined the distinguished triangles in $\mathfrak{S}(\mathcal{S})$ to be the colimits in $\mathfrak{S}(\mathcal{S}) \subset \mathrm{Mod}\text{–}\mathcal{S}$ of Cauchy sequences of distinguished triangles in $\mathcal{S}$.

With this definition of distinguished triangles, the category $\mathfrak{S}(\mathcal{S})$ is triangulated.

Theorem (second general theorem about approximable categories)

Let $\mathcal{T}$ be an approximable triangulated category. For a suitable choice of metric on $\mathcal{T}^c$ we have

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}_c^b.$$

If we further assume that $\mathcal{T}$ is noetherian, then for a suitable choice of metric on $\left[\mathcal{T}_c^b\right]^{\mathrm{op}}$ we have

$$\mathfrak{S}\Big(\left[\mathcal{T}_c^b\right]^{\mathrm{op}}\Big) = \left[\mathcal{T}^c\right]^{\mathrm{op}}.$$

Theorem (second general theorem about approximable categories)

Let $\mathcal{T}$ be an approximable triangulated category. For a suitable choice of metric on $\mathcal{T}^c$ we have

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If we further assume that $\mathcal{T}$ is noetherian, then for a suitable choice of metric on $\left[\mathcal{T}^b_c\right]^{\mathrm{op}}$ we have

$$\mathfrak{S}\Big(\left[\mathcal{T}^b_c\right]^{\mathrm{op}}\Big) = \left[\mathcal{T}^c\right]^{\mathrm{op}}.$$

Noetherian triangulated categories

The notion of noetherian triangulated categories is new, and motivated by the theorem. It is a slight relaxation of the assertion that there is, in the preferred equivalence class, a t–structure $\left(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0}\right)$ such that

$$\left(\mathcal{T}^-_c \cap \mathcal{T}^{\leq 0}\ ,\ \mathcal{T}^-_c \cap \mathcal{T}^{\geq 0}\right)$$

is a t–structure on $\mathcal{T}^-_c$.

The case $\mathcal{T} = \mathsf{D}(R)$

Let R be a coherent ring and let $\mathcal{T} = \mathsf{D}(R)$. Then

$$\mathcal{T}^c = \mathsf{D}^b(R\text{–proj}), \qquad\qquad \mathcal{T}^b_c = \mathsf{D}^b(R\text{–mod}).$$

The theorem now gives

$$\mathfrak{S}\big[\mathsf{D}^b(R\text{–proj})\big] = \mathsf{D}^b(R\text{–mod})$$

and

$$\mathfrak{S}\Big(\big[\mathsf{D}^b(R\text{–mod})\big]^{\mathrm{op}}\Big) = \big[\mathsf{D}^b(R\text{–proj})\big]^{\mathrm{op}} .$$

The case $\mathcal{T} = \mathbf{D}_{\mathsf{qc}}(X)$

Let X be a noetherian, separated scheme. Then

$$\mathcal{T}^c = \mathbf{D}^{\mathrm{perf}}(X), \qquad\qquad \mathcal{T}^b_c = \mathbf{D}^b_{\mathsf{coh}}(X)$$

The theorem now gives

$$\mathfrak{S}\big[\mathbf{D}^{\mathrm{perf}}(X)\big] = \mathbf{D}^b_{\mathsf{coh}}(X)$$

and

$$\mathfrak{S}\Big(\big[\mathbf{D}^b_{\mathsf{coh}}(X)\big]^{\mathrm{op}}\Big) = \big[\mathbf{D}^{\mathrm{perf}}(X)\big]^{\mathrm{op}} .$$

And now for a totally different example

Example

Let $\mathcal{T}$ be the homotopy category of spectra. Then $\mathcal{T}$ is approximable and noetherian.

For the purpose of the formulas that are about to come: $\pi_i(t)$ stands for the ith stable homotopy group of the spectrum t. It can be computed that

1. $$\mathcal{T}^- = \{t \in \mathcal{T} \mid \pi_i(t) = 0 \text{ for } i \ll 0\}$$
2. $$\mathcal{T}^+ = \{t \in \mathcal{T} \mid \pi_i(t) = 0 \text{ for } i \gg 0\}$$
3. $$\mathcal{T}^b = \{t \in \mathcal{T} \mid \pi_i(t) = 0 \text{ for all but finitely many } i \in \mathbb{N}\}$$

4. $\mathcal{T}^c$ is the subcategory of finite spectra.
5. $$\mathcal{T}_c^- = \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \pi_i(t) = 0 \text{ for } i \ll 0, \text{ and} \\ \pi_i(t) \text{ is a finite } \mathbb{Z}\text{–module for all } i \in \mathbb{Z} \end{array} \right\}$$
6. $$\mathcal{T}_c^b = \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \pi_i(t) = 0 \text{ for all but finitely many } i \in \mathbb{Z}, \text{ and} \\ \pi_i(t) \text{ is a finite } \mathbb{Z}\text{–module for all } i \in \mathbb{Z} \end{array} \right\}$$

The general theory applies, telling us (for example)

$$\mathfrak{S}(\mathcal{T}^c) = \mathcal{T}_c^b\,, \qquad\qquad \mathfrak{S}\Big(\big[\mathcal{T}_c^b\big]^{\mathrm{op}}\Big) = \big[\mathcal{T}^c\big]^{\mathrm{op}}\,.$$

- Amnon Neeman, *Strong generators in* $\mathrm{D}^{\mathrm{perf}}(X)$ *and* $\mathrm{D}^b_{\mathrm{coh}}(X)$, Ann. of Math. (2) **193** (2021), no. 3, 689–732.
- Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, https://arxiv.org/abs/1804.02240.
- Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, https://arxiv.org/abs/1806.05342.
- Amnon Neeman, *The category* $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ *as functors on* $\mathcal{T}^b_c$, https://arxiv.org/abs/1806.05777.
- Amnon Neeman, *The categories* $\mathcal{T}^c$ *and* $\mathcal{T}^b_c$ *determine each other*, https://arxiv.org/abs/1806.06471.

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Jesse Burke, Amnon Neeman, and Bregje Pauwels, *Gluing approximable triangulated categories*, https://arxiv.org/abs/1806.05342.

Amnon Neeman, *The category* $\left[\mathcal{T}^c\right]^{\mathrm{op}}$ *as functors on* $\mathcal{T}^b_c$, https://arxiv.org/abs/1806.05777.

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Thank you!