Conference in celebration of the work of Bill Crawley-Boevey

1–10 Sept 2021

Sibylle Schroll, Cologne

Algebra and Module Varieties

jt w EL Green & L Hille

jt in progress with EL Green & E Marcos

1) Algebra Varieties

Idea: Non-commutative Gröbner basis theory

$\rightsquigarrow$

affine algebraic variety $\mathcal{V}$

- every pt in $\mathcal{V}$ is an algebra kQ/I
- $\exists$ a distinguished pt in $\mathcal{V}$ given by a monomial algebra A_{mon}

Theorem: (Green - Hille - S21) $\forall A \in V$

1) $\dim A = \dim A_{mon}$

2) Cartan matrix C_A of $A = C_{A_{mon}}$

3) $\operatorname{gldim} A_{mon} \geq \operatorname{gldim} A$

4) If $\operatorname{gldim} A_{mon} < \infty$ then the Cartan determinant conj holds for A.

5) A_{mon} (D)-Koszul $\Rightarrow$ A (D)-Koszul

6) A_{mon} quasihereditary $\Rightarrow$ A quasihereditary

Rk: (Green-S) For kQ/I monomial, then there is a comb. criterion on (Q, I) to determine whether (Q, I) is quasihereditary or not.

Construction of γ

$A = kQ/I$, $I \subset \langle \text{arrows} \rangle^2$ $\qquad \mathcal{B} = \{ \text{finite paths in } Q \}$

$<$ well-order on $\mathcal{B}$ admissible

Eg length lexicographical

Def 1) • $x = \sum_{p \in \mathcal{B}} \alpha_p \, p \in kQ$

tip x $= p$ largest p st $\alpha_p \neq 0$.

• $X \subseteq kQ$

tip X $= \{ \text{tip } x \mid x \in X \}$

2) $\mathcal{G} \subseteq I$ is a Gröbner basis for I (wrt $<$)

cf

$$_{kQ}\langle \operatorname{tip} \mathcal{G} \rangle_{kQ} = {}_{kQ}\langle \operatorname{tip} I \rangle_{kQ}$$

Rk:

$$\langle \mathcal{G} \rangle = I$$

3) $A_{mon} = kQ / \langle \operatorname{tip} I \rangle$ associated monomial algebra

T min gen set of $\langle \operatorname{tip} I \rangle$

4) Nontips: $\mathcal{N} = \mathcal{B} \setminus T$

Fundamental lemma:

$$kQ \underset{\text{vector space}}{\cong} I \oplus \operatorname{Span}_k \mathcal{N}$$

$\Rightarrow \forall\, t \in T \leq kQ, \ \exists!\, g_t \in I \ \exists!\, n_t \in \operatorname{Span}_k \mathcal{N}$

$$n_t = \sum c_n n$$

$$\mathcal{N}_t \begin{cases} n \in \mathcal{N} \\ n \nmid t \\ n < t \end{cases}$$

$$t = g_t + n_t$$

$$\mathcal{G}_{(c_{t,n})} = \left\{ g_t = t - \sum_{n \in \mathcal{N}_t} c_{t,n}\, n \;\middle|\; t \in T \right\}$$

is a Gröbner basis for I — Reduced Gröbner basis

Def

$$V_T = \left\{ (c_{t,n}) \in k^{\mathcal{N}} \;\middle|\; \mathcal{H}_{(c_{t,n})} = \left\{ t - \sum_{n \in \mathcal{N}_t} c_{t,n}\, n \;\middle|\; t \in T \right\} \text{ is a Gröbner basis for } \langle \mathcal{H}_{(c_{t,n})} \rangle \right\}$$

Theorem (Green - Hille - S)

V_T is an affine algebraic variety.

Example

$$Q: \quad \begin{tikzcd} 1 \arrow[r, "c"] \arrow[d, "a"'] & 2 \arrow[r, "f"] \arrow[d, "d"'] & 3 \arrow[d, "g"] \\ 4 \arrow[r, "b"'] & 5 \arrow[r, "e"'] & 6 \end{tikzcd}$$

$<$ length lexicographical

$a > \dots > g$

$$I = \langle ab - cd, de - fg, be \rangle$$

$$T = \{ab, de, be\} \leq \mathcal{B}$$

$$\mathcal{V}_T = \{ (\alpha, \beta) \in k^2 \mid \mathcal{X}_{(\alpha,\beta)} = \{ab - \underset{C_{ab,cd}}{\alpha} cd, de - \underset{C_{de,fg}}{\beta} fg, be\}$$

is a Gröbner basis for $\langle \mathcal{X}_{(\alpha,\beta)} \rangle \}$

Reduce $g_{(\alpha,\beta)}$ by T: $abe \rightsquigarrow -\alpha cde \rightsquigarrow -\beta\alpha cfg \Rightarrow \alpha\beta = 0$

$$\mathcal{V}_T \simeq \{ kQ / \langle \mathcal{X}_{(\alpha,\beta)} \rangle \mid \alpha \cdot \beta = 0 \quad \alpha, \beta \in k \}$$

$$(0,0) \leftrightsquigarrow kQ / \langle ab, de, be \rangle = A_{mon}$$

2) Module varieties

Idea $\quad A = kQ/I \qquad M \in kQ/I - \mathrm{Mod}$

Right Gröbner basis theory

$\rightsquigarrow$

affine algebraic variety $\mathcal{V}_M$

- pts in $\mathcal{V}_M$ are in $kQ/I - \mathrm{Mod}$
- two modules in $\mathcal{V}_M$ share homological properties

Prop (Green - Marcos - S): $\quad \forall M' \in \mathcal{V}_M$

$$\underline{\dim}\, M = \underline{\dim}\, M'$$

Green - Solberg algorithmic construction of a projective resolution $Q_M^{\bullet}$ of M

order resolution

Def: Θ-pd $M := \sup\{ n \mid Q_M^n \neq 0\}$

Θ-fin dim $A := \sup\{ \Theta\text{-pd}\, M \mid \forall M \in kQ/I\text{-mod st } \Theta\text{-pd}\, M < \infty\}$

Theorem (GMS):

1) $\forall M' \in \mathcal{V}_M$ $\quad$ opd $M' =$ opd M $\quad$ odim $\mathcal{V}$

2) odim $\mathcal{V}_M \geq$ pd M', $\forall M' \in \mathcal{V}_M$

3) o findim $A < \infty$

4) findim $A < \infty \iff$

$\sup\{\operatorname{pd} M \mid \operatorname{pd} M < \infty \ \& \ \operatorname{opd} M = \infty\} < \infty$

Construction of $\mathcal{V}_M$

$$\bigoplus_{i \in I} w_i kQ = P \twoheadrightarrow M \qquad P \in kQ\text{-Mod projective}$$

$<$ admissible order on $\mathcal{B}$ & fix order on I

$\rightsquigarrow$ $<$ on a basis $\mathcal{B}^*$ of P

$$L_M = \ker \pi \hookrightarrow P \xrightarrow{\pi} M$$

Def: $\mathcal{G}^* \subseteq L_M$ is a Right Gröbner basis for L_M

if $$\underbrace{\langle \operatorname{tip} \mathcal{G}^* \rangle_{kQ}}_{T^*} \cong \langle \operatorname{tip} L_M \rangle_{kQ}$$

Rk: $\langle \mathcal{G}^* \rangle_{kQ} = L_M$

Def:

$$V_M = \{ L \hookrightarrow P \mid \langle \operatorname{tip} L \rangle_{kQ} = \langle \operatorname{tip} L_M \rangle_{kQ} \text{ and } PI \subset L \}$$

Theorem (GMS):

V_M is an affine algebraic variety.

Example 1

$$Q: \quad \begin{tikzcd} 1 \arrow[r, "a"] & 4 \arrow[dr, "b"] & & \\ 2 \arrow[r, "c"] & 5 \arrow[r, "d"] & 7 \arrow[r, "g"] & 8 \\ 3 \arrow[r, "e"] & 6 \arrow[ur, "f"] & & \end{tikzcd}$$

$I = \langle abg \rangle$

$a > b > \dots > g$

$$M = \begin{tikzcd} k \arrow[r, "1"] & k \arrow[dr, "0"] & & \\ k \arrow[r, "1"] & k \arrow[r, "\binom{1}{0}"] & k^2 \arrow[r, "(1,-1)"] & k \\ k \arrow[r, "1"] & k \arrow[ur, "\binom{0}{1}"] & & \end{tikzcd}$$

$$L_M = \boxed{\begin{matrix} 7 \\ 8 \end{matrix}} \oplus \boxed{8} \hookrightarrow P_1 \oplus P_2 \oplus P_3 \twoheadrightarrow M$$

$$\begin{matrix} P_1 & P_2 & P_3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ \boxed{\begin{matrix} 7 \\ 8 \end{matrix}} & \begin{matrix} 7 \\ \boxed{8} \end{matrix} & \begin{matrix} 7 \\ \boxed{8} \end{matrix} \end{matrix}$$

$\mathcal{H}^*_M = \{ ab, \ cdg - efg \}$

$\mathcal{T}^* = \{ ab, \ cdg \}$

$\mathcal{H}^* = \{ ab - Xcd - Yef, \ cdg - Zefg \}$

Reduce I by $\mathcal{H}^*$

$abg \rightsquigarrow -Xcdg - Yefg$

$\rightsquigarrow +ZXefg - Yefg$

$\Rightarrow (ZX - Y) = 0$

$$V_M = \{ L \hookrightarrow P \ / \ \langle \text{tip } L \rangle_{kQ} = \langle ab, cdg \rangle_{kQ} \ \& \ PI \subset L \}$$

$$\cong \{ (X, Y, Z) \in k^3 \ / \ XZ - Y = 0 \}$$

Example 2

$$Q: \quad \begin{tikzcd} & 2 \arrow[dr, "b"] & \\ 1 \arrow[ur, "a"] \arrow[rr, "c"] & & 3 \end{tikzcd} \qquad M_\lambda = \begin{tikzcd} & k \arrow[dr, "1"] & \\ k \arrow[ur, "1"] \arrow[rr, "\lambda"] & & k \end{tikzcd}, \ \lambda \in k$$

$$3 = L_{M_1} \hookrightarrow P_1 \twoheadrightarrow M_\lambda$$

$a > b > c$

$$\mathcal{R}^* = \{ab - \lambda c\}$$

$$T^* = \{ab\}$$

$$V_{M_\lambda} = k$$

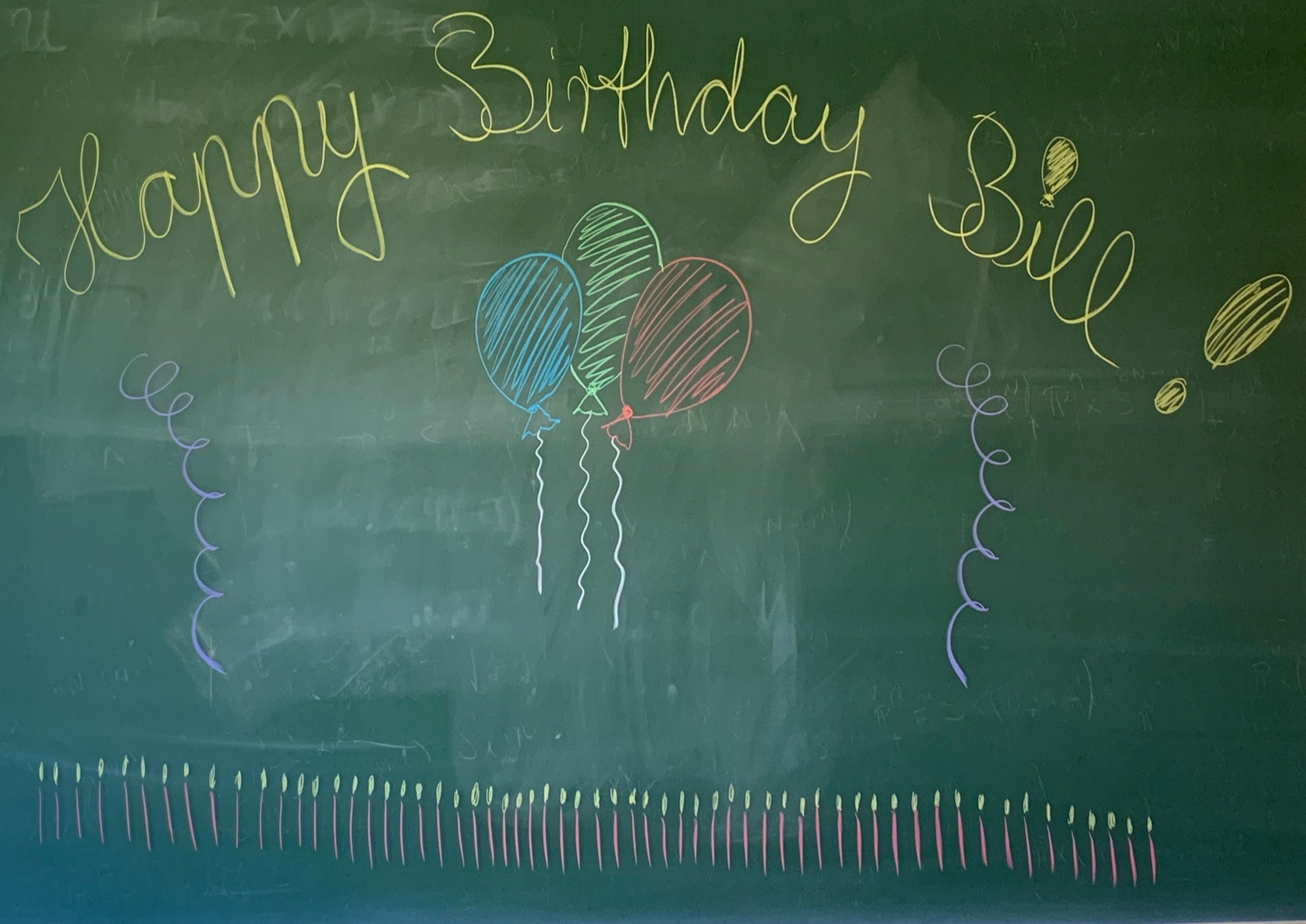
$\forall \mu \in k \quad M_\mu \in V_{M_\lambda}$ and $M_\lambda \neq M_\mu$

Distinguished point in V_{M_λ} : "monomial module"

$\lambda = 0$

$$\begin{tikzcd} & k \arrow[dr, "1"] & \\ k \arrow[ur, "1"] \arrow[rr, "0"] & & k \end{tikzcd} \in V_{M_\lambda}$$

seems to play a similar role for V_M as monomial alg for alg. var.


Happy Birthday Bill!!