HMS symmetries and hypergeometric systems

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0. Motivation

X CY manifold $\xrightarrow{\text{HMS}}$ $\mathcal{K}_X$

$\boxed{\text{Conj. } \pi_1(\mathcal{K}_X) \curvearrowright D^b(X)}$

Cor. $\pi_1(\mathcal{K}_X) \curvearrowright K_0(X)_{\mathbb{C}}$

$\longleftrightarrow$ loc. system (loc. constant sheaf of fin. dim. vec. sp.) on $\mathcal{K}_X$

$\xleftrightarrow{RH}$ reg. connection ("system of lin. diff. equations") on $\mathcal{K}_X$

loc. system $\rightsquigarrow$ perverse sheaf

$\mathcal{Y} = \bigsqcup \mathcal{Y}_\alpha$ a stratification

${}^p D_\alpha^{\leq 0} = \{ \mathcal{F} \in D^b_{loc}(\mathcal{Y}_\alpha) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i > -\dim \mathcal{Y}_\alpha \}$

${}^p D_\alpha^{\geq 0} = \{ \mathcal{F} \in D^b_{loc}(\mathcal{Y}_\alpha) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for } i < -\dim \mathcal{Y}_\alpha \}$

${}^p D_\alpha^{\leq 0} \cap {}^p D_\alpha^{\geq 0} = \mathrm{Loc}(\mathcal{Y}_\alpha)[\dim \mathcal{Y}_\alpha]$

${}^p D^{\leq 0} = \{ \mathcal{F} \in D^b_{loc}(\mathcal{Y}) \mid i_\alpha^* \mathcal{F} \in {}^p D_\alpha^{\leq 0} \ \forall \alpha \}$

${}^p D^{\geq 0} = \{ \mathcal{F} \in D^b_{loc}(\mathcal{Y}) \mid i_\alpha^! \mathcal{F} \in {}^p D_\alpha^{\geq 0} \ \forall \alpha \}$

${}^p D^{\leq 0} \cap {}^p D^{\geq 0} = \mathrm{Perv}(\mathcal{Y})$

$\mathrm{Perv}(\mathcal{Y})$ is abelian $\left(= R\mathcal{H}om_{D_\mathcal{Y}}(-, \mathcal{O}_\mathcal{Y}) \, (\mathrm{mod}^{rh} D_\mathcal{Y})[\dim \mathcal{Y}] \right)$

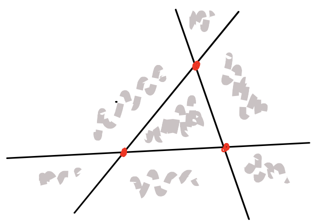
— Extend $\pi_1(\mathcal{K}_X) \curvearrowright D^b(X)$ to a "~~perverse sheaf of categories~~" perverse schober on a partial compactification of $\mathcal{K}_X$.

1. Setting

$\mathcal{X}_X = V_{\mathbb{C}} \setminus \mathcal{H}_{\mathbb{C}}$, V real vec. sp., $\mathcal{H}$ real (affine) hyperplane arrangement

$\mathcal{H} \longrightarrow$ stratification $\mathcal{C}$ of V into locally closed sets "faces"

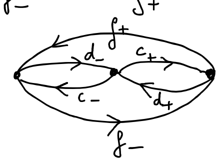
$C' \leq C \iff C' \subset \overline{C}$



2. Perverse sheaves on hyperplane arrangements

<u>Ex.</u> $\mathrm{Perv}(\mathbb{C}, \bullet) \cong \left(E_- \underset{\gamma_-}{\overset{\delta_-}{\rightleftarrows}} E_0 \underset{\delta_+}{\overset{\gamma_+}{\rightleftarrows}} E_+ \;\middle|\; \gamma_\pm \delta_\pm = id_{E_\pm},\ \gamma_\pm \delta_\mp \text{ iso.} \right)$

$\cong$ rep (



$\left| \; c_\pm d_\pm = e_\pm,\ c_\pm d_\mp f_\pm = e_\pm,\ f_\pm c_\pm d_\mp = e_\mp \right)$

$\cong \mathrm{rep}\left(k\langle x_-, x_0, x_+ \rangle / (x_{\pm 0}^2 - x_{\pm 0},\ x_\pm x_0 - x_\pm,\ x_0 x_\pm - x_\pm) \left[(x_\pm x_\mp x_\pm + 1 - x_\pm)^{-1} \right] \right)$

<u>Thm</u> [Kapranov, Schechtman; 2016]

$\mathrm{Perv}(V_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}}) \cong$

$(E_C, \delta_{CC'}, \gamma_{C'C})_{C, C' \in \mathcal{C}, C' \leq C}$ s.t.

- $E_{C'} \xleftarrow{\delta_{CC'}} E_C$ rep. of $(\mathcal{C}, \geq)$ in $\mathrm{vec}(\mathbb{C})$
- $E_{C'} \xrightarrow{\gamma_{C'C}} E_C$ rep. of $(\mathcal{C}, \leq)$ in $\mathrm{vec}(\mathbb{C})$

(m) $\gamma_{C'C} \delta_{CC'} = id_{E_C}$ $\left(\Rightarrow \phi_{C_1 C_2} := \gamma_{C' C_2} \delta_{C_1 C'},\ C' \leq C_1, C_2, \text{ well defined} \right)$

(i) $\phi_{C_1 C_2}$ iso. $\forall C_1, C_2$. $\dim C_1 = \dim C_2 = \dim C_1 \wedge C_2 + 1$, in the same $\dim C_1$-dim. vec-sp.

(t) $\phi_{C_1 C_3} = \phi_{C_2 C_3} \phi_{C_1 C_2}$ $\forall$ collinear triple C_1, C_2, C_3 which share a face $(\exists c_i \in C_i \cdot c_2 \in [c_1, c_3])$

<u>Prop</u> [Bapat; 2018]

$\mathrm{Perv}(V_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}}) \cong \mathrm{rep}\Big(\mathbb{C}\langle x_C \mid C \in \mathcal{C} \rangle / (x_C^2 - x_C,\ x_C x_{C'} x_C - x_{C'} \mid C', C \in \mathcal{C}, C' \leq C,$
$x_{C_1} x_{C_3} - x_{C_1} x_{C_2} x_3 \mid (C_1, C_2, C_3) \text{ collinear}) \left[(1 - x_{C_1} + x_{C_1} x_{C_2} x_{C_1})^{-1} \mid C_1, C_2 \text{ as in (i)} \right] \Big)$

3. Perverse schobers on hyperplane arrangement

<u>Def</u> [Bondal, Kapranov, Schechtman; 2018]
$\mathrm{Perv}(V_{\mathbb{C}}, \mathcal{H}_{\mathbb{C}}) =$

$(\mathcal{E}_C, \delta_{CC'}, \gamma_{C'C})_{C, C' \in \mathcal{C},\, C' \leq C}$ s.t.

- $\mathcal{E}_{C'} \xleftarrow{\delta_{CC'}} \mathcal{E}_C$ rep. of $(\mathcal{C}, \geq)$ in $\mathcal{C}at_\Delta$
- $\mathcal{E}_{C'} \xrightarrow{\gamma_{C'C}} \mathcal{E}_C$ rep. of $(\mathcal{C}, \leq)$ in $\mathcal{C}at_\Delta$

$\delta_C \vdash \gamma_C$

(m) $\gamma_{C'C}\, \delta_{CC'} \xleftarrow{\sim} id_{\mathcal{E}_C}$ $\quad (\Rightarrow \phi_{C_1C_2} := \gamma_{C'C_2}\delta_{C_1C'}$, $C' \leq C_1, C_2$, well defined)

(i) $\phi_{C_1C_2}$ equ. $\forall C_1, C_2$. $\dim C_1 = \dim C_2 = \dim C_1 \wedge C_2 + 1$, in the same dim. C_1-dim. vec. sp.

(t) $\phi_{C_1C_3} \xrightarrow{\sim} \phi_{C_2C_3}\phi_{C_1C_2}$ $\quad \forall$ collinear triple C_1, C_2, C_3 $\quad (\exists c_i \in C_i \cdot c_2 \in [c_1, c_3])$

4. GIT perverse schobers

X crepant (stacky) res. of affine Gorenstein toric var. Y

Y – represented by a cone ($\subseteq \mathbb{R}^k$) over a polytope with vertices in $A \subseteq \mathbb{Z}^{k-1} \times \{1\}$

$|A| = d \overset{*}{\Longrightarrow} Y \cong \mathbb{C}^d /\!/ (\mathbb{C}^*)^n$, $n = d-k$, $0 \longrightarrow \mathbb{Z}^k \xrightarrow{A} \mathbb{Z}^d \xrightarrow{B} \mathbb{Z}^n \longrightarrow 0$

$\mathcal{K}_X$ = complement of a hypersurface given by "principal A-determinant"

<u>Thm</u> [Kite; 2017]

If $(\mathbb{C}^*)^n \curvearrowright \mathbb{C}^d$ is "quasi-symmetric" then $\mathcal{K}_X$ is a complement of a hyperplane arrangement in $\mathbb{C}^n$ (in logarithmic coordinates).

<u>Ex</u>. $\mathbb{C}^* \curvearrowright \mathbb{C}^4$, $-1, -1, 1, 1$ $\quad$ (Y conifold)
$\mathcal{K}_X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, $\frac{1}{2\pi i}\log \mathcal{K}_X = \mathbb{C} \setminus \mathbb{Z}$

<u>Thm</u> [Halpern-Leistner, Sam; 2016]

If X is quasi-symmetric, then $\pi_1(\mathcal{K}_X) \curvearrowright D^b(X)$.

Thm [ŠVdB; 2019]

If X is quasi-symmetric, then $\pi_1(\mathcal{J}d_X) \curvearrowright D^b(X)$ extends a perverse schober on $\mathbb{C}^n$.

Sketch. $X = [(\mathbb{C}^d)^{\chi, ss} / (\mathbb{C}^*)^n]$ $\quad$ ($\chi \in X(({\mathbb{C}^*})^n)_{\mathbb{R}}$ generic)

- $\mathcal{E}_C \subseteq D^b([\mathbb{C}^d / (\mathbb{C}^*)^n])$
- $\delta_{CC'} : \mathcal{E}_C \hookrightarrow \mathcal{E}_{C'}$, $\quad C' \leq C$, admissible

(If C is a chamber, then $\mathcal{E}_C \cong$ NCCR of Y, $\mathcal{E}_C \cong$ NCR of Y $\forall C$.)

$\Rightarrow \gamma_{C'C}$ right adjoint of $\delta_{CC'}$

(m) ✓

(i) $\mathcal{E}_C = \langle \mathcal{E}_{C,C_1}, \mathcal{E}_{C_1} \rangle = \langle \mathcal{E}_{C_1}, \mathcal{E}_{C,C_2} \rangle = \langle \mathcal{E}_{C,C_2}, \mathcal{E}_{C_2} \rangle = \langle \mathcal{E}_{C_2}, \mathcal{E}_{C,C_1} \rangle$

$\Longrightarrow \mathcal{E}_{C_1} \cong \mathcal{E}_{C_2}$ $\quad$ ($\underset{\text{[Bodzenta, Bondal, 2015]}}{\Longrightarrow}$ $\mathcal{E}_{C_1}, \mathcal{E}_{C_2} \subset \mathcal{E}_C$ (mutation) spherical pair)

(t) ...

Ex [Donovan] $\mathbb{C}^* \curvearrowright \mathbb{C}^4$, $-1, -1, 1, 1$

$\frac{1}{2\pi i} \log \mathcal{J}d_X = \mathbb{C} \setminus \mathbb{Z}$, $\quad \mathcal{C} = \{ (m, m+1) \mid m \in \mathbb{Z} \} \cup \mathbb{Z}$

$\mathcal{E}_{(m,m+1)} = \langle \chi_m \otimes \mathcal{O}_{\mathbb{C}^4}, \chi_{m+1} \otimes \mathcal{O}_{\mathbb{C}^4} \rangle \subseteq D^b([\mathbb{C}^4 / \mathbb{C}^*])$ $\quad$ ($\chi_m = \mathbb{C}$, $t \cdot v = t^m v$)

$\mathcal{E}_m = \langle \chi_{m-1} \otimes \mathcal{O}_{\mathbb{C}^4}, \chi_m \otimes \mathcal{O}_{\mathbb{C}^4}, \chi_{m+1} \otimes \mathcal{O}_{\mathbb{C}^4} \rangle \subseteq D^b([\mathbb{C}^4 / \mathbb{C}^*])$

$$\gamma_{(m,m+1),m} : \mathcal{E}_{(m,m+1)} \longrightarrow \mathcal{E}_m$$

$$\chi_{m(+1)} \otimes \mathcal{O}_{\mathbb{C}^4} \longmapsto \chi_{m(+1)} \otimes \mathcal{O}_{\mathbb{C}^4}$$

$$\chi_{m-1} \otimes \mathcal{O}_{\mathbb{C}^4} \longmapsto \left(\chi_{m+1} \otimes \mathcal{O}_{\mathbb{C}^4} \longrightarrow (\chi_m \otimes \mathcal{O}_{\mathbb{C}^4})^{\oplus 2} \right)$$

5. Decategorification

$\pi_1(\mathcal{M}_X) \curvearrowright K_0(X)_{\mathbb{C}}$? $\qquad K_0\left((\mathcal{E}_c, S_{cc'}, \gamma_{c'c})_{c' \leq c \in \mathcal{C}} \right)_{\mathbb{C}}$?

Ex. $\mathbb{C}^* \curvearrowright \mathbb{C}^4$, $-1, -1, 1, 1$

$\pi_1(\mathcal{M}_X) \curvearrowright K_0(X)_{\mathbb{C}}$ coincides with the monodromy of the Gauss hypergeometric equation.

$z(1-z)\, y'' + (c - (a+b+1) z)\, y' - ab\, y = 0 \qquad$ for $a = b = c = 0$

Intermezzo Can we get Gauss h.e. also for other parameters ?

$(\mathbb{C}^*)^4 \curvearrowright \mathbb{C}^4$ coordinate-wise, $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^4$, $t \longmapsto (t^{-1}, t^{-1}, t, t)$

$\hookrightarrow$ splits, $(\mathbb{C}^*)^4 = \mathbb{C}^* \times (\mathbb{C}^*)^3$

Replace $\mathcal{E}_c \rightsquigarrow \widetilde{\mathcal{E}}_c \subset D^b([\mathbb{C}^4 / (\mathbb{C}^*)^4])$

$\mathcal{E}_{(0,1)} = \langle \chi_0 \otimes \mathcal{O}_{\mathbb{C}^4}, \chi_1 \otimes \mathcal{O}_{\mathbb{C}^4} \rangle \rightsquigarrow$

$$\left(\begin{array}{ccccc} \mathbb{Z}^4 & \xrightarrow{B} & \mathbb{Z} & \longrightarrow & 0 \\ \| & & \| & & \\ X((\mathbb{C}^*)^4) & & X(\mathbb{C}^*) & & \end{array} \right)$$

$\widetilde{\mathcal{E}}_{(0,1)} = \langle \mu \otimes \mathcal{O}_{\mathbb{C}^4} \mid \mu \in X((\mathbb{C}^*)^4), B\mu \in \{0,1\} \rangle \circlearrowleft X((\mathbb{C}^*)^3)$

$\Rightarrow K_0(\widetilde{\mathcal{E}}_c)_{\mathbb{C}}$ module over $\mathbb{C}\{X((\mathbb{C}^*)^3)\} = \mathbb{C}[(\mathbb{C}^*)^3]$

$\Rightarrow$ specialising at (generic) $h \in (\mathbb{C}^*)^3$ $\left(K_0(\widetilde{\mathcal{E}}_c)_{\mathbb{C}} \otimes_{\mathbb{C}[(\mathbb{C}^*)^3]} \mathbb{C}_h, S_{cc'}, \gamma_{c'c} \right)_{c' \leq c \in \mathcal{C}}$

gives the Gauss h.e. with parameters $\sim \frac{1}{2\pi i} \log h$

5.1 GKZ hypergeometric systems [Gelfand, Kapranov, Zelevinsky, 1988]

$A \subset \mathbb{Z}^{d-n}$, $|A| = d$

$\alpha \in \mathbb{C}^{d-n}$,

- $H_i = \sum_j a_{ij} \cdot x_j \partial_j$ homogeneity relations
- $\Box_\ell = \prod_{l_i > 0} \partial_i^{l_i} - \prod_{l_i < 0} \partial_i^{-l_i}$ $\quad \ell \in \ker A \quad (A : \mathbb{Z}^d \longrightarrow \mathbb{Z}^{d-n})$

GKZ system (of parameter α)

$(H_i - \alpha_i)\, \Phi = 0 \quad \forall\ 1 \le i \le d-n$

$\Box_\ell\, \Phi = 0 \quad \forall\ \ell \in \ker A$

Rem GKZ system on $(\mathbb{C}^*)^d$, due to $(H_i)_i$ it descends to $(\mathbb{C}^*)^n$.

Ex (Gauss h.e.)

$$A = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\ker A = \mathbb{Z} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \alpha = \begin{pmatrix} c-1 \\ -a \\ -b \end{pmatrix}$$

$$(-x_1 \partial_1 + x_2 \partial_2)\, \Phi = (c-1)\, \Phi$$

$$(x_1 \partial_1 + x_3 \partial_3)\, \Phi = -a\, \Phi$$

$$(x_1 \partial_1 + x_4 \partial_4)\, \Phi = -b\, \Phi$$

$$(\partial_1 \partial_2 - \partial_3 \partial_4)\, \Phi = 0$$

$$\left(x_3^{-1} x_4^{-1} \left(x_1 \partial_1^2 - (1+a+b) x_1 \partial_1 - ab \right) - x_2^{-1} \left(x_1 \partial_1^2 - c \partial_1 \right) \right) \Phi = 0$$

$F(x) := \Phi(x, 1, 1, 1)$ sol. of Gauss h.e.

(by homogeneity F determines Φ)

"<u>Thm</u>" [ŠVdB ; 2020]

$\boxed{K_0\left((\mathcal{E}_c, \delta_{cc'}, \gamma_{c'c})_{c' \leq c \in \mathcal{C}} \right) \text{ is a GKZ perverse sheaf}}$

which parameter ?!

We can only do a tweaked version.

Replace $\mathcal{E}_c \rightsquigarrow \tilde{\mathcal{E}}_c \subset D^b([\mathbb{C}^d/(\mathbb{C}^*)^d])$

$\Rightarrow K_0(\tilde{\mathcal{E}}_c)_{\mathbb{C}}$ module over $\mathbb{C}\{X((\mathbb{C}^*)^{d-n})\} = \mathbb{C}[(\mathbb{C}^*)^{d-n}]$

$\Rightarrow$ specialising at (generic) $h \in (\mathbb{C}^*)^{d-n}$ $\left(K_0(\tilde{\mathcal{E}}_c)_{\mathbb{C}} \otimes_{\mathbb{C}[(\mathbb{C}^*)^{d-n}]} \mathbb{C}_h, \delta_{cc'}, \gamma_{c'c} \right)_{c' \leq c \in \mathcal{C}}$

gives the GKZ h.s. with parameters $\sim \frac{1}{2\pi i} \log h$

App Monodromy of GKZ h.s. for quasi-symmetric B.