# Induction-restriction adjunction in $2$-representation theory

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# Classical representation theory – algebras and representations

Let $\Bbbk$ be a(n algebraically closed) field.

**Algebra over $\Bbbk$**: A $\Bbbk$-linear category $\mathcal{A}$ with one (or finitely many) object(s), say $\bullet$.
**Representation of $\mathcal{A}$**: A $\Bbbk$-linear functor from $\mathcal{A}$ to $\mathcal{V}ect_{\Bbbk}$.

Observe:

- $A := \mathrm{End}_{\mathcal{A}}(\bullet)$ is an associative $\Bbbk$-algebra.
- If the functor describing a representation is given by $\bullet \mapsto V$, $\mathrm{End}_{\mathcal{A}}(\bullet) \ni a \mapsto \rho(a) \in \mathrm{End}_{\mathcal{V}ect_{\Bbbk}}(V)$, $V$ is an $A$-module and $\rho$ is a representation of $A$.
- If $\mathcal{A}$ has several objects $\mathtt{1}, \dots, \mathtt{n}$, their identities are idempotents in the algebra $A = \mathrm{End}_{\mathcal{A}}(\bigoplus_{\mathtt{i}=\mathtt{1}}^{\mathtt{n}} \mathtt{i})$.

# Classical representation theory - induction and restriction

Let $A, B$ be two associative $\Bbbk$-algebras, $\phi\colon B \to A$ an algebra morphism.
$\rightsquigarrow$ two functors $\mathrm{Ind}_B^A = A \otimes_B -$, $\mathrm{Res}_B^A = - \circ \phi$, and an adjunction

$$(\mathrm{Ind}_B^A, \mathrm{Res}_B^A)$$

Categorically:
$\mathcal{A}, \mathcal{B}$ algebras ($\Bbbk$-linear categories with one object), $\phi$ a $\Bbbk$-linear functor

$$\phi\colon \mathcal{B} \to \mathcal{A}$$
$$\bullet_{\mathcal{B}} \mapsto \bullet_{\mathcal{A}}$$
$$\mathrm{End}_{\mathcal{B}}(\bullet_{\mathcal{B}}) = B \xrightarrow{\phi} A = \mathrm{End}_{\mathcal{A}}(\bullet_{\mathcal{A}})$$

**Categorify?**

# $2$-categories

A $2$-**category** $\mathscr{C}$ is a category enriched over the monoidal category $\mathbf{Cat}$ of small categories, i.e. it consists of

- a class (or set) $\mathscr{C}$ of objects;
- for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms from $\mathtt{i}$ to $\mathtt{j}$
  - objects in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called $1$-**morphisms**
  - morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called $2$-**morphisms**;
- functorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity $1$-morphisms $\mathbb{1}_{\mathtt{i}}$ for every $\mathtt{i} \in \mathscr{C}$;
- natural (strict) axioms.

**Remark.** Everything I will say has a bicategorical analogue.

# 2-categories

**Examples.**

- The $2$-category $\mathbf{Cat}$:
  - objects are small categories;
  - $1$-morphisms functors;
  - $2$-morphisms are natural transformations.
- The $2$-category $\mathfrak{A}_{\Bbbk}^{f}$:
  - objects are small idempotent complete $\Bbbk$-linear additive categories with finitely many indecomposable objects up to isomorphism and finite-dimensional morphism spaces
    (that is, equivalent to the category of finitely generated projective modules over a finite-dimensional $\Bbbk$-algebra);
  - $1$-morphisms are $\Bbbk$-linear (additive) functors;
  - $2$-morphisms are natural transformations.

# 2-categories

A 2-category $\mathscr{C}$ is **multifinitary** over $\Bbbk$ if

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is in $\mathfrak{A}_{\Bbbk}^{f}$;
- composition is biadditive and $\Bbbk$-bilinear;

**Moral:** Multifinitary 2-categories are 2-analogues of finite dimensional algebras.

A 2-category $\mathscr{C}$ is **multifiat** (**f**initary - **i**nvolution - **a**djunction - **t**wo-category) if

- it is multifinitary;
- there is a weak involutive equivalence $(-)^* \colon \mathscr{C} \to \mathscr{C}^{\mathrm{op},\mathrm{op}}$ such that there exist adjunction morphisms $F \circ F^* \to \mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{j}} \to F^* \circ F$.

# 2-categories

**Example.** Let $A$ be a finite-dimensional $\Bbbk$-algebra. The 2-category $\mathscr{C}_A$ has

- one object $\bullet$ (identified with $A$-proj);
- 1-morphisms are endofunctors of $\varnothing$ isomorphic to tensoring with bimodules in the additive closure of $A \oplus A \otimes_{\Bbbk} A$;
- 2-morphisms are natural transformations (bimodule homomorphisms).

**Observe:**

- $\mathscr{C}_A$ is multifinitary.
- If $A$ is basic with complete set of idempotents $e_1, \dots, e_n$, the indecomposable 1 morphisms correspond to the bimodules $A$ and $Ae_i \otimes_{\Bbbk} e_j A$, for $i, j = 1, \dots n$.
- If $A$ is weakly symmetric, $\mathscr{C}_A$ is multifiat with involution given by $(Ae_i \otimes_{\Bbbk} e_j A)^* \cong Ae_j \otimes_{\Bbbk} e_i A$.

# 2-representations

A **finitary 2-representation** $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is a (strict) 2-functor $\mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$, i.e.

- $\mathbf{M}(\mathtt{i}) \approx B_{\mathtt{i}}\text{-proj}$ for some algebra $B_{\mathtt{i}}$;
- for $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $\mathbf{M}(\mathrm{F})\colon \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{j})$ is an additive functor;
- for $\alpha\colon \mathrm{F} \to \mathrm{G}$, $\mathbf{M}(\alpha)\colon \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{G})$ is a natural transformation.

**Examples.**

- For $\mathtt{i} \in \mathscr{C}$, we have $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$, the **principal** 2-representation.
- $\mathscr{C}_A$ was defined via its **natural** 2-representation on $A\text{-proj}$.

# Restriction

From now on:

- $\mathscr{C}$, $\mathscr{D}$ multifiat 2-categories
- $\Phi\colon \mathscr{C} \to \mathscr{D}$ a 2-functor.

**Definition.** If $\mathbf{N}$ is a 2-representation of $\mathscr{D}$, set $\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{N} = \mathbf{N} \circ \Phi$.

**Question.** How to define induction? No notion of relative tensor product.

**Answer.** Internalise 2-representations!

Need **injective abelianisation** $\underline{\mathscr{C}}$ of $\mathscr{C}$: 2-category where each $\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ is abelian, and the injective objects in $\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ are precisely the objects of $\mathscr{C}(\mathtt{i},\mathtt{j})$.

# Internal cohom and coalgebras

Let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$.

**Definition.** Let $X \in \mathbf{M}(\mathtt{i}), Y \in \mathbf{M}(\mathtt{j})$. Then there exists a 1-morphism ${}_{\mathbf{M}}[X,Y]$ in $\underline{\mathscr{C}}$, the **internal cohom from $X$ to $Y$**, such that for all $\mathrm{F} \in \mathscr{C}(\mathtt{i},\mathtt{j})$

$$\mathrm{Hom}_{\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})}({}_{\mathbf{M}}[X,Y],\mathrm{F}) \cong \mathrm{Hom}_{\mathbf{M}(\mathtt{j})}(Y,\mathbf{M}(\mathrm{F})\,X).$$

**Fact.** ${}_{\mathbf{M}}[X,X] =: \mathrm{C}^X$ has a natural coalgebra structure.

So we can consider $\mathrm{C}^X$-comodules in $\underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$, i.e. those $\mathrm{T} \in \underline{\mathscr{C}}(\mathtt{i},\mathtt{j})$ with structure map $\mathrm{T} \to \mathrm{T}\mathrm{C}^X$, such that comodule axioms hold.

# Internalisation

Let $\mathbf{M}$ be a finitary $2$-representation of $\mathscr{C}$.

**Definition.** $\mathbf{M}$ is **generated by** $X \in \mathbf{M}(\mathtt{i})$ if for each $\mathtt{j} \in \mathscr{C}$

$$\mathbf{M}(\mathtt{j}) \simeq \mathsf{add}\{\mathbf{M}(\mathrm{F})\, X \,\|\, \mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})\}.$$

**Fact.** If $\mathbf{M}$ is generated by $X$, there is an equivalence

$$\mathbf{M} \simeq \mathbf{inj}_{\underline{\mathscr{C}}}\mathrm{C}^X$$

with $\mathbf{inj}_{\underline{\mathscr{C}}}\mathrm{C}^X(\mathtt{j})$ the category of injective right $\mathrm{C}^X$-comodules in $\underline{\mathscr{C}}(\mathtt{i}, \mathtt{j})$.

# Induction

Assume $\mathbf{M}$ is generated by $X$, so $\mathbf{M} \simeq \textbf{inj}_{\underline{\mathscr{C}}}\mathrm{C}^X$.

Recall the 2-functor $\Phi\colon \mathscr{C} \to \mathscr{D}$ (which extends to $\Phi\colon \underline{\mathscr{C}} \to \underline{\mathscr{D}}$).

**Fact.** $\Phi(\mathrm{C}^X) \in \underline{\mathscr{D}}$ has a natural coalgebra structure.

**Definition.** Define $\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{M} := \textbf{inj}_{\underline{\mathscr{D}}}\Phi(\mathrm{C}^X)$.

**Problem.** If we want to prove a 2-adjunction $(\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}, \mathbf{Res}_{\mathscr{C}}^{\mathscr{D}})$, we need to also formulate $\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}$ in terms of coalgebras.

# Restriction revisited

Let $\mathbf{N}$ be a finitary 2-representation of $\mathscr{D}$.

WLOG, we can assume that $\mathscr{D}$ only has one object $\bullet$ and that there is a coalgebra $\mathrm{D} \in \mathscr{D}(\bullet, \bullet)$ such that $\mathbf{N} = \mathbf{inj}_{\underline{\mathscr{D}}}\mathrm{D}$.

Then $\mathrm{D}$ generates $\mathbf{N} = \mathbf{inj}_{\underline{\mathscr{D}}}\mathrm{D}$ as a 2-representation of $\mathscr{D}$ but **not**, in general, its restriction to $\mathscr{C}$.

**Question.** How to uniformly produce a generator for $\mathbf{Res}^{\mathscr{D}}_{\mathscr{C}}\mathbf{N}$?

Recall the principal 2-representation of $\mathscr{D}$, namely $\mathbf{P}_\bullet = \mathscr{D}(\bullet, -)$.

Pick a generator $\mathrm{G}$ of $\mathbf{Res}^{\mathscr{D}}_{\mathscr{C}}\mathbf{P}_\bullet$.

Then $\mathrm{GD}$ is a generator of $\mathbf{Res}^{\mathscr{D}}_{\mathscr{C}}\mathbf{inj}_{\underline{\mathscr{D}}}\mathrm{D}$ for any coalgebra $\mathrm{D} \in \underline{\mathscr{D}}(\bullet, \bullet)$.

**Definition.** Define $\mathcal{R}\mathrm{D} = {}_{\mathbf{Res}^{\mathscr{D}}_{\mathscr{C}}\mathbf{N}}[\mathrm{GD}, \mathrm{GD}]$. Then

$$\mathbf{Res}^{\mathscr{D}}_{\mathscr{C}}\mathbf{N} \simeq \mathbf{inj}_{\underline{\mathscr{C}}}\ \mathcal{R}\mathrm{D}.$$

# 2-adjunction

**Theorem.** *[M.–Powell–Zhang]*
Let $\mathbf{M} = \mathbf{inj}_{\underline{\mathscr{C}}}\mathrm{C}$ be a finitary 2-representation of $\mathscr{C}$ and $\mathbf{N} = \mathbf{inj}_{\underline{\mathscr{D}}}\mathrm{D}$ be a finitary 2-representation of $\mathscr{D}$. Then we have morphisms of 2-representations

$$\varepsilon_{\mathbf{N}} \colon \mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{N} = \mathbf{inj}_{\underline{\mathscr{D}}}\Phi(\mathcal{R}\mathrm{D}) \to \mathbf{N}$$

$$\eta_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{M} = \mathbf{inj}_{\underline{\mathscr{C}}}\mathcal{R}\Phi(\mathrm{C})$$

such that the triangles

$$\begin{tikzcd}
\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{M} \arrow[r, "\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\eta_{\mathbf{M}}"] \arrow[dr, equal] & \mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{M} \arrow[d, "\varepsilon_{\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{M}}"] \\
 & \mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{N} \arrow[r, "\eta_{\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{N}}"] \arrow[dr, equal] & \mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{Ind}_{\mathscr{C}}^{\mathscr{D}}\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{N} \arrow[d, "\mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\varepsilon_{\mathbf{N}}"] \\
 & \mathbf{Res}_{\mathscr{C}}^{\mathscr{D}}\mathbf{N}
\end{tikzcd}$$

commute up to invertible modifications.

# Example

Let $\mathscr{C} = \mathscr{C}_A$ and note that $\mathscr{C}_{\Bbbk} = \mathscr{V}ect_{\Bbbk}$.

Consider the embedding
$\mathscr{C}_{\Bbbk} \to \mathscr{C}_A \colon \bullet_{\Bbbk} \mapsto \bullet_A, \quad \mathbb{1}_{\bullet_{\Bbbk}} \mapsto \mathbb{1}_{\bullet_A}, \quad \mathrm{id}_{\mathbb{1}_{\bullet_{\Bbbk}}} \mapsto \mathrm{id}_{\mathbb{1}_{\bullet_A}}$.

Let $\mathbf{M} = \mathbf{inj}_{\underline{\mathscr{C}_{\Bbbk}}} \mathbb{1}_{\bullet_{\Bbbk}}$ be the trivial $2$-representation of $\mathscr{C}_{\Bbbk}$.

Then $\mathbf{Ind}_{\mathscr{C}_{\Bbbk}}^{\mathscr{C}_A} \mathbf{M} = \mathbf{inj}_{\underline{\mathscr{C}_A}} \mathbb{1}_{\bullet_A} = \mathbf{P}_{\bullet_A}$.

Moreover

$$\mathbf{Res}_{\mathscr{C}_{\Bbbk}}^{\mathscr{C}_A} \mathbf{Ind}_{\mathscr{C}_{\Bbbk}}^{\mathscr{C}_A} \mathbf{M} \cong \mathbf{inj}_{\underline{\mathscr{C}_{\Bbbk}}} \mathrm{C}$$

where $\mathrm{C}$ is the coalgebra

$$\mathrm{C} = {}_{\mathbf{Res}_{\mathscr{C}_{\Bbbk}}^{\mathscr{C}_A} \mathbf{P}_\bullet}[A \oplus A \otimes_{\Bbbk} A, A \oplus A \otimes_{\Bbbk} A] \cong \mathrm{End}_{A\text{-mod-}A}(A \oplus A \otimes_{\Bbbk} A)^*.$$

In general, computing internal cohoms is hard!

# Thank you!

Thank you for your attention!