Maximal Cohen-Macaulay modules over complete intersections

6/5/2011

Ragnar O. Buchweitz

(dedicated to D. Quillen)

We assume

S regular local ring (eg. $S = k[[x_0, \dots, x_n]]$)

$R = S/I$ Gorenstein (eg. $I = (f)$; $I = (\underline{f_1, \dots, f_c})$)

Note: $\dim(S) = n+1$, for $S = k[[x_0, \dots, x_n]]$; $\dim(S/(f)) = n$, a regular sequence

Review: M is maximal Cohen-Macaulay over R (where (R, m, k) local) if (*) $\exists\, y_1, \dots, y_{\dim(R)} \in m$ s.th. y_1 is a nonzerodivisor on M and y_{i+1} is a nonzero divisor on $M_i = M/(y_1, \dots, y_i)M$ for $i = 1, \dots, \dim(R) - 1$.

Equivalently, $\mathrm{Ext}^i_R(k, M) = 0$ for $0 \leq i < \dim(R)$.

In general, the maximum d s.th. $\exists\, y_1, \dots, y_d \in m$ a regular sequence on a f.g. R-mod N is called the "depth of N". Also, $\inf\{i \mid \mathrm{Ext}^j_R(k, N) = 0, \forall\, j < i\} \in \mathbb{N} \cup \{\infty\}$ is called the "grade" of N. The grade and the depth of N coincide.

If

$$0 \to N' \to N \to N'' \to 0$$

is a s.e.s. of f.g. R-modules, then $\mathrm{depth}(N') = \min(\mathrm{depth}(N), \mathrm{depth}(N'') + 1)$. In particular, if R is Cohen-Macaulay (i.e. $\mathrm{depth}(R) = \dim(R)$) and N is free

$$\mathrm{depth}(N') = \min(\dim(R), \mathrm{depth}(N'') + 1).$$

If N is any f.g. R-module and you resolve

$$\begin{tikzcd}
 & & & & 0 & N_2 \arrow[l] \arrow[dl] & 0 \arrow[l] & & & & \\
0 & N \arrow[l] & F_0 \arrow[l] & & F_1 \arrow[ll] \arrow[dl] & & F_2 \arrow[ll] \arrow[ul] & \cdots \arrow[l] & F_d \arrow[l] & M \arrow[l] & 0 \arrow[l] \\
 & & & N' \arrow[ul] \arrow[dl] & & & & & & & \\
 & & 0 & & 0 \arrow[ul] & & & & & &
\end{tikzcd}$$

$\mathrm{depth}(M) = \dim(R) = d$

perfect!

The Auslander-Buchsbaum-Serre Thm

"If the projective dimension of M is finite, then

$$\mathrm{pdim}_T M = \mathrm{depth}(T) - \mathrm{depth}(M),$$

where M is a f.g. module over T, some local ring."

In particular, for $T = R$, and M a MCM module, then $\mathrm{pdim}_R(M) < +\infty$ implies M is projective.

If $T = S$, then $\mathrm{pdim}_S(M) < +\infty$ is Hilbert's Syzygy Thm; so

$$\mathrm{pdim}_S(M) = \dim(S) - \mathrm{depth}(M)$$

so MCM modules have minimal projective dimension (over S).

Example: 1) Let $R = S/(f)$. $M \in MCM(R)$.

Then $\mathrm{pdim}_S(M) = 1 \iff \exists\ 0 \to F_1 \xrightarrow{A} F_0 \to M \to 0$

2) If $R = S/(f_1, , f_c)$, and $M \in MCM(\underline{R})$, then

$$\mathrm{pdim}_S(M) = c \iff \exists\ 0 \to \overbrace{F_c \xrightarrow{A} . \to F_1 \xrightarrow{A} F_0}^{\mathbb{F}} \to M \to 0$$

$$(\text{so } A : \mathbb{F} \to \mathbb{F}[-1])$$

Recall that

$$\begin{tikzcd}
0 \arrow[r] & F_1 \arrow[r, "A"] \arrow[d, "f"] & F_0 \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\exists B'"'] & M \arrow[r] \arrow[d, "f=0"] & 0 \\
0 \arrow[r] & F_1 \arrow[r, "A"'] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

On the other hand,

$$0 \leftarrow R \leftarrow \underbrace{S \xleftarrow{f\frac{\partial}{\partial\sigma}} S.\sigma}_{} \leftarrow 0$$

$\Delta := (\bigwedge_S(S.\sigma), \partial = f\frac{\partial}{\partial\sigma})$ is a graded commutative d.g. algebra over S

Thus,

$\begin{cases} A \text{ "corresponds" to the differential on } F_1 \to F_0, \\ B \text{ "corresponds" to the action of } \sigma \text{ on } F_1 \oplus F_0, \end{cases}$

so $(F_0 \oplus F_1, A)$ is a dg. Δ-module exactly when B turns (A, B) into a matrix factorization of f.

(Quillen) If $(\mathcal{A},\partial)$, $(\mathcal{A}',\partial')$ are quasi-isomorphic d.g. algebras, then $D(\mathcal{A},\partial) \simeq D(\mathcal{A}',\partial')$. $\boxed{2}$

If $\rho : \mathcal{A} \to \mathcal{A}'$ is a d.g. algebra homomorphism giving the quasi-isomorphism, then

$$\begin{tikzcd} D(\mathcal{A},\partial) \arrow[r, shift right, "(-)\otimes^{L}_{\mathcal{A}}\mathcal{A}'"'] & D(\mathcal{A}',\partial') \arrow[l, shift right, "\rho_*"'] \end{tikzcd}$$

are quasi-inverse equivalences.

Example: For $\Lambda = (\Lambda(S\sigma),\partial) \xrightarrow{\sim} (R,0)$ we get

$$\begin{tikzcd} D^b(\Lambda,\partial) \arrow[r, "\sim"] \arrow[d] & D^b(R) \arrow[d] \\ D^b_{sg}(\Lambda,\partial) \arrow[r, "\sim"] & D^b_{sg}(R) \end{tikzcd}$$

defined such that it gives the equivalence $\to$

$\square$

Now for the matrix factorization $F_1 \underset{B}{\overset{A}{\rightleftarrows}} F_0$, consider

$$\nabla : A + B.s : \mathbb{F} \otimes_S \underbrace{S[s]}_{\text{polynomial algebra}} \to \mathbb{F} \otimes_S \underbrace{S[s]}\underset{\text{shift}}{[-1]}, \text{ where } \deg(s) = 2$$

Note

$$\nabla^2 = (A+Bs)(A+Bs) = \underset{\overset{\shortparallel}{0}}{A^2} + (AB+BA)s + \underset{\overset{\shortparallel}{0}}{B^2} s^2$$

$$= f.s,$$

so $(A,B) \longleftrightarrow (\nabla = A + Bs,\ \nabla^2 = f.s)$.

Also, note that $S[s] = \bigoplus_{i,\nu} H^i(\mathrm{Proj}_S\, S[s], \mathcal{O}_S(\nu))$, so $S[s]$ is the cohomology ring of $\mathbb{P}^0_S$. Furthermore, one also has $S[s] \cong \mathrm{Ext}^{\bullet}_{(\Lambda,\partial)}(S,S)$, and $R[s] \simeq \mathrm{Ext}^{\bullet}(\Lambda,\Lambda)$.

Consider now $R = S/(f)$

$$\begin{tikzcd} \underline{\mathrm{modgr}}_{[0,1]}(\Lambda,\partial) \arrow[rrrr, "\sim\ (\Lambda,\partial)\otimes^{L}_{S}(\Lambda,0)"] & & & & D^b_{sg}(\Lambda,\partial) \arrow[d, "\wr"] \\ \underline{\mathrm{MF}}_S(f) \arrow[u, "\wr"] \arrow[r] & \mathrm{Hot}^2(\mathrm{proj}\,R) \arrow[r] & K_{ac}(\mathrm{proj}\,R) \arrow[r, "\sim"] & \underline{\mathrm{MCM}}(R) \arrow[r] & D^b_{sg}(R) \\ (A,B) \arrow[r, mapsto] & (\dots \xrightarrow{A} \xrightarrow{B} \dots) \arrow[r, mapsto] & (\dots \xrightarrow{B} \xrightarrow{A} \dots) \arrow[r, mapsto] & \mathrm{Coker}(A) \arrow[r, mapsto] & \mathrm{Coker}(A)[0] \end{tikzcd}$$

for any Gorenstein ring

Recall that $\mathrm{mod}_{[0,c]}(\Lambda, \partial)$ has objects

$$\mathbb{F} = \bigoplus_{i=0}^{c} F_i \quad , F_i \text{ are finite rank free } S\text{-modules}$$: Terms in the proj. S-resol.

$$A : \mathbb{F} \longrightarrow \mathbb{F}[1] \qquad A\ S\text{-linear map}$$: differentiati of the resol.

$$B_i : \mathbb{F} \longrightarrow \mathbb{F}[\qquad B_i \text{ action of } \sigma_i \text{ on } \mathbb{F}$$

Let $s_1, \dots, s_c$ be variables of degree $+2$ and set

$$\nabla = A + \sum_{i=1}^{c} B_i c_i \quad \text{with} \quad \nabla^2 = \sum_{i=1}^{c} f_i s_i$$

$$\Big(\nabla^2 = A^2 + \sum_{i=1}^{c} \underbrace{(AB_i + B_i A)}_{\downarrow \text{ d.g. module structure}} s_i + \underbrace{\sum_{i<j} (B_i B_j + B_j B_i) s_i s_j + \sum_{i=1}^{c} B_i^2 s_i^2}_{\downarrow\ \Lambda\text{-module structure}}$$

$$= 0 + \sum_{i=1}^{c} f_i s_i + 0$$

Now, for $\left(\mathbb{F}, A, B_i, \nabla = A + \sum_{i=1}^{c} B_i s_i \text{ s.th. } \nabla^2 = \sum_{i=1}^{c} f_i s_i \right)$

we consider $\mathrm{Coker}(A : F_1 \to F_0)$

Exercise: $(\mathbb{F}, A)$ is a resolution of a module, which is a MCM R-module. □

Thm (jt with T. Pham, C. Roberts)

* $$M = \mathrm{coker}(A : F_1 \longrightarrow F_0) \simeq \mathrm{Im}\Big(\underbrace{B_1 \circ \dots \circ B_c}_{\text{action of the socle of } \Lambda!} : F_0 \otimes_S R \longrightarrow F_c \otimes_S R\Big).$$

So, consider

$$\underline{\mathrm{modgr}}_{[0,c]}(\Lambda, \partial) \xrightarrow{\ \sim\ } K_{ac}(\mathrm{proj}\, R)$$

$$\begin{tikzcd}
(\mathbb{F}, A) \arrow[r, "\sim"', "S"] & M \arrow[r] & 0 \\
\mathbb{F} \otimes_S \Gamma_R\Big(\bigoplus_{i=1}^{c} R\sigma_i\Big) \arrow[r] \arrow[rr, bend left] & M & \mathbb{F} \otimes_S \mathrm{Sym}_R\Big(\bigoplus_{i=1}^{c} Rs_i\Big) \otimes_{R[s_1, \dots, s_c]} R
\end{tikzcd}$$

$\uparrow$ divided power algebra $\deg(\sigma) = -2$

$\deg(s_i) = 2$

Now, replace $\mathbb{T}$ by $\underline{\Lambda}$:

$$\begin{tikzcd}
\Lambda \otimes_S \Gamma_S\left(\bigoplus_{i=1}^{c} S\sigma_i\right) \otimes_S \circledR \arrow[dr] & & \underline{\Lambda} \otimes_S \mathrm{Sym}_S\left(\bigoplus_{i=1}^{c} S s_i\right) \otimes_S \circledR \\
& \circledR \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] & & 0
\end{tikzcd}$$

replace by $\underline{\Lambda}$

replace by $\underline{\Lambda}$

replace by $\underline{\Lambda}$

is a complete resolution of Λ as ${}^{o}\Lambda \otimes_S \Lambda$-module!

The graded module $\mathbb{T} \otimes_S S[s_1, \ldots, s_c]$ over $S[s_1, \ldots, s_c]$, with $\deg(s_i) = 2$ ($\rightsquigarrow$ gr. matrix factorization of $\Sigma f_i s_i$).

We see it as a $((\mathbb{Z}/2))$-graded sheaves on $\mathbb{P}^{c-1}_S$.

If "S is a field"

$$\underline{\mathrm{modgr}}_{[0,c]} \underline{\Lambda} \underset{[\text{Beilinson}]}{\simeq} D^b(\mathbb{P}^{c-1}_S) \xrightarrow[\text{[BGG]}]{\sim} D^b_{sg}(\Lambda) \simeq \underline{\mathrm{modgr}}\, \underline{\Lambda}$$

Hence, one may think

$$\underline{\mathrm{modgr}}_{[0,c]}(\Lambda, \partial) \to K_{ac}(\mathrm{proj}\, R)$$

is a kind of "Beilinson" construction.