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Representation theory of matrix factorizations

$S = k[\![x_0, x_1, \dots, x_d]\!] \supset \mathfrak{m} = (x_0, \dots, x_d) \supset \mathfrak{m}^2 \ni f$

$MF(f) = \{ (\varphi, \psi) \in \mathrm{Mat}_{t\times t}(S) \times \mathrm{Mat}_{t \times t}(S) \mid \varphi \cdot \psi = f \cdot 1\!\!1_{t\times t} = \psi \varphi \}$

matrix factorizations of f

<u>Example</u> (Dirac ~ 1930)

$$\Delta = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] \cdot 1\!\!1_{2\times 2} = \Big[\underbrace{\frac{\partial}{\partial x}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{\nabla}\Big]^2$$

$$\Delta = \nabla^2$$

i.e. for $f = x^2 + y^2 + z^2 \in \mathbb{C}[\![x, y, z]\!]$ $\quad \varphi = \begin{pmatrix} x & y - iz \\ y + iz & -x \end{pmatrix}$

$\varphi^2 = f \cdot 1\!\!1_{2\times 2}$

<u>Fact</u> (φ, φ) is "the only non-trivial" MF of f.

<u>Observation</u> (Knörrer). $f^\# = f + y^2 + z^2 \in \mathbb{C}[\![x_0, \dots, x_d, y, z]\!]$.

Then,
$$\begin{array}{ccc} MF(f) & \longrightarrow & MF(f^\#) \\ \rotatebox{90}{$\in$} & & \\ (\varphi, \psi) & \longmapsto & \left(\begin{pmatrix} \varphi & (y-iz)1\!\!1 \\ (y+iz)1\!\!1 & -\psi \end{pmatrix}, \begin{pmatrix} \psi & (y-iz)1\!\!1 \\ (y+iz)1\!\!1 & -\varphi \end{pmatrix} \right) \end{array}$$

is "essentially" a bijection.

Morphisms in $MF(f)$

$$\begin{tikzcd} S^p \arrow[r, "\psi"] \arrow[d, dashed, "\alpha"'] & S^p \arrow[r, "\varphi"] \arrow[d, "\beta"] \arrow[dl, dashed, "\delta"'] & S^p \arrow[d, "\alpha"] \arrow[dl, dashed, "\gamma"'] \\ S^q \arrow[r, "\psi'"] & S^q \arrow[r, "\varphi'"] & S^q \end{tikzcd}$$

$[\alpha, \beta] \sim [0, 0]$

$\iff \begin{cases} \beta = \gamma\varphi + \psi'\delta \\ \alpha = \varphi'\gamma + \delta\psi \end{cases}$

$\Rightarrow (1, f) \cong 0 \cong (f, 1)$ in <u>$MF(f)$</u>

<u>Theorem</u>. $A = S/f$.

- <u>$MF(f)$</u> is triangulated, $\Omega(\varphi, \psi) = (-\psi, -\varphi) \cong (\psi, \varphi)$

$\Omega^2 \cong \mathrm{Id}$.

- (Eisenbud) $\underline{MF}(f) \xrightarrow{\sim} \mathrm{Hot}^2_{ac}(\mathrm{pro}(A))$

$$= \{ \xrightarrow{\overline{\varphi}} A^p \xrightarrow{\overline{\psi}} A^p \xrightarrow{\overline{\varphi}} A^p \longrightarrow \}$$

hom. category of 2-periodic acyclic free complexes

$$\begin{array}{ccc} & & \cdots \xrightarrow{\overline{\varphi}} A^p \xrightarrow{\overline{\psi}} A^p \xrightarrow{\overline{\varphi}} \cdots \\ & \Big\downarrow ? & \Big\downarrow \\ & \underline{MCM}(A) & M = \mathrm{Cok}(\overline{\varphi}) \end{array}$$

$\mathrm{Ext}^p_A(k, M) = 0 \ \forall\, 0 \le p < d$

- (Knörrer) $\underline{MF}(f) \longrightarrow \underline{MF}(f^\#)$ is an equivalence.

Lemma. $\underline{MF}(f)$ is Hom-finite $\Leftrightarrow$ f is isolated singularity

$$\Leftrightarrow \dim_k \Bigg(\underbrace{\frac{k[|x_0, \ldots, x_d|]}{(f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_d})}}_{= T_f} \Bigg) < \infty$$

Tjurina algebra

Proof. "$\Leftarrow$": enough to show $\dim_k \underline{\mathrm{End}}(\varphi, \psi) < \infty$

$$\begin{array}{ccccc} S^p & \xrightarrow{\psi} & S^p & \xrightarrow{\varphi} & S^p \\ g\downarrow & & g\downarrow & & \downarrow g \\ S^p & \xrightarrow[\psi]{} & S^p & \xrightarrow[\varphi]{} & S^p \end{array}$$

$\varphi \cdot \psi = f \cdot 1\!\!1_{t \times t}$

$(\partial_i \varphi)\psi + \varphi \partial_i \psi = \frac{\partial f}{\partial x_i} \cdot 1\!\!1 \quad \forall i$

$g \in k[|x_0, \ldots, x_d|]$

$$\begin{array}{ccc} S & \xrightarrow{\text{finite}} & \underline{\mathrm{End}}(\varphi, \psi) \\ & \searrow \quad \nearrow & \\ & T_f & \end{array}$$

$Z = \mathrm{Sing}(f)$. $\exists\ p \in \mathrm{Supp}(Z)$, $p \neq m$.

$\dim_k \mathrm{Ext}^p(A/p, A/p) = \infty \quad \forall\ p$

$M = \mathrm{Syz}^d(A/p) \qquad \dim \underline{\mathrm{Hom}}(M, M) = \infty$.

Thm (Auslander). f is isolated $\Rightarrow$ $\exists$ a bifunctorial isomorphism

$$\underline{\mathrm{Hom}}(M,N) \xrightarrow{\sim} \underline{\mathrm{Hom}}(N, M[\sigma])^* \qquad \sigma \equiv d+1 \bmod 2$$

$\exists$ explicit formula

$$\underline{\mathrm{Hom}}(M,N) \times \underline{\mathrm{Hom}}(N, M[1]) \longrightarrow k \qquad \text{Kapustin-Li Murfet}$$

Corollary. $M \in \underline{MF}(f)$ indecomposable

$\Rightarrow$ $\Lambda = \underline{\mathrm{End}}(M)$ is local

$\underline{\mathrm{End}}(M)^* \cong \underline{\mathrm{Hom}}(M, M[\sigma]) \ni w \neq 0$

$\langle w \rangle_k = \mathrm{Soc}(\underline{\mathrm{Hom}}(M, M[\sigma]))$

$$\Rightarrow \quad \begin{array}{ccccccc} M[\sigma-1] & \longrightarrow & Y_w & \longrightarrow & M & \xrightarrow{w} & M[\sigma] \\ \| & & \nwarrow & & \uparrow \text{non-isom} & & \\ \tau(M) & & \exists & & N \ \text{ind} & & \end{array}$$

is almost split

$\tau(M)$ Auslander-Reiten translate of M

Example. Let $f = x^3 + y^4$ E_6-singularity.

Goal: describe all indecomposables in $\underline{MF}(f)$.

(I) $\alpha = \begin{pmatrix} x & y \\ y^3 & -x^2 \end{pmatrix}$ $\begin{pmatrix} x^2 & y \\ y^3 & -x \end{pmatrix} = \beta$ $\qquad \mathrm{cok}(\alpha) = \langle x^2, y \rangle \subseteq A$

$\gamma = \begin{pmatrix} x & y^2 \\ y^2 & -x^2 \end{pmatrix}$ $\begin{pmatrix} x^2 & y^2 \\ y^2 & -x \end{pmatrix} = \delta$ $\qquad \mathrm{cok}(\beta) = \langle x, y \rangle \subseteq \mathfrak{m}$

$\mathrm{cok}(\gamma) = \langle y^2, x \rangle \cong \mathrm{cok}(\delta)$

$$\begin{array}{ccccc} A^2 & \xrightarrow{\beta} & A^2 & \xrightarrow{\alpha} & A^2 \\ \rho \downarrow & & \downarrow \begin{pmatrix} 0 & -x \\ y^2 & 0 \end{pmatrix} = \theta & & \downarrow \begin{pmatrix} 0 & 1 \\ -xy^2 & 0 \end{pmatrix} = \rho \\ A^2 & \xrightarrow{\beta} & A^2 & \xrightarrow{\alpha} & A^2 \end{array}$$

socle element of $\underline{\mathrm{Hom}}((\beta, \alpha), (\beta, \alpha))$

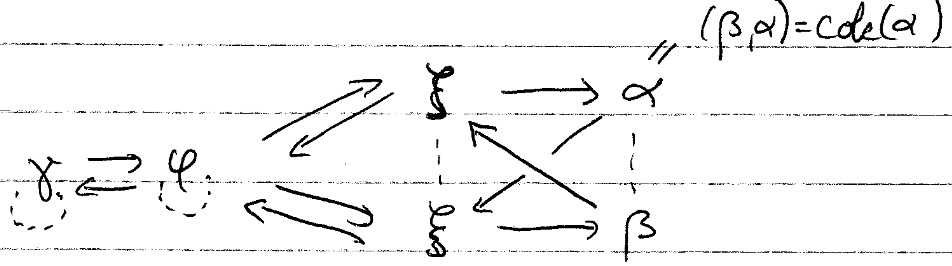
get $\Rightarrow \begin{pmatrix} \beta & \theta \\ 0 & -\alpha \end{pmatrix}, \begin{pmatrix} \alpha & \rho \\ 0 & -\beta \end{pmatrix} \cong \underbrace{\left(\underbrace{\begin{pmatrix} y & 0 & x \\ x & -y^2 & 0 \\ 0 & x & -y \end{pmatrix}}_{=\varsigma}, \underbrace{\begin{pmatrix} y^3 & x^2 & xy^2 \\ xy & -y^2 & x^2 \\ x^2 & -xy & -y^3 \end{pmatrix}}_{=\xi} \right)}_{\varsigma = \qquad \cong \xi} \oplus \underset{\substack{\cong \\ 0}}{(f, 1)}$

$\mathrm{Cok}(\zeta) = \langle x^2, -xy, -y^3 \rangle$

$\Rightarrow$ almost split triangle

$(\alpha, \beta) \longrightarrow (\xi, \zeta) \longrightarrow (\beta, \alpha) \longrightarrow (\beta, \alpha)$

$(\beta, \alpha) \longrightarrow (\zeta, \xi) \longrightarrow (\alpha, \beta) \longrightarrow (\alpha, \beta)$



<u>Thm</u> (Dieterich, Yoshino)
If AR quiver of <u>MCM(A)</u> has a finite component
$\Rightarrow$ it is the whole quiver.

<u>Thm</u> (Drozd - Roiter, ..., Buchweitz - Greuel - Schreyer)
<u>MF</u>(f) has finitely many indecomposables
$\Leftrightarrow$ f is a simple hypersurface singularity.