Cohen-Macaulay modules over non-isolated surface singularities

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joint with Yuriy Drozd

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See the book of Yoshino and a paper of Kajiura, Saito and Takahashi for explicit lists.

Indecomposable Cohen–Macaulay modules over *minimally elliptic* singularities

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$$T_{p,q,r}(\lambda): x^p + y^q + z^r + \lambda xyz$$
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Here $X \in \mathsf{Mat}_{n \times n}(K)$ and $S, T \in \mathsf{GL}_n(D)$ are such that S(0) = T(0) for $K = \mathsf{k}((t))$ and $D = \mathsf{k}[\![t]\!]$.

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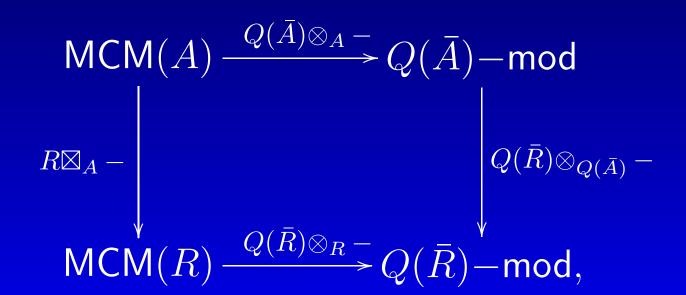
Hence, I is Cohen–Macaulay viewed as A– or R–module.

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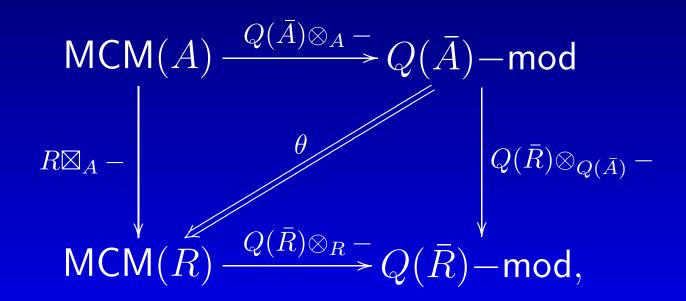


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 - Gluing map $Q(\bar{R}) \otimes_{Q(\bar{A})} V \stackrel{\theta}{\longrightarrow} Q(\bar{R}) \otimes_R \widetilde{M}$.

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<u>Definition</u>. Consider the category Tri(A) whose

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 - such that $V \stackrel{\widetilde{\theta}}{\longrightarrow} Q(\bar{R}) \otimes_R \widetilde{M}$ is injective.

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such that the diagram

$$Q(\bar{R}) \otimes_{Q(\bar{A})} V \xrightarrow{\theta} Q(\bar{R}) \otimes_{R} \widetilde{M}$$

$$1 \otimes g \downarrow \qquad \qquad \downarrow 1 \otimes f$$

$$Q(\bar{R}) \otimes_{Q(\bar{A})} V' \xrightarrow{\theta'} Q(\bar{R}) \otimes_{R} \widetilde{M}'$$

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Once again: we have a diagram of categories

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Theorem (Burban–Drozd). The functor

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- Hence, we have:

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 - θ is given by a pair of matrices $(\Theta_1, \Theta_2) \in \mathsf{Mat}_{p \times t}(K) \times \mathsf{Mat}_{q \times t}(K)$.

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Remark. This problem of linear algebra is close to classification of representations of $\bullet \leftarrow \bullet \rightarrow \bullet$ over the field K.

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Matrix Problem for $\mathsf{MCM}(T_{32\infty})$

Let $T = (\widetilde{M}, V, \theta)$ be an object of Tri(A).

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Canonical Forms: Continuous Series

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$$\Theta_1 = egin{pmatrix} z^{a_1}I & 0 & 0 & \dots & 0 \ 0 & z^{a_2}I & 0 & \dots & 0 \ dots & dots & \ddots & \ddots & dots \ 0 & 0 & \dots & \ddots & 0 \ 0 & 0 & \dots & 0 & z^{a_t}I \end{pmatrix} \ \Theta_2 = egin{pmatrix} 0 & z^{b_2}I & 0 & \dots & 0 \ 0 & 0 & z^{b_3}I & \dots & 0 \ 0 & 0 & 0 & \dots & z^{b_t}I \ z^{b_1}J & 0 & 0 & \dots & 0 \end{pmatrix}$$

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ight)$$

$$\Theta_2 = \left(egin{array}{ccccc} 0 & z^{n_1} & 0 & \dots & 0 \\ 0 & 0 & z^{n_2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & z^{n_{t+1}} \end{array}
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and any indecomposable maximal Cohen–Macaulay A–module has such form.

$\overline{\mathbf{MCM}}$ modules over $T_{32\infty}$

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Example. In the terms of matrix factorizations

$$J_{2,\lambda} \cong \operatorname{Cok}\left(A^2 \xrightarrow{\left(egin{array}{ccc} x + \mu(\mu+1)z^2 & y + \mu xz \\ y - (\mu+1)xz & -x^2 \end{array}
ight)} A^2 \right).$$

$$\begin{pmatrix}
 u & 0 \\
 v^p + \lambda w^q & vw
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Any $M \in \mathsf{MCM}^{lf}(A)$, $\mathsf{rk}(M) = 1$ is isomorphic to

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$$M(\omega, \lambda) := \langle J, I(\omega, \lambda) \rangle_A^{\vee \vee} \subseteq A.$$

Example. In the above notations, let

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$$m_1 = n_1 = p_1 = q_1 = 1$$

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Then the corresponding Cohen–Macaulay module is

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Moreover, its presentation is

$$A^{8} \xrightarrow{\begin{pmatrix} y & 0 & 0 & 0 & v & u^{n} & 0 & 0 \\ 0 & v & 0 & 0 & 0 & x & y^{p} & 0 \\ 0 & 0 & x & 0 & 0 & 0 & u & v^{q} \\ 0 & 0 & 0 & u & \lambda x^{m} & 0 & 0 & y \end{pmatrix}} A^{4} \longrightarrow M(\omega, \lambda) \longrightarrow 0.$$

Thank you for your attention!