# Graded matrix factorizations and functor categories

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# Attributions

Based on joint work with Matthew Ballard (Upenn) and Ludmil Katzarkov (Miami and Wien).

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**Theorem (Dyckerhoff)**

Let $f$ and $f'$ define isolated singularities in regular local rings, $R, R'$. The full sub(dg)category of compact objects in the category of functors from $\mathrm{MF}(R,f)$ to $\mathrm{MF}(R',f')$ is equivalent to $\mathrm{MF}(R \otimes R', f \otimes 1 - 1 \otimes f')$. We write, $(\mathrm{MF}(R,f) \hat{\otimes} \mathrm{MF}(R',f'))_{\mathrm{pe}} \cong \mathrm{MF}(R \otimes R', f \otimes 1 + 1 \otimes f')$.

## Definition

A semi-orthogonal decomposition of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:

$$\begin{tikzcd}
0 \arrow[rr] & & T_{m-1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & T_2 \arrow[rr] & & T_1 \arrow[rr] \arrow[dl] & & T \arrow[dl] \\
& A_m \arrow[ul, "\shortmid" marking] & & & & A_2 \arrow[ul, "\shortmid" marking] & & A_1 \arrow[ul, "\shortmid" marking] &
\end{tikzcd}$$

where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We shall denote a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$.

## Theorem (Orlov)

Let $X$ be a hypersurface in $\mathbb{P}^n$ which is the zero locus of a homogeneous polynomial, $f$, of degree, $d$.

1. If $n+1-d > 0$, there is a semi-orthogonal decomposition,
$$\mathrm{D^b}(\operatorname{coh} X) = \langle \mathcal{O}_X(d-n), \ldots, \mathcal{O}_X, \mathrm{MF}(R, f, \mathbb{Z})\rangle.$$
2. If $n+1-d = 0$, there is an equivalence of triangulated categories,
$$\mathrm{D^b}(\operatorname{coh} X) = \langle \mathrm{MF}(R, f, \mathbb{Z})\rangle.$$
3. If $n+1-d < 0$, there is a semi-orthogonal decomposition,
$$\mathrm{MF}(R, f, \mathbb{Z}) \cong \left\langle k, \ldots, k(n+2-d), \mathrm{D^b}(\operatorname{coh} X)\right\rangle.$$

**Question**

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What is the analog of Dyckerhoff's result in the case of graded matrix factorizations?

## Theorem

Let $M, M'$ be finitely generated abelian groups. Let $R = k[x_0, ..., x_n], R' = k[y_0, ..., y_{n'}]$ be $M, M'$ graded rings with $x_i, y_i$ homogeneous. Let $f \in R_d, f' \in R_{d'}$ be homogeneous functions such that $f \in df, f' \in df'$ and $d \in M, d' \in M'$ are not torsion. The full sub(dg)category of compact objects in the category of functors from $\mathrm{MF}(R, f, M)$ to $\mathrm{MF}(R', f', M')$ is equivalent to $\mathrm{MF}(R \otimes R', f \otimes 1 - 1 \otimes f', M \oplus M'/(d, -d'))$.

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## Remark

Indepedently, Polishchuk and Vaintrob prove this theorem in the case where singularities are isolated and $M \otimes_{\mathbb{Z}} \mathbb{Q}, M' \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

We have some corollaries:

## Corollary

If $M/(d)$ is finite, let $G$ be the finite group with $\mathrm{char}(G) = M/(d)$. Let $A = \mathrm{Sym}\, V$ with $V$ an $M$-graded vector space. Let $w_g$ be the restriction of $w$ to the fixed locus of $g$ on $V$ and $A^g = \mathrm{Sym}\, V^g$. The space of derived natural transformations, $\mathrm{Id} \to (m)[t]$, is

$$\begin{cases} \bigoplus_{g\in G} \bigoplus_{p=2q} \mathrm{H}^{p-c_g}(dw_g; A^g)_{m+d(l-q)-v_g} & t = 2l \\ \bigoplus_{g\in G} \bigoplus_{p=2q+1} \mathrm{H}^{p-c_g}(dw_g; A^g)_{m+d(l-q)-v_g} & t = 2l+1 \end{cases}$$

where $\mathrm{H}^*(dw_g; A^g)$ is the Koszul cohomology of the ideal $(dw_g)$ in $A_g$, $c_g = \mathrm{codim}\, V^g \subset V$, and $v_g$ the degree of the graded rank one vector space $\Lambda^{\mathrm{top}} W^\vee$ with $V = V^g \oplus W$ a splitting as an $M$-graded vector space.

We can describe complete intersection categories in similar manner: let $R = k[x_1, \ldots, x_n]$ and $(f_1, \ldots, f_c)$ a regular sequence. Set $S = R/(f_1, \ldots, f_c)$. Let $P$ be the $\mathbb{Z}$-graded ring $R[u_1, \ldots, u_c]$ with $\deg u_i = 1$ and let $w = u_1 f_1 + \cdots + u_c f_c$. Isik gives a useful equivalence:

**Theorem (Isik)**

There is an equivalence, $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} S) \cong \mathrm{MF}(P, w, \mathbb{Z})$. Moreover, this equivalence restricts to an equivalence, $\operatorname{Perf} S \cong \mathrm{MF}_u(P, w, \mathbb{Z})$, where $\mathrm{MF}_u(P, w, \mathbb{Z})$ is the subcategory of $(u_1, \ldots, u_c)$-torsion matrix factorizations. So, $\mathrm{D}_{\mathrm{sg}}(S)$ is equivalent to the quotient $\mathrm{MF}(P, w, \mathbb{Z}) / \mathrm{MF}_u(P, w, \mathbb{Z})$.

What is the (dg) category of functors $\mathrm{D}_{\mathrm{sg}}(S) \to \mathrm{D}_{\mathrm{sg}}(S')$?

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## Theorem

The (compact objects in the homotopy category of the derived dg) category of (colimit preserving) functors $\mathrm{D}_{\mathrm{sg}}(S) \to \mathrm{D}_{\mathrm{sg}}(S')$ is equivalent to the (idempotent completion of the) quotient $\mathrm{MF}(P \otimes_k P', w \otimes 1 - 1 \otimes w', \mathbb{Z}) / \langle \mathrm{MF}_u(w) \boxtimes \mathrm{MF}(-w'), \mathrm{MF}(w) \boxtimes \mathrm{MF}_u(-w') \rangle$.

## Theorem

The (compact objects in the homotopy category of the derived dg) category of (colimit preserving) functors $\mathrm{D_{sg}}(S) \to \mathrm{D_{sg}}(S')$ is equivalent to the (idempotent completion of the) quotient $\mathrm{MF}(P \otimes_k P', w \otimes 1 - 1 \otimes w', \mathbb{Z}) / \langle \mathrm{MF}_u(w) \boxtimes \mathrm{MF}(-w'), \mathrm{MF}(w) \boxtimes \mathrm{MF}_u(-w') \rangle$.

## Corollary

- $\mathrm{D_{sg}}(S)$ is smooth.
- There is a spectral sequence

$$E_2^{pq} = \mathbf{R}^p Q_{(u_1,\ldots,u_c)} \mathrm{HH}^q(\mathrm{MF}(P, w, \mathbb{Z})) \Longrightarrow \mathrm{HH}^{p+q}(\mathrm{D_{sg}}(S)).$$

where $Q_{(u_1,\ldots,u_c)}$ is the ideal transform associated to $(u_1, \ldots, u_c)$.

## Question

How does this compare with the standard interpretation of functors between $\mathrm{D^b}(\mathrm{coh}\, X)$ and $\mathrm{D^b}(\mathrm{coh}\, Y)$ as $\mathrm{D^b}(\mathrm{coh}\, X \times Y)$, for hypersurfaces, $X$, $Y$?

To answer this question, first we will need to gather the setup a bit more.

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Consider a collection of hypersurfaces, $X_i \subseteq \mathbb{P}^{n_i}$ defined by polynomials $f_i$ of degree $d_i$ for $1 \leq i \leq s$. Let $R_i$ be the coordinate rings of the $\mathbb{P}^{n_i}$. Consider the free abelian group of rank $s$, $\mathbb{Z}^s$, with basis $\mathbf{e}_i$, $1 \leq i \leq s$. Let $L$ be the subgroup generated by $d_i \mathbf{e}_i = d_j \mathbf{e}_j$ and $M := \mathbb{Z}^s / L$. Denote by $H$ the torsion subgroup of $M$. Explicitly, letting $d_{ij}$ be the greatest common divisor of $d_i$ and $d_j$, $H$ is the finite subgroup of $M$ generated by the images of $\frac{d_i}{d_{ij}}\mathbf{e}_i - \frac{d_j}{d_{ij}}\mathbf{e}_j$. One has $M/H \cong \mathbb{Z}$. Let $m$ be the least common multiple of the $d_i$. In this setting the degree map $\deg : M \to \mathbb{Z}$ can be identified with the mapping which takes $\mathbf{e}_i$ to $\frac{d}{d_i}$. Let $\delta$ be an element of degree 1.

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The dual group to $M$ can be identified with the set, $D := \{(\lambda_1, ..., \lambda_s) | \lambda_i^{d_i} = \lambda_j^{d_j} \forall i, j\} \subseteq (k^*)^s$ and acts on $\mathbb{A}^{n_1 + ... n_s + s} \backslash 0$ by multiplication by $\lambda_i$ on the coordinates, $x_{d_1 + ... + d_{i-1}}$ through $x_{d_1 + ... + d_i}$. Let $Y$ denote the hypersurface in $\mathbb{A}^{n_1 + ... n_s + s} \backslash 0$ defined by the zero locus of $f_1 + ... f_s$ and consider the global quotient stack, $Z := [Y/D]$.

## Theorem (Orlov)

Let $\mathcal{A} = \mathrm{MF}(R_1 \otimes \ldots \otimes R_s, f_1 + \ldots + f_s, M)$. (which by our theorem is equivalent to $(\mathrm{MF}(R_1, f_1, \mathbb{Z}) \hat{\otimes}_k \ldots \hat{\otimes}_k \mathrm{MF}(R_s, f_s, \mathbb{Z}))_{\mathrm{pe}}$).

1. If $a > 0$, there is a semi-orthogonal decomposition,
$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) \cong \langle \bigoplus_{h \in H} \mathcal{O}_Z((-a+1)\delta h), \ldots, \bigoplus_{h \in H} \mathcal{O}_Z(h), \mathcal{A} \rangle.$$
2. If $a = 0$, there is an equivalence of triangulated categories,
$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) \cong \mathcal{A}.$$
3. If $a < 0$, there is a semi-orthogonal decomposition,
$$\mathcal{A} \cong \langle \bigoplus_{h \in H} k(h), \ldots, \bigoplus_{h \in H} k((a+1)\delta h), \mathrm{D}^{\mathrm{b}}(\operatorname{coh} Z) \rangle.$$

## Example

In the simple case of one variable, Orlov's theorem in the context of algebras yields an equivalence between $\mathrm{MF}(k[x], x^d, \mathbb{Z})$ and $\mathrm{D}^{\mathrm{b}}(A_{d-1})$. Therefore,

$$\begin{aligned} &(\mathrm{MF}(k[x], x^p, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[y], y^q, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[z], z^r, \mathbb{Z}))_{\mathrm{pe}} \\ \cong &(\mathrm{D}^{\mathrm{b}}(A_{p-1}) \hat{\otimes}_k \mathrm{D}^{\mathrm{b}}(A_{q-1}) \hat{\otimes}_k \mathrm{D}^{\mathrm{b}}(A_{r-1}))_{\mathrm{pe}} \\ \cong &\,\mathrm{D}^{\mathrm{b}}(A_{p-1} \otimes_k A_{q-1} \otimes_k A_{r-1}). \end{aligned}$$

The stack, $Z$, defined by this data is the weighted projective line corresponding to the weight sequence $(p, q, r)$, as introduced by Geigle and Lenzing where they also show that this is equivalent to the derived category of a quiver with $p + q + r - 1$ vertices. This equivalence was discussed in the talks of Kussin and Lenzing.

## Example

Consider the weight sequence of Dynkin type (this means that $a > 0$), $(2, 3, 5)$. We have, $a = 30(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} - 1) = 1$ and, via Orlov's theorem, we can compare $\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathbb{P}(2:3:5))$ with $\mathrm{D}^{\mathrm{b}}(A_1 \otimes_k A_2 \otimes_k A_5) \cong \mathrm{D}^{\mathrm{b}}(E_8)$. We get:

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, \mathbb{P}(2:3:5)) = \langle \mathcal{O}, \mathrm{D}^{\mathrm{b}}(E_8) \rangle.$$

This matches with the result of Kajiura, Saito, and Takahashi (discussed by Iyama). Specifically, this is similar to the construction in their appendix written by Ueda.

### Example

Consider the weight sequence $(3,3,3)$. We have, $a = 3(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} - 1) = 0$ hence we obtain:

$$\mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{P}(3:3:3)) = \mathrm{D}^{\mathrm{b}}(A_2 \otimes A_2 \otimes A_2)\rangle.$$

## Example

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## Example

Consider the weight sequence $(4,4,4)$. We have $H \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ and $a = 4(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}-1) = -1$ hence we obtain:

$$\mathrm{D}^{\mathrm{b}}(A_3 \otimes A_3 \otimes A_3)\rangle \cong \langle \bigoplus_{h \in H} k(h), \mathrm{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{P}(4:4:4))\rangle.$$

Counting vertices we have 27 on the left hand side and 16+11 on the right hand side.

## Example

Let $f(x,y,z) = x(x-z)(x-\lambda z) - zy^2$ and $g(u,v,w) = u(u-w)(u-\gamma w) - wv^2$ define two smooth elliptic curves, $E$ and $F$ respectively. Then $f+g$ defines a smooth cubic fourfold containing at least three planes by setting $z = w = 0$. By work of Kuznetsov, the category $\mathrm{MF}(k[x,y,z,u,v,w], f+g, \mathbb{Z})$ is, in this case, equivalent to the derived category of a certain gerby $K3$ surface, $Y$. On the other hand, letting $M = \mathbb{Z} \oplus \mathbb{Z}/(3,-3)$ with $x,y,z$ in degree $(1,0)$ and $u,v,w$ in degree $(0,1)$, we have $\mathrm{MF}(k[x,y,z,u,v,w], f+g, M) \cong (\mathrm{MF}(k[x,y,z], f, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[u,v,w], g, \mathbb{Z}))_{\mathrm{pe}}$. From Orlov, ee have $\mathrm{MF}(k[x,y,z], f, \mathbb{Z}) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} E)$ and $\mathrm{MF}(k[u,v,w], g, \mathbb{Z}) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} F)$. Hence $\mathrm{MF}(k[x,y,z,u,v,w], f+g, M) \cong \mathrm{D}^{\mathrm{b}}(\operatorname{coh} E \times_k F)$. In a moment we will discuss how $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} E \times_k F)$ is a $\mathbb{Z}_3$-cover of $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} Y)$.

## Hodge diamond of cubic fourfold and a $K3$ surface

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

## Hodge diamond of cubic fourfold and a $K3$ surface

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & {\color{green}1} & & 0 & & \\
 & 0 & & {\color{green}0} & & {\color{green}0} & & 0 & \\
0 & & {\color{green}1} & & {\color{green}20} & & {\color{green}1} & & 0 \\
 & 0 & & {\color{green}0} & & {\color{green}0} & & 0 & \\
 & & 0 & & {\color{green}1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

## Remark

Furthermore, on each elliptic curve, $E, F$ the autoequivalence $(1)$ is a composition of Dehn twists. Hence this autoequivalence can be viewed as a symplectic automorphism of the mirror. The action of $\mathbb{Z}_3$ on $\mathrm{D}^{\mathrm{b}}(\operatorname{coh} E \times_k F)$ is given by $(1, -1)$. This can therefore be considered as a product of sympletic automorphisms of the product of the two mirrors. The relationship between the surfaces $E \times_k F$ and $Y$ can then be seen by viewing the mirror of $E \times_k F$ as a three to one symplectic cover of the mirror of $Y$.

# Orbit Categories

## Definition

Let $\Gamma$ be a finitely generated abelian group of rank at most one which is a subgroup of the automorphism group of a triangulated category $\mathcal{T}$. The orbit category of $\mathcal{T}$ by $\Gamma$, denoted $\mathcal{T}/\Gamma$ has the same objects as $\mathcal{T}$ with morphisms from $A$ to $B$ given by

$$\mathrm{Hom}_{\mathcal{T}/\Gamma}(A,B) = \bigoplus_{g \in \Gamma} \mathrm{Hom}_{\mathcal{T}}(A, g(B)).$$

Composition of morphisms is defined in the obvious way.

# Orbit Categories

## Definition

Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and $\Gamma$ a group of triangulated automorphisms of $\mathcal{T}$. We say that $\mathcal{T}$ is a $\Gamma$-cover of $\mathcal{S}$ if there is a fully faithful functor,

$$F : \mathcal{T}/\Gamma \to \mathcal{S},$$

such that every object in $\mathcal{S}$ is a summand of the essential image of $F$.

# Orbit Categories

The following proposition is inspired by work of Keller, Murfet, and van den Bergh:

**Proposition**

Let $M$ be a finitely generated abelian group and $L$ be a finite subgroup of $M$ of order $n$. Let $S$ be an $M$-graded ring and assume that $n$ is a unit in $S$. Denote by $T$ the ring $S$ with the $M/L$ grading given by $S_{[m]} := \bigoplus_{l \in L} S_{lm}$. The category, $\mathrm{MF}(S, f, M)$ is an $L$-cover of $\mathrm{MF}(T, f, M/L)$.

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**Example**

Let $R = k[x_0, \dots, x_n]$ and $R = k[x_0, \dots, x_m]$ are $\mathbb{Z}$-graded rings over $k$ and $w, w'$ be homogeneous polynomials with $\deg w = d$ and $\deg w' = d'$. Let $m$ be the least common multiple of $d$ and $d'$. Equip $R \otimes_k R'$ with the $\mathbb{Z}$ grading $(R \otimes_k R')_s := \bigoplus_{d'i+dj=s} R_i \otimes R_j$. The category, $(\mathrm{MF}(R, w, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(R', w', \mathbb{Z}))_{\mathrm{pe}}$ is a $\mathbb{Z}_m$-cover of $\mathrm{MF}(R \otimes_k R', w \otimes 1 + 1 \otimes w', \mathbb{Z})$.

# Orbit Categories

## Example

Consider a quartic $K3$ surface, $Y$, defined by $f(x,y,z,w)$ in $\mathbb{P}^3$. Let $t^2 - f$ be the quartic double solid, $Q$, in weighted projective space $\mathbb{P}(2:1:1:1:1)$. Notice that $\mathrm{MF}(k[t], t^2, \mathbb{Z}) = \mathrm{D^b}(A_1)$ is equivalent to the derived category of vector spaces. Hence, $(\mathrm{MF}(k[t], t^2, \mathbb{Z}) \hat{\otimes}_k \mathrm{MF}(k[x,y,z,w], f, \mathbb{Z}))_{\mathrm{pe}} \cong \mathrm{MF}(k[x,y,z,w], f, \mathbb{Z}) \cong \mathrm{D^b}(\mathrm{coh}\, Y)$. Therefore, $\mathrm{D^b}(\mathrm{coh}\, Y)$ is a $\mathbb{Z}_2$-cover of $\mathrm{MF}(k[t,x,y,z,w], t^2 - f, \mathbb{Z})$, an admissible subcategory of $\mathrm{D^b}(\mathrm{coh}\, Q)$. By work of Ingalls and Kuznetsov, in certain special cases, this admissible subcategory is equivalent to the derived category of an Enriques surface obtained from a $\mathbb{Z}_2$ action on the $K3$.

### Example

Let $A$ be the free abelian group generated by $\mathbf{e}_0, ..., \mathbf{e}_n$ and $B$ be the subgroup generated by $d\mathbf{e}_i - d\mathbf{e}_j$ for all $i, j$. Let $M := A/B$. Let $R$ be the polynomial algebra $k[x_0, ..., x_n]$ which its natural $\mathbb{Z}$ grading, and let $f := x_0^d + \cdots + x_n^d$ be the Fermat polynomial. We have:

$$\mathrm{D}^{\mathrm{b}}(A_{d-1})^{\otimes n+1}) \cong (\mathrm{MF}(k[x], x^d, \mathbb{Z})^{\hat{\otimes} n+1})_{\mathrm{pe}} \cong \mathrm{MF}(R, f, M).$$

Let $C \cong \mathbb{Z}_d^{\oplus n}$ be the subgroup generated by $\mathbf{e}_i - \mathbf{e}_j$ for all $i, j$. Then $(A/B)/C \cong \mathbb{Z}$. Hence one realizes $\mathrm{D}^{\mathrm{b}}(A_{d-1}^{\otimes n+1})$ as a $\mathbb{Z}_d^{\oplus n}$-cover of $\mathrm{MF}(R, f, \mathbb{Z})$.

# Orbit Categories

More generally, consider a partition, $\mathcal{P} = \{0, \cdots i_0\} \cdots \{i_{m-1}+1, \cdots, i_m\}$ of the set $\{0, ..., n\}$ into $m$ parts. Let $D_{\mathcal{P}} \cong \mathbb{Z}_d^{\oplus n+1-m}$ be the subgroup generated by $\mathbf{e}_i - \mathbf{e}_j$ for all $i, j$ in the same part of the partition and $M_{\mathcal{P}} := M/D_{\mathcal{P}}$. One obtains, $\mathrm{D^b}(\mathrm{mod} - (A_{d-1})^{\otimes n+1})$ as a $\mathbb{Z}_d^{\oplus n+1-m}$-cover of $\mathrm{MF}(R, f, M_{\mathcal{P}})$ which is equivalent to

$$(\mathrm{MF}(k[x_0, ..., x_{i_0}, x_0^d + \cdots + x_{i_0}^d, \mathbb{Z})\hat{\otimes}_k \cdots$$
$$\hat{\otimes}_k \mathrm{MF}(k[x_{i_{m-1}+1}, ..., x_{i_m}], x_{i_{m-1}+1}^d + \cdots + x_{i_m}^d), \mathbb{Z}))_{\mathrm{pe}},$$

Notice that varying the partitions, one gets a partially ordered collection of covers with maximal element $\mathrm{D^b}(A_{d-1}^{\otimes n+1})$ and minimal element, $\mathrm{MF}(R, f, \mathbb{Z})$.

# Some applications to the Hodge conjecture

**Theorem**

Let $Y$ be the unique $K3$ surface of Picard rank 20 with polarization of degree 14. The Hodge conjecture over $\mathbb{Q}$ holds for $n$-fold products of $Y$.

# Some applications to the Hodge conjecture

## Theorem

Let $Y$ be the unique $K3$ surface of Picard rank 20 with polarization of degree 14. The Hodge conjecture over $\mathbb{Q}$ holds for $n$-fold products of $Y$.

## Idea of the proof:

Similarly to the previous example, due to Orlov's theorem and results of Kuznetsov, $\mathrm{D^b}(\mathrm{coh}\, Y) \cong \mathrm{MF}(k[x_0, \ldots, x_5], x_0^3 + \ldots + x_5^3, \mathbb{Z})$. Therefore by our theorem, $\mathrm{D^b}(\mathrm{coh}\, Y^n)$ is a $\mathbb{Z}_3^{n-1}$-cover of $\mathrm{MF}(k[x_0, \ldots, x_{6n-1}, x_0^3 + \ldots + x_{6n-1}^3, \mathbb{Z})$. By work of Shioda and Ran (which we also reproduce using a matrix factorization argument), over $\mathbb{C}$, all $(p, p)$-cycles in the cohomology of a cubic hypersurface are algebraic. Using grading changes, we use this to deduce that all $(p, p)$-cycles in the cohomology of $Y^n$ are algebraic.

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For two subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$ we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of objects $X \in \mathcal{T}$ such that there is a distinguished triangle $X_1 \to X \to X_2 \to X_1[1]$ with $X_i \in \mathcal{I}_i$.

# Some applications to dimensions of triangulated categories

Roughly, the dimension of a triangulated category $\mathcal{T}$ is the minimal number of triangles it takes to produce any object from a fixed object. More precisely the definition is as follows:

For a subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the full subcategory of $\mathcal{T}$ whose objects are summands of direct sums of shifts of objects in $\mathcal{I}$.

For two subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$ we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of objects $X \in \mathcal{T}$ such that there is a distinguished triangle $X_1 \to X \to X_2 \to X_1[1]$ with $X_i \in \mathcal{I}_i$.

Further set $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$. By setting $\langle \mathcal{I} \rangle_1 := \langle \mathcal{I} \rangle$ we are able to inductively define $\langle \mathcal{I} \rangle_n := \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle$.

**Summary**

$\mathcal{I}_1 \diamond \mathcal{I}_2$ is the full subcategory of objects $X \in \mathcal{T}$ such that there is a distinguished triangle $X_1 \to X \to X_2 \to X_1[1]$ with $X_i \in \mathcal{I}_i$ closed under summands. Define $\langle \mathcal{I} \rangle_n := \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle$.

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**Definition**

Let $X$ be an object in $\mathcal{T}$. The generation time of $X$, denoted $\circleddash(X)$, is

$$\circleddash(X) := \min \{ n \in \mathbb{N} \,|\, \mathcal{T} = \langle X \rangle_{n+1} \}.$$

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**Definition**

The dimension of a triangulated category $\mathcal{T}$ is the minimal generation time among the strong generators.

**Theorem (Rouquier)**

For a separated scheme of finite type over a perfect field, $X$, the dimension of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is finite.

## Some applications to dimensions of triangulated categories

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1. $\dim(X) \leq \dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$
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Let $X$ be a smooth variety. Then $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \dim(X)$.

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**Theorem (Orlov)**

The above conjecture holds for smooth curves. More generally, if $C$ is a smooth curve, then the spectrum of $\mathrm{D}^{\mathrm{b}}(C)$ contains $\{1, 2\}$ with equality if and only if $C = \mathbb{P}^1$.

# Some applications to dimensions of triangulated categories

**Proposition**

Let $L \subseteq M$ be a finite subgroup. The categories, $\mathrm{MF}(R, f, M)$ and $\mathrm{MF}(R, f, M/L)$ have the same Rouquier dimension.

# Some applications to dimensions of triangulated categories

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**Example**

Let $(d_0, .., d_n)$ be a weight sequence with $\sum_{i=1}^{s} \frac{1}{d_i} \leq 1$ containing either $\{2\}$, $\{3, 3\}$, $\{3, 4\}$, or $\{3, 5\}$. Let $k$ be a field whose characteristic does not divide any of the $d_i$ then Orlov's Conjecture holds for the weighted fermat hypersurface defined by $f$. Similarly, the Rouquier dimension of $\mathrm{D}^{\mathrm{b}}(A_{d_0-1} \otimes ... \otimes A_{d_n-1})$ is equal to $n-2$.

# Some applications to dimensions of triangulated categories

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**Example**

Orlov's Conjecture holds for the product, $E \times F$ of two elliptic curves and the infamous $K3$ surface obtained as a $\mathbb{Z}_3$ quotient.