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Lutz Hille (Münster) **Hochschild cohomology and tilting objects**

§ 1 Tilting theory

X smooth projective algebraic variety

T tilting bundle, vector bundle on X

1) $T = \bigoplus_{i=1}^{t} T_i$, $T_i \not\cong T_j$ $\forall i \neq j$, $\langle T \rangle = D^b(X)$

2) $\mathrm{Ext}^l(T,T) = 0$ $\forall l > 0$, $A = \mathrm{End}(T)$

Theorem (Hille/Perling) X rational surface, then there exists a tilting bundle on X.

Example $\mathbb{P}^2$ $\mathcal{O} \Rrightarrow \mathcal{O}(1) \Rrightarrow \mathcal{O}(2)$

$$\mathbb{P}^1 \times \mathbb{P}^1 \quad \begin{tikzcd} & \mathcal{O}(1,0) \arrow[dr, shift left] \arrow[dr, shift right] & \\ \mathcal{O} \arrow[ur, shift left] \arrow[ur, shift right] \arrow[dr, shift left] \arrow[dr, shift right] & & \mathcal{O}(1,1) \\ & \mathcal{O}(0,1) \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

$\mathbb{F}_a$ $\mathcal{O} \rightrightarrows \mathcal{O}(P) \begin{smallmatrix}\longrightarrow \\ \vdots \\ \longrightarrow\end{smallmatrix} \mathcal{O}(Q+aP) \rightrightarrows \mathcal{O}(Q+(a+1)P)$

$P^2 = 0$, $Q^2 = a$

Relative Example

$$\begin{tikzcd} X \supseteq \mathcal{O}(-a) \arrow[r, "\pi"] \arrow[d] & \mathbb{P}^1 \\ Y = \mathrm{Spec}\, R & \end{tikzcd} \qquad a > 0 \qquad \pi^* T \text{ tilting?}$$

$E^2 = -a$ $\mathrm{Ext}^l(\pi^* T, \pi^* T) = 0$ $\forall l > 0$?

$$\begin{tikzcd} Z = \mathcal{L}^{-1} \arrow[r, shift left] \arrow[d, "\mathrm{proj}"] & X \arrow[l, dashed, shift left] \\ Y = \mathrm{Spec}\, R & \end{tikzcd} \qquad \mathcal{L} \text{ ample}$$

Theorem (Hille/Perling) X toric, $\mathcal{L}$ ample, then there exists T such that $\pi^* T$ tilting on Z.

$\Phi = \mathbb{R}\mathrm{Hom}(T, -) : D^b(X) \xrightarrow{\sim} D^b(A)$

§ 2 Hochschild (co)homology

$X \overset{\Delta}{\underset{i}{\hookrightarrow}} X \times X$ $D^b(X \times X) \longrightarrow D^b(\mathrm{mod}\, A^{env})$ $A^{env} = A \otimes A^{op}$

$\mathcal{O}_\Delta \longmapsto A$ C^l_{ij} $1 \leq i,j \leq t$

$\bar{\Phi} = \mathbb{R}\mathrm{Hom}(T^\vee \boxtimes T, -)$

<u>Theorem</u> Resolution of the diagonal

$$T^\vee \boxtimes C^\bullet \boxtimes T \longrightarrow \mathcal{O}_\Delta$$
$$\| $$
$$(T_i^\vee \boxtimes T_j)^{c_{ij}^\ell}$$

For A, $A_0 = A/\mathrm{rad}\,A$

$$A \otimes_{A_0} C^\bullet \otimes_{A_0} A \xrightarrow{\mathrm{q.is}} A$$

minimal resolution $\bigoplus P(i,j)^{c_{ij}}$

<u>Theorem</u> (Buchweitz/Hille)

$$HH^\ell(X) := \mathrm{Ext}^\ell_{X\times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \mathrm{Ext}^\ell_{A^{env}}(A, A) = HH^\ell(A)$$

$$HH_\ell(X) := H^{-\ell}(X, \mathbb{L} i^* \mathcal{O}_\Delta) = \mathrm{Tor}_\ell^{A^{env}}(A, A) = HH_\ell(A)$$

<u>Theorem</u> (Buchweitz/Hille) $\quad T^\vee \otimes C^\bullet \otimes T \overset{\mathrm{q.is}}{\simeq} \bigoplus \Omega^p[p] \qquad T_i^\vee \otimes T_j$

$$V = H^0(\mathcal{O}(1))$$

$H^q(X, \Omega^p) = 0$
$\forall p \neq q$

$$\Lambda^2 V \otimes \mathcal{O}(-2) \otimes \mathcal{O} \qquad (V \otimes \mathcal{O}(-1) \otimes \mathcal{O})^2 \longrightarrow \begin{matrix} \mathcal{O} \otimes \mathcal{O} \\ \mathcal{O}(-1) \otimes \mathcal{O}(1) \\ \mathcal{O}(-2) \otimes \mathcal{O}(2) \end{matrix}$$

$$\begin{matrix} \cong & \cong & \| \\ \omega & \Omega^1_{\mathbb{P}^2} & \mathcal{O}^{\oplus 3} \end{matrix} \qquad H^2(\omega_X)$$

$$\begin{tikzcd} & D^b(A) \arrow[dl, "H"'] \arrow[dr] & \\ \mathrm{mod}^{\mathbb{Z}} TA \arrow[rr, dashed] & & D^b(X) \end{tikzcd}$$

$$\mathcal{O}(-3) \longrightarrow \Lambda^2 \mathcal{O}(-2) \longrightarrow V \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O}$$