

Bielefeld, May 8, 2011

Lutz Hille (Münster) Hochschild cohomology and filtered objects

§ 1 Filtrierung Theorie

X smooth projective algebraic variety

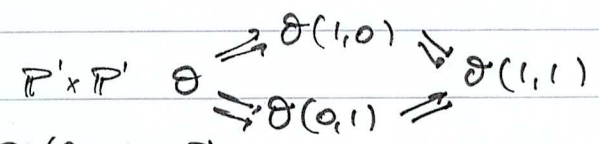
T filtered bundle, vector bundle on X

1)  $T = \bigoplus_{i=1}^t T_i$ ,  $T_i \not\cong T_j \ \forall i \neq j$ ,  $\langle T \rangle = \mathcal{D}^b(X)$

2)  $\text{Ext}^l(T, T) = 0 \ \forall l > 0$ ,  $A = \text{End}(T)$

Theorem (Hille/Parling) X rational surface, then there exists a filtered bundle on X.

Example  $\mathbb{P}^2 \quad \mathcal{O} \cong \mathcal{O}(1) \cong \mathcal{O}(2)$

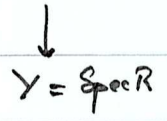


$\mathbb{P}^1 \quad \mathcal{O} \cong \mathcal{O}(P) \xrightarrow{\cong} \mathcal{O}(Q+aP) \cong \mathcal{O}(Q+(a+1)P)$

$P^2 = 0, Q^2 = a$

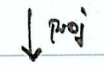
Relative Example  $X \cong \mathcal{O}(-a) \xrightarrow{\pi} \mathbb{P}^1 \quad a > 0 \quad \pi^* T \text{ filtered?}$

$E^2 = -a$



$\text{Ext}^l(\pi^* T, \pi^* T) = 0 \ \forall l > 0?$

$Z = \mathcal{L}^{-1} \xrightarrow{\dots} X \quad \mathcal{L} \text{ ample}$



$Y = \text{Spec } R$

Theorem (Hille/Parling) X toric,  $\mathcal{L}$  ample, then there exists T s.t.  $\pi^* T$  filtered on Z.

$\Phi = \mathbb{R} \text{Hom}(T, -) : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(A)$

§ 2 Hochschild (co)homology

$X \xrightarrow{\Delta} X \times X \quad \mathcal{D}^b(X \times X) \longrightarrow \mathcal{D}^b(\text{Mod } A^{e, \nu}) \quad A^{e, \nu} = A \otimes A^{\text{op}}$

$\mathcal{O}_X \longmapsto A \quad C_{i,j} \quad 1 \leq i, j \leq t$

$\overline{\Phi} = \mathbb{R} \text{Hom}(T^{\vee} \boxtimes T, -)$

Theorem Resolution of the diagonal

$$\begin{array}{c} \Gamma^v \boxtimes C^\bullet \boxtimes \Gamma \longrightarrow \mathcal{O}_\Delta \\ \parallel \\ (\Gamma_i^v \boxtimes \Gamma_j) \xrightarrow{c_{ij}} \end{array}$$

For  $A$ ,  $A_0 = A/\text{rad } A$

$$A \otimes_{A_0} C^\bullet \otimes_{A_0} A \xrightarrow{g_{ij}} A$$

Minimal resolution:  $\bigoplus P(c_{ij})^{c_{ij}}$

Theorem (Buchsatz/Hille)

$$HH^l(X) := \text{Ext}_{X \times X}^l(\mathcal{O}_\Delta, \mathcal{O}_\Delta) = \text{Ext}_{A^{euv}}^l(A, A) = HH^l(A)$$

$$HH_e(X) := H^{-l}(X, \mathbb{L}i^* \mathcal{O}_\Delta) = \text{Tor}_l^{A^{euv}}(A, A) = HH_e(A)$$

Theorem (Buchsatz/Hille)

$$\Gamma^v \otimes C^\bullet \otimes \Gamma \cong \bigoplus \Omega^p[\rho] \quad \Gamma_i^v \otimes \Gamma_j$$

$$H^q(X, \Omega^p) = 0 \quad \forall p \neq q$$

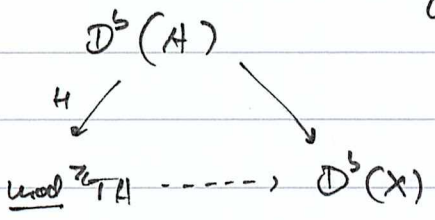
$$V = H^0(\mathcal{O}(1))$$

$$\Lambda^2 V \otimes \mathcal{O}(-2) \otimes \mathcal{O} \xrightarrow{\cong} \omega$$

$$(V \otimes \mathcal{O}(-1) \otimes \mathcal{O})^2 \xrightarrow{\cong} \Omega_{\mathbb{P}^2}^1$$

$$\begin{array}{c} \mathcal{O} \otimes \mathcal{O} \\ \mathcal{O}(1) \otimes \mathcal{O}(1) \\ \mathcal{O}(-2) \otimes \mathcal{O}(2) \\ \parallel \\ \mathcal{O}^{\oplus 3} \end{array}$$

$$H^2(\omega_X)$$



$$\mathcal{O}(-3) \rightarrow \Lambda^2 \mathcal{O}(-2) \rightarrow V \otimes \mathcal{O}(-1) \rightarrow \mathcal{O}$$