Osamu Iyama: TILTING AND CLUSTER TILTING FOR STABLE CATEGORY OF CM MODULES

Workshop "Matrix factorisations"
Bielefeld, 2011/05/07

R local Gorenstein of dim d (e.g. $K[[x_0, x_1, \dots, x_d]]/(f)$)
$\mathrm{MCM}(R)$ cat. of maximal Cohen-Macaulay modules
$\underline{\mathrm{MCM}}(R)$ is triangulated category ($\simeq D^b_{sg}(R)$, $\underline{\mathrm{MF}}(f)$)

$\underline{\mathrm{MCM}}(R) \overset{?}{\longleftrightarrow}$ (rep. theory of f.d. algebras)

Auslander '78: order

- [Auslander] R is an isolated singularity $\Longleftrightarrow$ $\underline{\mathrm{MCM}}(R)$ is Hom-finite
- $\underline{\mathrm{Hom}}_R(X, Y) \simeq D\underline{\mathrm{Hom}}_R(Y, X[d-1])$ AR duality, $(d-1)$-Calabi-Yau

A: fin. dim. K-algebra, gl.dim $A \leq n$
$D^b(A) = \mathrm{perf}\, A \circlearrowleft \nu := - \overset{L}{\underset{A}{\otimes}} DA$ is a Serre functor

$D^b(A)/\nu \circ [-n]$ orbit category (not triangulated), $n \geq 1$
$\mathcal{C}_n(A) =$ triangulated hull $(D^b(A)/\nu \circ [-n])$ <u>n-cluster category</u> of A

- If $\mathcal{C}_n(A)$ is Hom-finite, the $\mathcal{C}_n(A)$ is n-Calabi-Yau [BMRRT, Amiot, Guo]

<u>Q1</u> When is $\underline{\mathrm{MCM}}(R)$ equivalent to $\mathcal{C}_{d-1}(\exists A)$?
Q2 $\underline{\mathrm{MCM}}^{\mathbb{Z}}(R)$ $D^b(\exists A)$?

$R = \bigoplus_{i \geq 0} R_i$ graded

$\mathrm{MCM}^{\mathbb{Z}}(R)$ cat. of graded maximal Cohen-Macaulay modules

$\underline{\mathrm{MCM}}^{\mathbb{Z}}(R)$ triangulated cat

$\underline{\mathrm{Hom}}_R(X, Y) \simeq D\underline{\mathrm{Hom}}_R(Y, X[d-1](a))$ a: Gorenstein parameter

<u>Tilting theory</u>

<u>Def</u> $\mathcal{T}$: triang. cat.

$\mathcal{T} \ni M$ tilting object $\overset{\text{def}}{\iff}$ $\begin{cases} \cdot \ \mathrm{Hom}_{\mathcal{T}}(M, M[i]) = 0 \ \ \forall i \neq 0 \\ \cdot \ \mathrm{thick}(M) = \mathcal{T} \end{cases}$

<u>Ex</u>: $\mathrm{perf}\, A \ni A$ is a tilting obj.

<u>Thm [Keller]</u> $\mathcal{T}$: triang cat, $M \in \mathcal{T}$ tilting obj., $\mathcal{T}$ algebraic

$\exists$ triang. equivalence $\mathcal{T} \xrightarrow{\simeq} \mathrm{perf}\, \mathrm{End}_{\mathcal{T}}(M)$

<u>Ex</u> (1) [Happel] A: fin. dim. alg., gl. dim. $A < \infty$

$TA = \underset{0}{A} \oplus \underset{1}{DA}$ trivial extension, $\underset{\text{tilting object}}{A} \in \underline{\mathrm{mod}}^{\mathbb{Z}}(TA) \simeq D^b(A)$

(2) [Yamaura] $A = \bigoplus_{i \geq 0} A_i$ fin. dim. selfinjective K-algebra

gl. dim $A_0 < \infty \iff \underline{\mathrm{mod}}^{\mathbb{Z}} A$ has a tilting object

(For example, if A is local, then $\bigoplus_{i \geq 0} A_i(i)$ is tilting object)

Cluster tilting theory

Def: $\mathcal{T}$: triang. category $n \geq 1$
$M \in \mathcal{T}$ n-CT $\overset{\text{def}}{\Longleftrightarrow}$ $\operatorname{add} M = \{X \in \mathcal{T} \mid \operatorname{Hom}_{\mathcal{T}}(M, X[i]) = 0 \ \ i = 1 \dots n-1\}$
$= \{ \text{———} (X, M[i]) \text{———} \}$

Ex: • $M \in \mathcal{T}$: 1-CT $\Leftrightarrow$ $\operatorname{add} M = \mathcal{T}$
• $\mathcal{C}_n(A)$ is Hom-finite $\Rightarrow$ $A \in \mathcal{C}_n(A)$ is n-CT

Thm $\operatorname{MCM}(R) \ni M^{\oplus > R}$, $n = d-1$, $\Gamma := \operatorname{End}_R(M)$
$\underline{\operatorname{MCM}}(R) \ni M$ n-CT $\Leftrightarrow \begin{cases} \bullet\ \Gamma \in \operatorname{MCM}(R) \\ \bullet\ \operatorname{gl.dim.} \Gamma = d \end{cases}$

Motivating ex: Thm [Auslander] char $K = 0$, $G \subset SL_2(K)$ finite
$S := K[[x,y]] \circlearrowleft G$, $R := S^G$

(AR quiver of R) $\simeq$ (McKay quiver of G)
$\simeq$ (double of extended Dynkin quiver Q)

$\underline{\operatorname{MCM}}(R) = D^b(KQ)/\tau = \mathcal{C}_1(KQ)$ Q = Dynkin quiver

$S * G = \operatorname{End}_R(S)$, $\operatorname{CM}(R) = \operatorname{add}(S)$

Setting $\Gamma = \bigoplus_{i \geq 0} \Gamma_i$ graded alg, bimodule d-Calabi-Yau alg (e.g. $\Gamma = S*G$) of Gorenstein parameter 1, $\Lambda = \Gamma_0 \ni e$ idempotent
$\operatorname{gl.dim.} \Gamma/(e) < \infty$, $e\Lambda(1-e) = 0$, $\dim_K \Gamma_i < \infty$, $R := e\Gamma e$,
$A := \Lambda/(e)$

$$\begin{array}{ccc} D^b(A) & \xrightarrow{\sim} & \underline{\operatorname{MCM}}^{\mathbb{Z}}(R) \\ \downarrow & & \downarrow \\ \mathcal{C}_{d-1}(A) & \xrightarrow{\sim} & \underline{\operatorname{MCM}}(R) \end{array}$$

Thm [Amiot-Iyama-Reiten]: $\mathcal{C}_{d-1}(A) \simeq \underline{\operatorname{MCM}}(R)$