Triangle singularities I

Kussin (1)

(jt. w. Lenzing & Meltzer)

$\mathbb{X}$ = weighted proj. line triple weight type (p_1, p_2, p_3) $(p_i \geq 2)$

$$S = \frac{k[X_1, X_2, X_3]}{(f)} \qquad f = X_1^{p_1} + X_2^{p_2} + X_3^{p_3} \qquad \mathbb{L}\text{- graded}$$

In $\operatorname{coh}\mathbb{X}$ (in $D^b \operatorname{coh} \mathbb{X}$)

$\exists$ tilting object consisting of line bundles

$$\underline{\operatorname{vect}}\,\mathbb{X} = \frac{\operatorname{vect}\mathbb{X}}{[\mathcal{L}]} \simeq \underline{\mathrm{CM}}^{\mathbb{L}} S \simeq \underline{\mathrm{MF}}^{\mathbb{L}}(f)$$

all line bundles

$$\simeq D^{b\,\mathbb{L}}_{sg}(S)$$

In $\underline{\operatorname{vect}}\,\mathbb{X}$ (indec) rank-2 bundles become important.

In particular Auslander bundles: $L \in \mathcal{L}$

almost split sequence $0 \to L(\vec{\omega}) \to E_L \to L \to 0$

(Rmk: t = # weights. E_L are exceptional in $D^b \operatorname{coh}\mathbb{X}$ $\iff t = 3$.)

THM: Case $(2, 3, p)$ $p \geq 2$ $L \in \mathcal{L}$, $E = E_L$

~~that the~~ then $T := \bigoplus_{\vec{x} \in M} E(\vec{x})$ is a tilting object in $\underline{\operatorname{vect}}\,\mathbb{X}$

$$M = \left\{ a\bar{x}_1 + b\bar{x}_3 \;\middle|\; \begin{array}{l} a = 0, 1; \\ b = 0, \dots, p-2 \end{array} \right\}$$

$$\bar{x}_i := \vec{x}_i + \vec{\omega}$$

$$\underline{\operatorname{End}}(T) = k\left(\underset{1}{\circ} \xrightarrow{x} \underset{2}{\circ} \xrightarrow{x} \cdots \quad \circ \xrightarrow{x} \underset{2(p-1)}{\circ} \right) \Big/ (x^3 = 0) = A_{2(p-1)}^{(3)}$$

In parti.: $\underline{\mathrm{vect}}\,\mathbb{X} \simeq D^b\left(\mathrm{mod}\, \mathbb{A}_{2(p-1)}(3)\right)$ Kussin (2)

THM $(2,3,p)$ $\underline{\mathrm{vect}}\,\mathbb{X} \simeq \underline{\mathcal{S}}^{\mathbb{Z}}(p)$

cat. of triples (V, U, f)

V fin. dim k-vectorspace

$U \subset V$ subsp.

$f \in \mathrm{End}_k(V)$, $f(U) \subseteq U$

$f^p = 0$

$\mathbb{Z}$-graded sense

Ringel-Schmidmeier

$\_$ = stabilize

General triple case

Extension bundles $L \in \mathcal{L}$, $\vec{x} \in \mathbb{L}$, $0 \le \vec{x} \le \vec{x}_{max}$

$$\vec{x}_{max} = \sum_{i=1}^{3} (p_i - 2)\vec{x}_i$$

$$\dim_k \mathrm{Ext}^1\left(L(\vec{x}), L(\vec{\omega})\right) = 1$$

$$0 \to L(\vec{\omega}) \to E_L\langle\vec{x}\rangle \to L(\vec{x}) \to 0$$

Prop: (1) $E \in \mathrm{vect}\,\mathbb{X}$, indec., of rank 2

$\Longrightarrow$ $E \simeq E_L\langle\vec{x}\rangle$, some $L \in \mathcal{L}$, $0 \le \vec{x} \le \vec{x}_{max}$

(2) E exceptional in $D^b(\mathrm{coh}\,\mathbb{X})$

(3) E exceptional in $\underline{\mathrm{vect}}\,\mathbb{X}$

Sketch (1) let $L \in \mathcal{L}$ of max. degree s.t. Kussin ③

$\mathrm{Hom}(L(\vec{\omega}), E) \neq 0$ $\quad 0 \to L(\vec{\omega}) \to E \to C \to 0$

$\uparrow$ bundle

$L(\vec{x})$

THM: $L \in \mathcal{L}$ fixed

$$T := \bigoplus_{0 \le \vec{x} \le \vec{x}_{max}} E_L\langle \vec{x} \rangle \quad \text{tilting object}$$

k **vect** $\mathbb{X}$ write

$$\underline{\mathrm{End}}(T) = kA_{p_1-1} \otimes kA_{p_2-1} \otimes kA_{p_3-1}$$

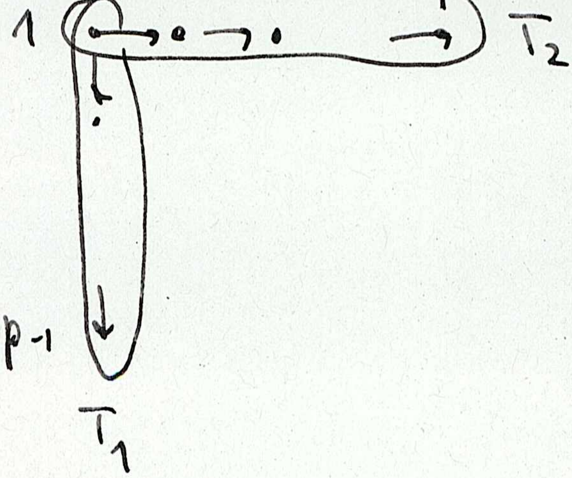
= incidence algebra of poset

$[0, p_1-2] \times [0, p_2-2] \times [0, p_3-2]$

"cuboid" of format $(p_1-1)\times(p_2-1)\times(p_3-1)$

$\underline{\mathrm{vect}}\, \mathbb{X} \simeq D^b(\cdots)$

Special case $(2, p, q)$ "rectangle" $(p-1)\times(q-1)$



T_1, T_2

part. tilt. objects

$S = \tau = [1] =$

Serre functor

$$T' = \bigoplus_{i=0}^{q-2} S^i T_1 , \qquad T'' = \bigoplus_{j=0}^{p-2} S^j T_2$$

tilting objects in $\underline{\text{vect}}\,\mathbb{X}$

$$\underline{\text{End}}(T') = \mathbb{A}_{(p-1)(q-1)}(p) \underset{\text{der.}}{\sim} \underline{\text{End}}(T'') = \mathbb{A}_{(p-1)(q-1)}(q)$$

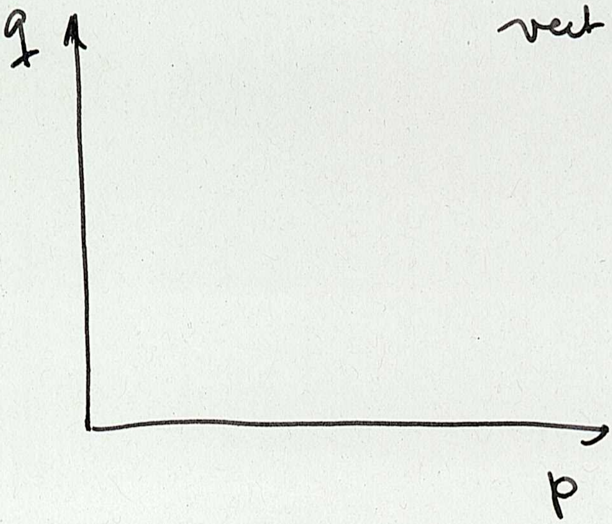
Happel-Seidel symmetry / Lidkoping

ADE-chains

$\underline{\text{vect}}\,\mathbb{X}\ (2,3,p)$ $\qquad p = 2, 3, 4, \ldots$

ADE chain of triangulated CY categories

$\text{vect}\,(2,4,p)$ $\qquad p$



$\text{vect}\,\mathbb{X}\,(2,p,q)$

$(2,3,p)$

p	2	3	4	5	6	7	8
CY-dim	$1/3$	$2/3$	$10/12$	$14/15$	$6/6$	$22/21$	$26/24$
	$\mathbb{A}_2$	$\mathbb{D}_4$	$\mathbb{E}_6$	$\mathbb{E}_8$	$(2,3,6)$	wild type	
		finite			tubular		
	>0				$=0$	<0	

← rep type (up to der. eq.; up to degr. shift)

← Gorenstein param. Euler char.