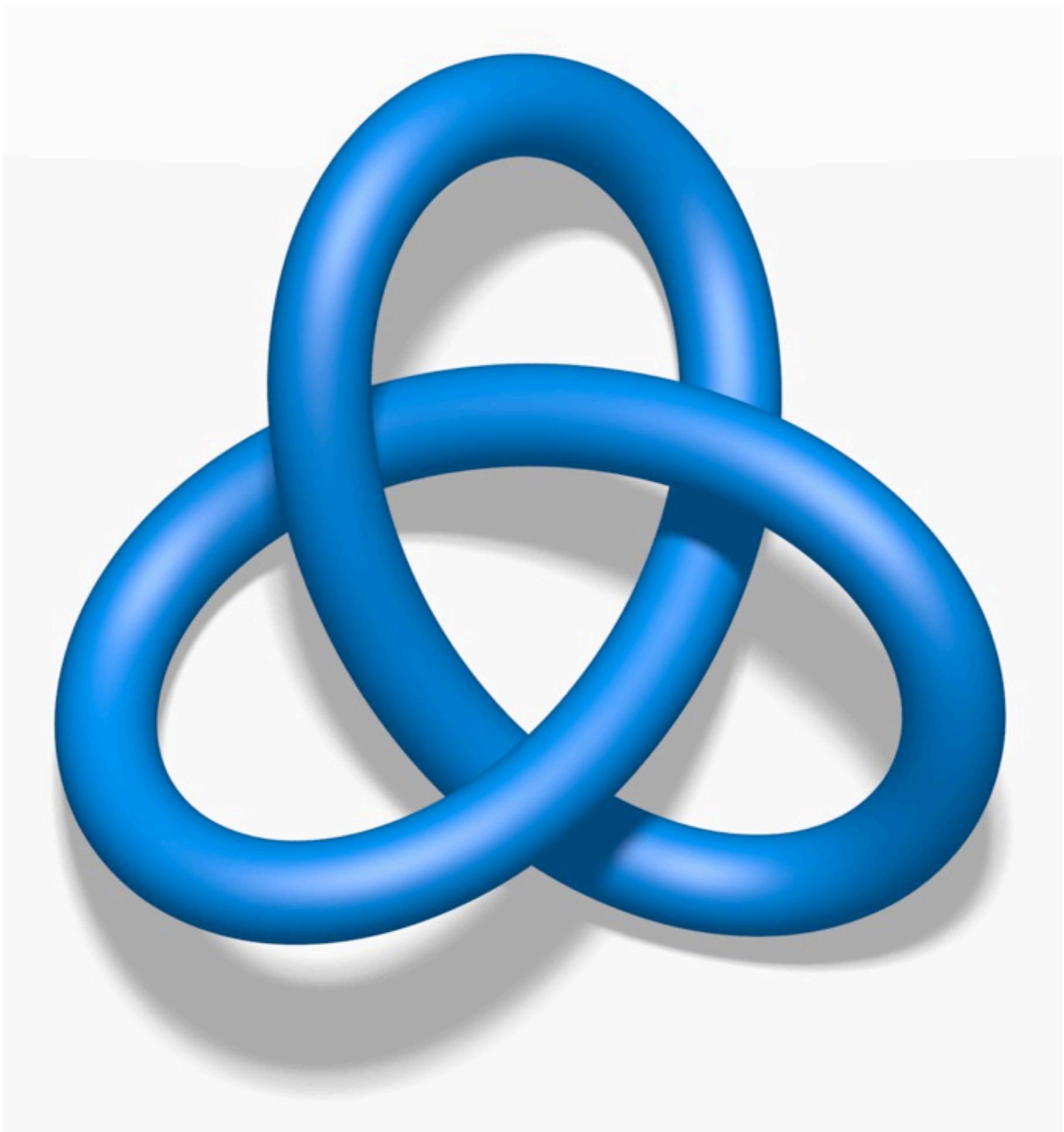


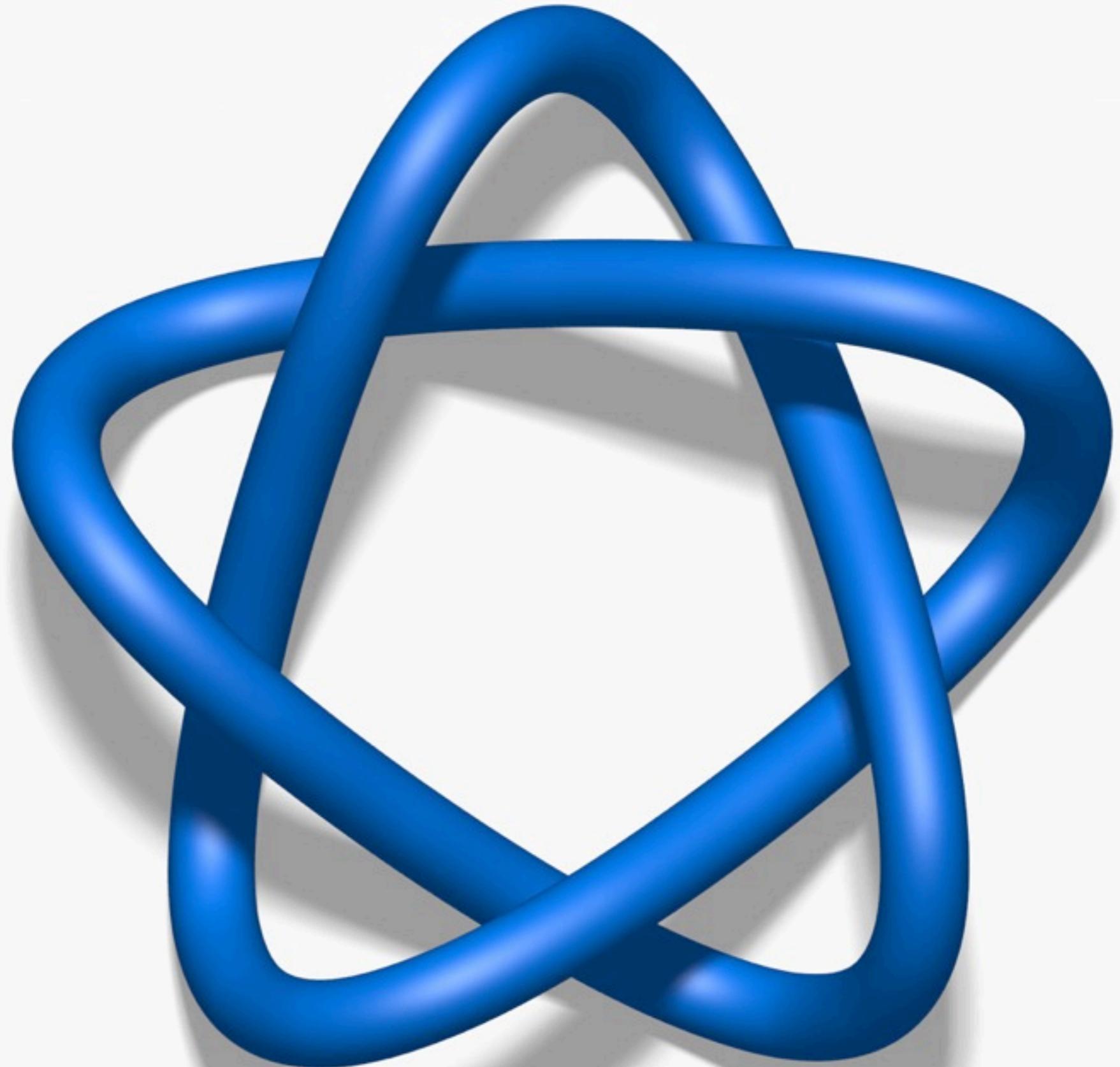
Link homology and convolution

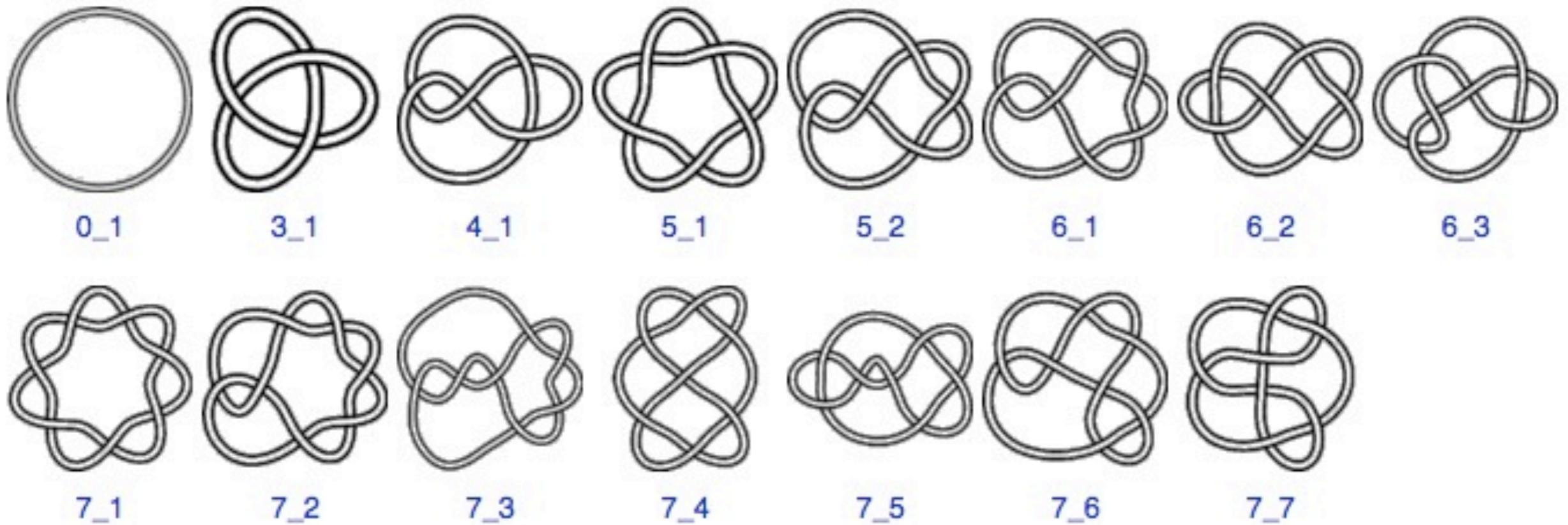
joint with Nils Carqueville

What is a knot?



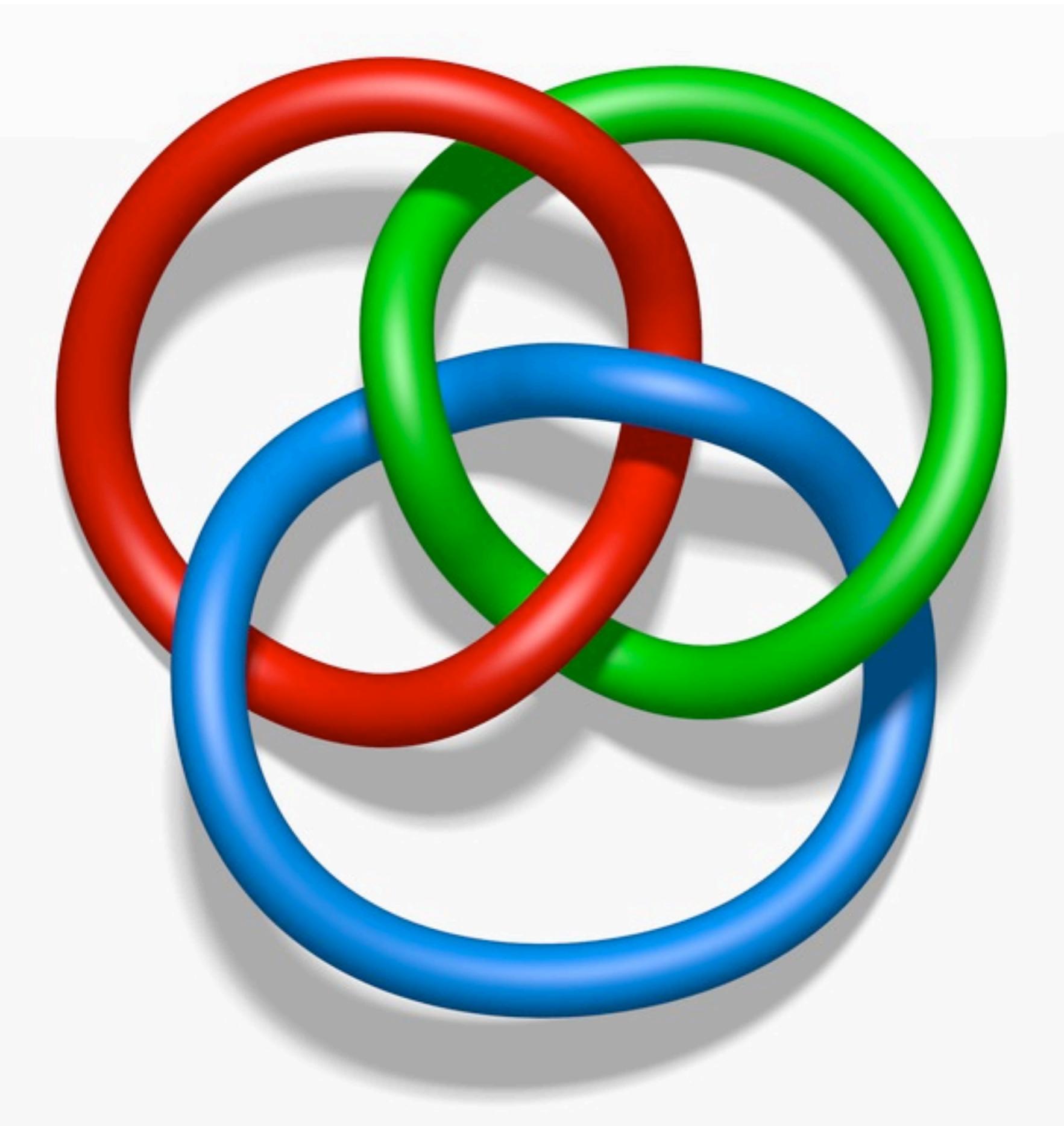


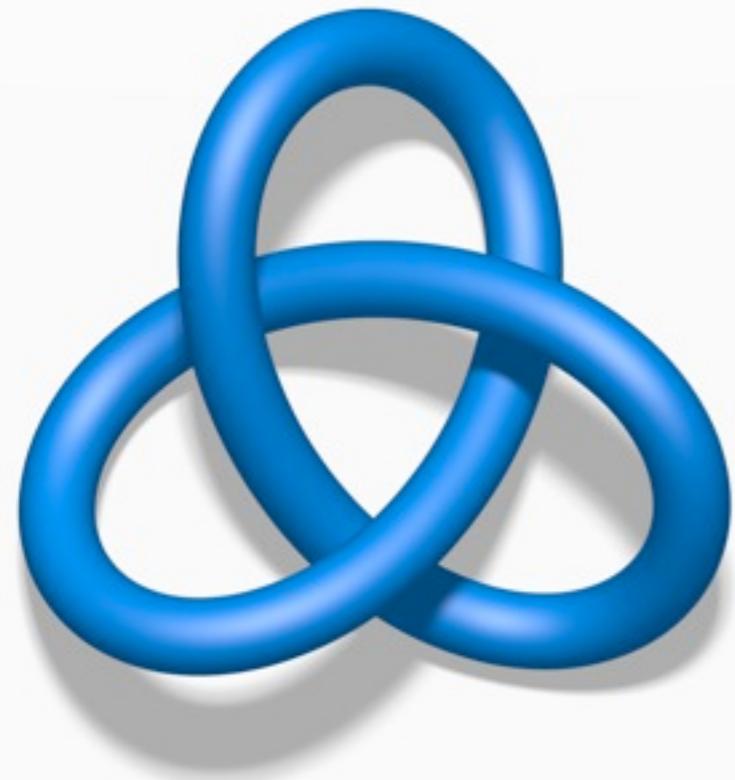




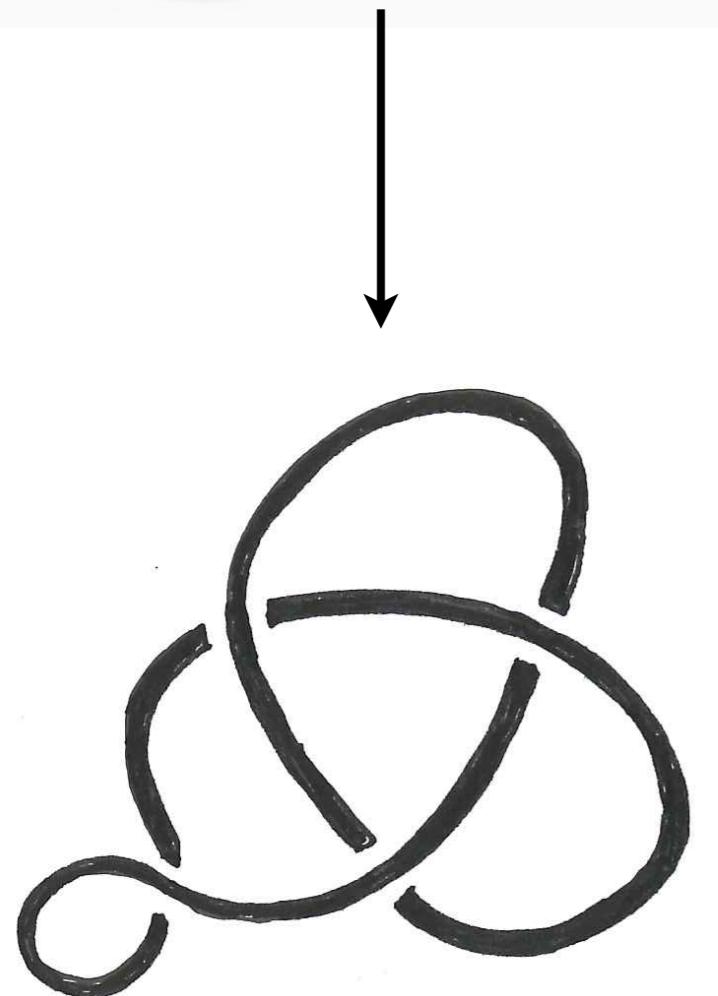
images courtesy of the Knot Atlas

What is a link?





planar diagram



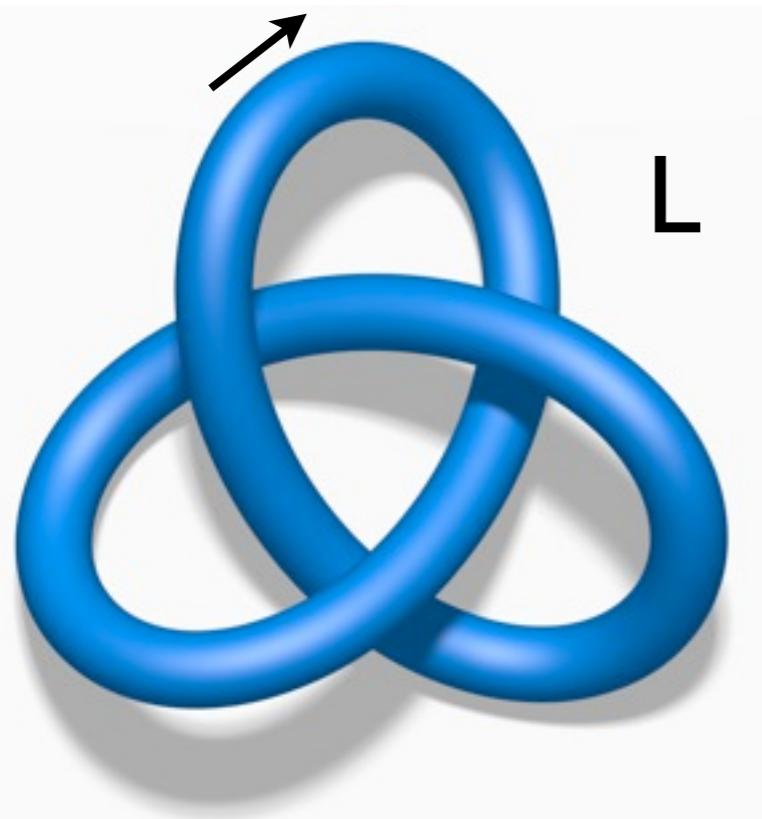
Reidemeister moves

recipe

invariant

Khovanov-Rozansky link homology

Fix an integer $N \geq 2$



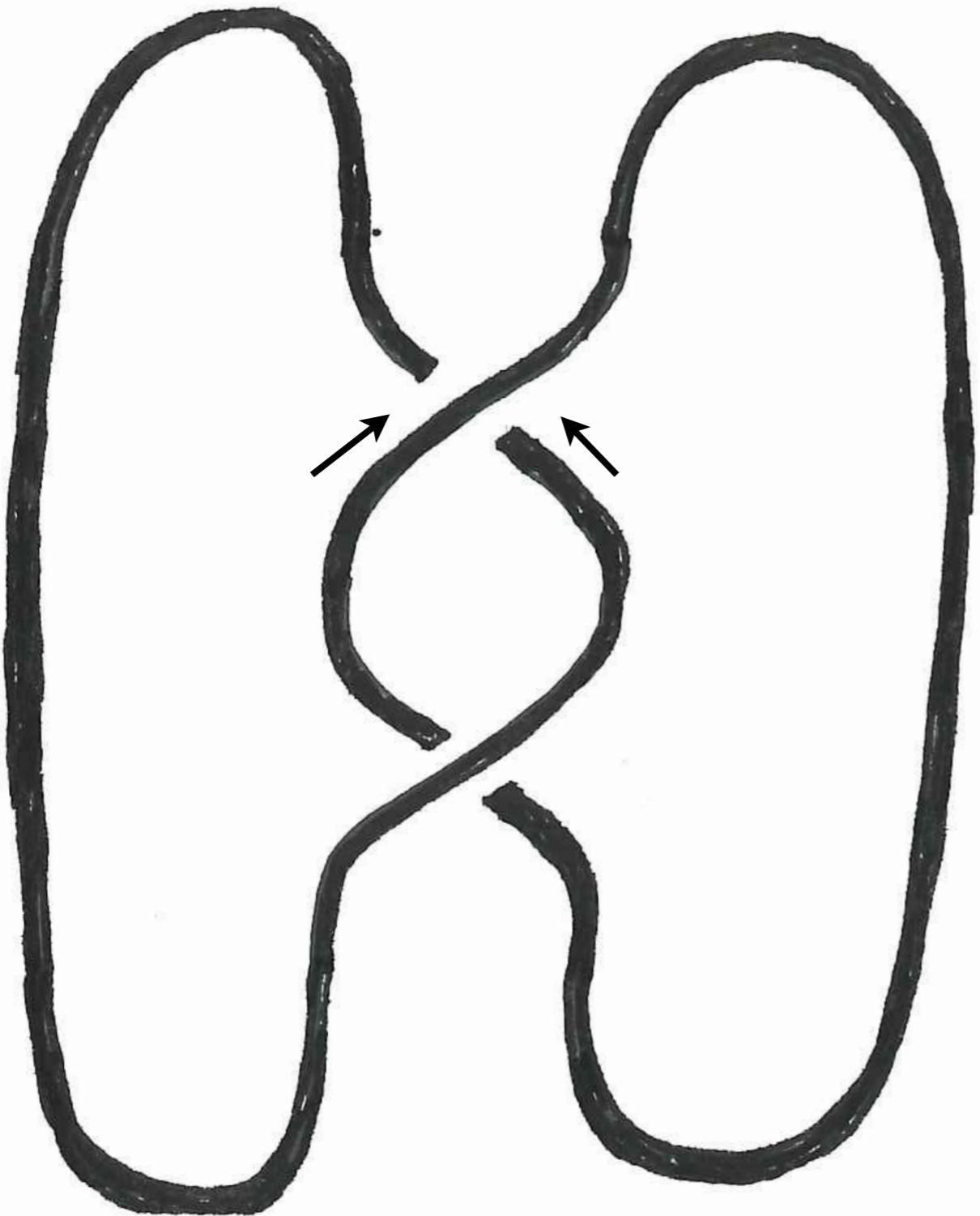
Khovanov-Rozansky '04

$$\longrightarrow \text{KR}_N(L) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}$$
$$\dim_{\mathbb{Q}} H^{i,j} < \infty$$

- Euler characteristic is a specialisation of HOMFLY

$$\chi(\text{KR}_N(L)) := \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim_{\mathbb{Q}}(H^{i,j})$$

- KR homology is functorial
- Constructed using matrix factorisations


$$\text{KR}_N(L)$$

L = Hopf link

\mathcal{T} a triangulated category (MFs)



$\mathcal{T}^{\otimes 2}$

F

$\text{KR}_N(L) = \text{HH}_*(F)$

$\mathcal{T}^{\otimes 2}$

\mathcal{T} a triangulated category (MFs)



$$\mathcal{T}^{\otimes 2}$$

$$c \uparrow$$

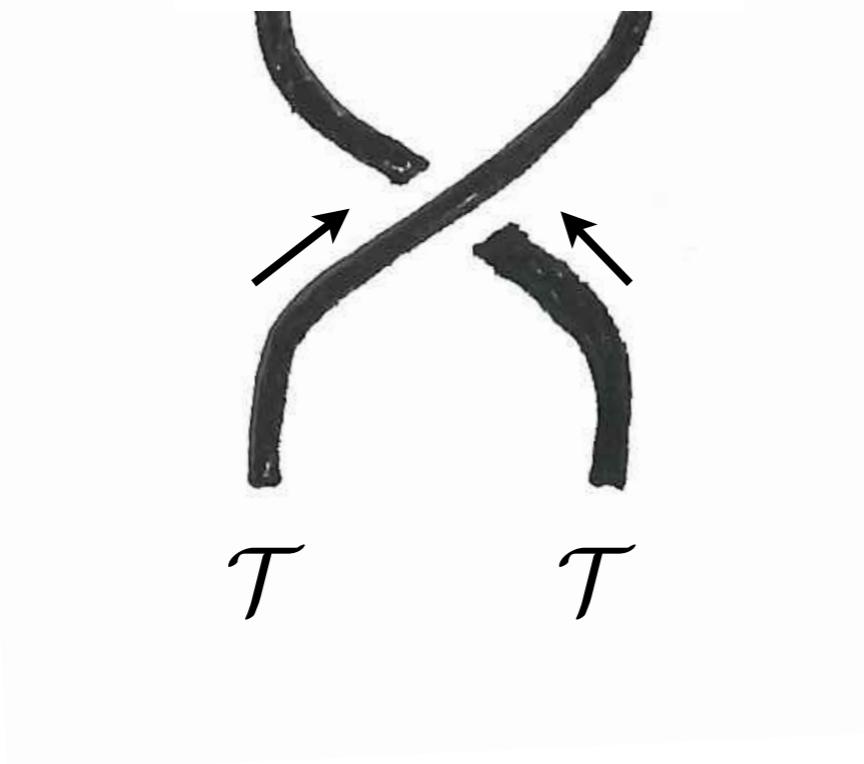
$$\mathcal{T}^{\otimes 2}$$

$$c \uparrow$$

$$\mathcal{T}^{\otimes 2}$$

$$F := c^2$$

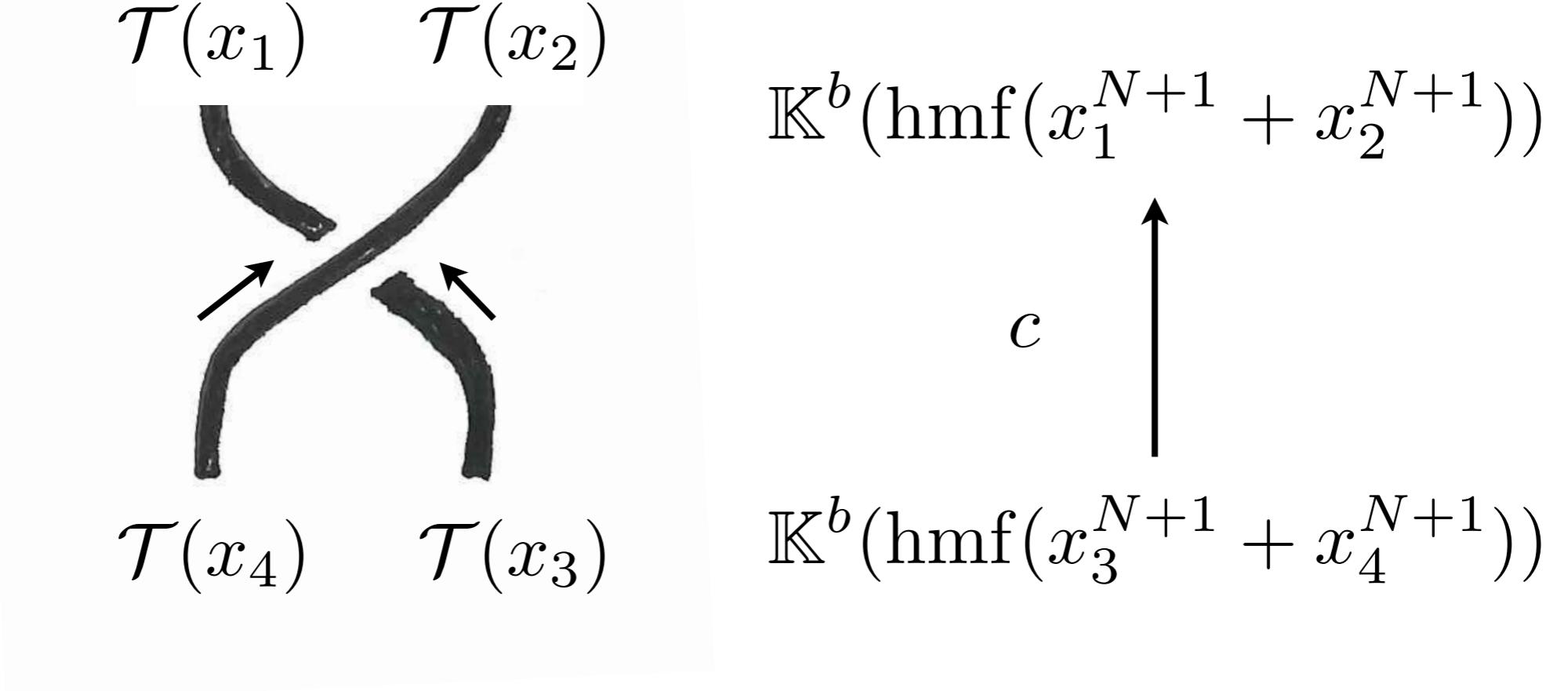
$$\text{KR}_N(L) = \text{HH}_*(F)$$



$$\begin{array}{c} \mathcal{T}^{\otimes 2} \\ \uparrow c \\ \mathcal{T}^{\otimes 2} \end{array}$$

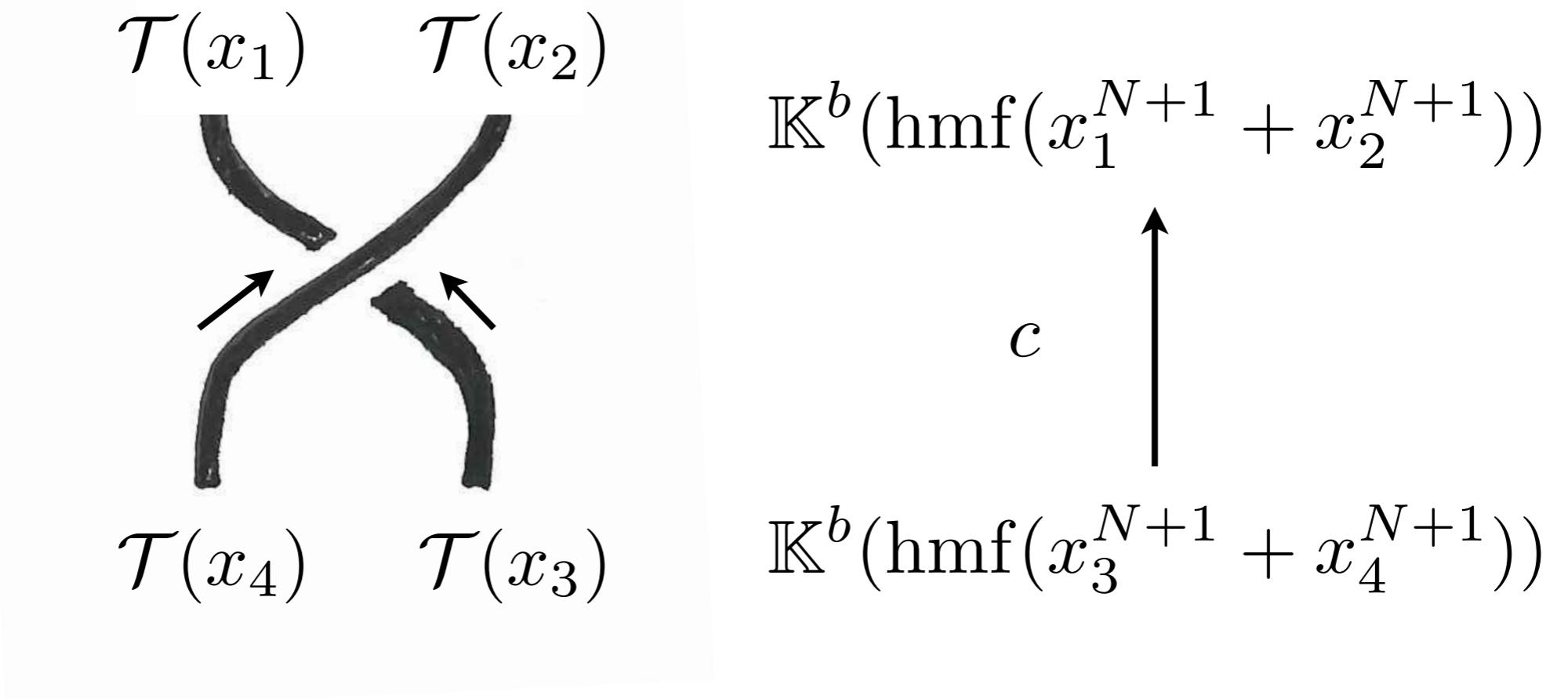
$$\mathcal{T} = \mathcal{T}(x) = \mathbb{K}^b(\mathrm{hmf}(\mathbb{Q}[x], x^{N+1}))$$

$$\mathcal{T}^{\otimes 2} = \mathcal{T}(x, y) = \mathbb{K}^b(\mathrm{hmf}(\mathbb{Q}[x, y], x^{N+1} + y^{N+1}))$$



$$\mathcal{T} = \mathcal{T}(x) = \mathbb{K}^b(\text{hmf}(\mathbb{Q}[x], x^{N+1}))$$

$$\mathcal{T}^{\otimes 2} = \mathcal{T}(x, y) = \mathbb{K}^b(\text{hmf}(\mathbb{Q}[x, y], x^{N+1} + y^{N+1}))$$



$$W = x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1}$$

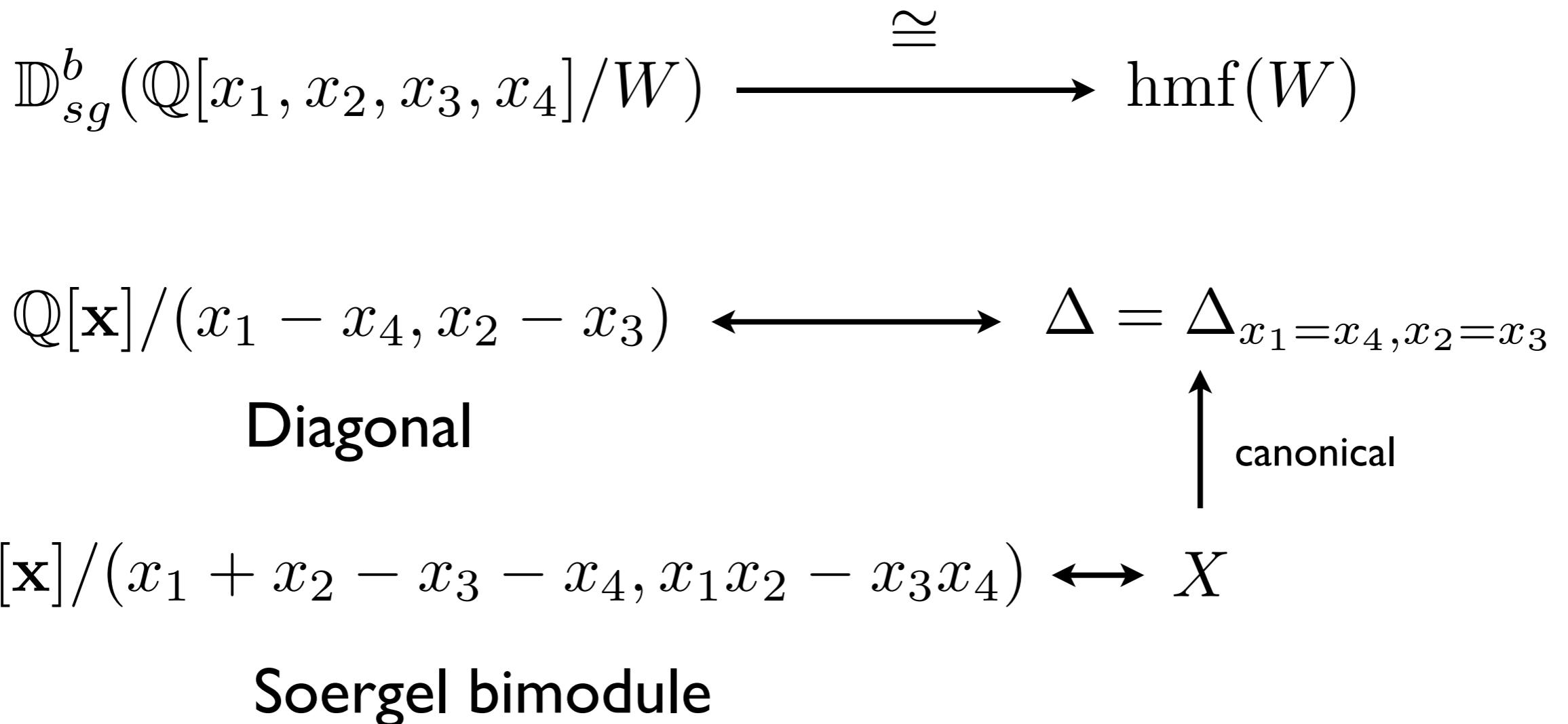
we will define $\Delta, X \in \text{hmf}(W)$

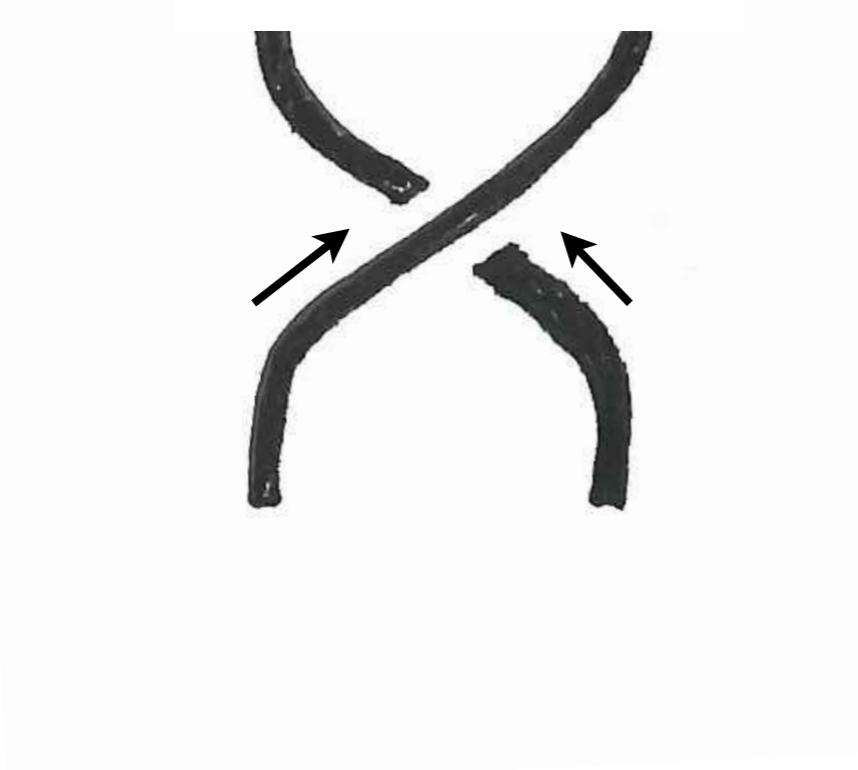
$$(0 \rightarrow X \rightarrow \Delta \rightarrow 0) \in \mathbb{K}^b(\text{hmf}(W))$$

$c = \Phi_{X \rightarrow \Delta}$ is an integral transform

$$\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_1, x_2, x_3, x_4]$$

$$W = x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1}$$





$$\mathbb{K}^b(\text{hmf}(x_1^{N+1} + x_2^{N+1}))$$

$$c \uparrow$$

$$\mathbb{K}^b(\text{hmf}(x_3^{N+1} + x_4^{N+1}))$$

$$W = x_1^{N+1} + x_2^{N+1} - x_3^{N+1} - x_4^{N+1}$$

$$(0 \longrightarrow X \longrightarrow \Delta \longrightarrow 0) \in \mathbb{K}^b(\text{hmf}(W))$$

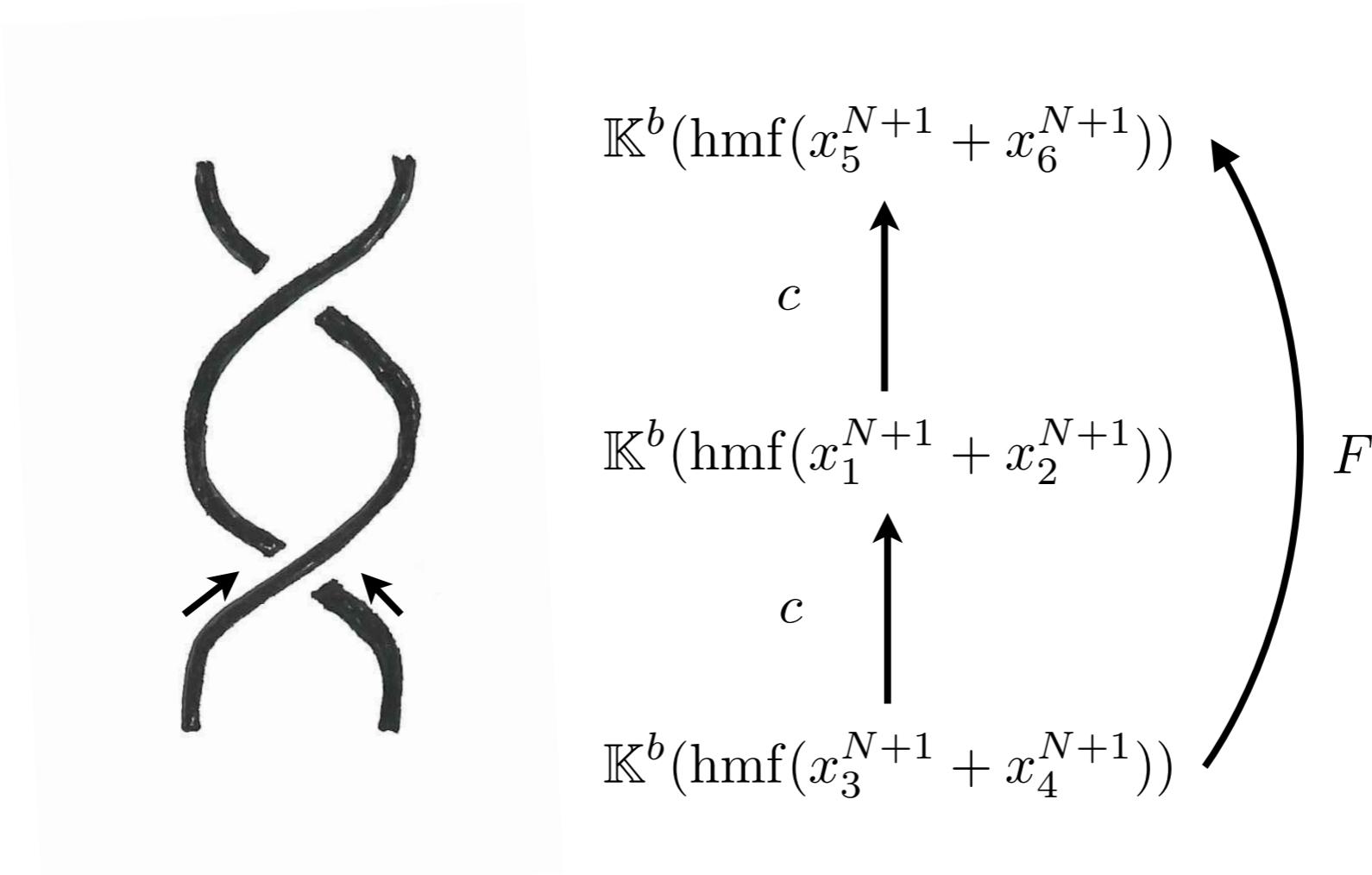
$c = \Phi_{X \rightarrow \Delta}$ is an **integral transform**



$$\begin{array}{c} \mathbb{K}^b(\text{hmf}(x_5^{N+1} + x_6^{N+1})) \\ \uparrow c \\ \mathbb{K}^b(\text{hmf}(x_1^{N+1} + x_2^{N+1})) \\ \uparrow c \\ \mathbb{K}^b(\text{hmf}(x_3^{N+1} + x_4^{N+1})) \end{array}$$

F

A commutative diagram showing three horizontal rows of mathematical expressions. Vertical arrows labeled c connect the middle row to the bottom row, and a curved arrow labeled F connects the top row to the middle row.



$$F = c^2 = \Phi_{X \rightarrow \Delta} \circ \Phi_{X \rightarrow \Delta} = \Phi_K$$

$K := (X \rightarrow \Delta) \otimes (X \rightarrow \Delta)$ is the convolution



$$\mathbb{K}^b(\text{hmf}(x_5^{N+1} + x_6^{N+1}))$$



$$F = \Phi_K$$

$$K = (X \rightarrow \Delta) \otimes (X \rightarrow \Delta)$$

$$\mathbb{K}^b(\text{hmf}(x_3^{N+1} + x_4^{N+1}))$$

$K \otimes \Delta_{x_5=x_4} \otimes \Delta_{x_6=x_3}$ is a complex over $\mathbb{Q}[x_1, \dots, x_6]$

Define: $\text{KR}_N(L) := H^*(K \otimes \Delta_{x_5=x_4} \otimes \Delta_{x_6=x_3})$

Theorem (KR): $\text{KR}_N(L)$ is f.d. over \mathbb{Q}
and is an invariant of links.

Computing Khovanov-Rozansky link homology

The problem

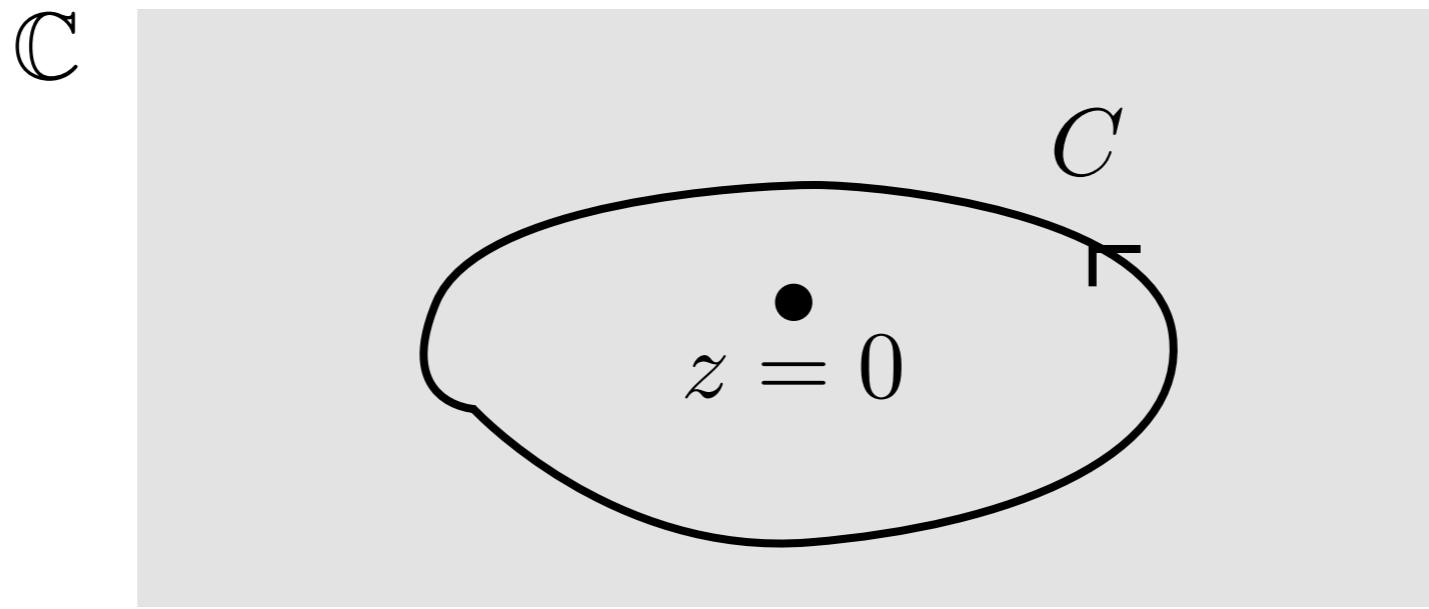
- $K \otimes \Delta_{x_5=x_4} \otimes \Delta_{x_6=x_3}$ is a **cpx** over $\mathbb{Q}[x_1, \dots, x_6]$
- The computer will not enjoy computing H^*
- Monomials x_i^N act null-homotopically, try to exploit this...

A toy example

- (X, D) a $\mathbb{Z}/2$ -graded complex of finite free $\mathbb{C}[z]$ -modules.
- Suppose $\lambda D + D\lambda = z^n \cdot 1_X$, for some homotopy λ .
- Then on $U = \mathbb{C} \setminus \{0\}$,

$$\frac{\lambda}{z^n}D + D\frac{\lambda}{z^n} = 1_X,$$

so X is contractible on U , $H^*(X)$ supported at $\{0\}$.



Contracting homotopy : meromorphic function

$$H^*(X)$$

$$\text{Res}_{z=0}(f)$$

$$\oint_C \frac{\lambda}{z^n} dX$$

$$\frac{1}{2\pi i} \oint_C f(z) dz$$

Theorem (M-Dyckerhoff)

$$\chi(H^*X) = \frac{1}{2\pi i} \oint_C \text{str} \left(\frac{\lambda dX}{z^n} \right) dz \quad dX := \partial_z(D)dz$$

To make sense of integrating without the supertrace:

$$\nabla^0 : \mathbb{C}[z] \longrightarrow \mathbb{C}[z] \otimes_{\mathbb{C}[z^n]} \Omega^1_{\mathbb{C}[z^n]/\mathbb{C}} \quad \text{a connection}$$

horizontal sections $\mathbb{C}[z]/z^n$

Lipman:

$$\frac{1}{2\pi i} \oint_C \frac{g(z)}{z^n} dz = \text{tr} \left(g[z, \nabla^0] \text{ on } \mathbb{C}[z]/z^n \right)$$

$$\frac{1}{2\pi i} \oint_C \frac{g(z)}{z^n} dz = \text{tr} \left(g[z, \nabla^0] \text{ on } \mathbb{C}[z]/z^n \right)$$

$$\nabla^0 : X \longrightarrow X \otimes_{\mathbb{C}[z^n]} \Omega^1_{\mathbb{C}[z^n]/\mathbb{C}}$$

$\oint_C \frac{\lambda}{z^n} dX := \lambda[D, \nabla^0]$ is an idempotent on $X \otimes \mathbb{C}[z]/z^n$

$[D, \nabla^0]$ is the Atiyah class

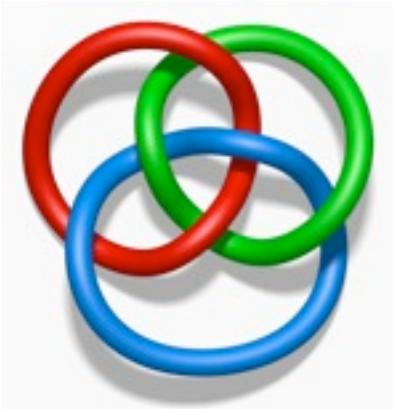
Theorem: $\text{Im}(\lambda[D, \nabla^0]) \cong H^* X$

Computations

- Using Singular to split idempotents
- Poincare polynomial

$$\text{KR}_N(L, q, t) = \sum_{i,j} t^i q^j \dim_{\mathbb{Q}} H^{i,j}$$

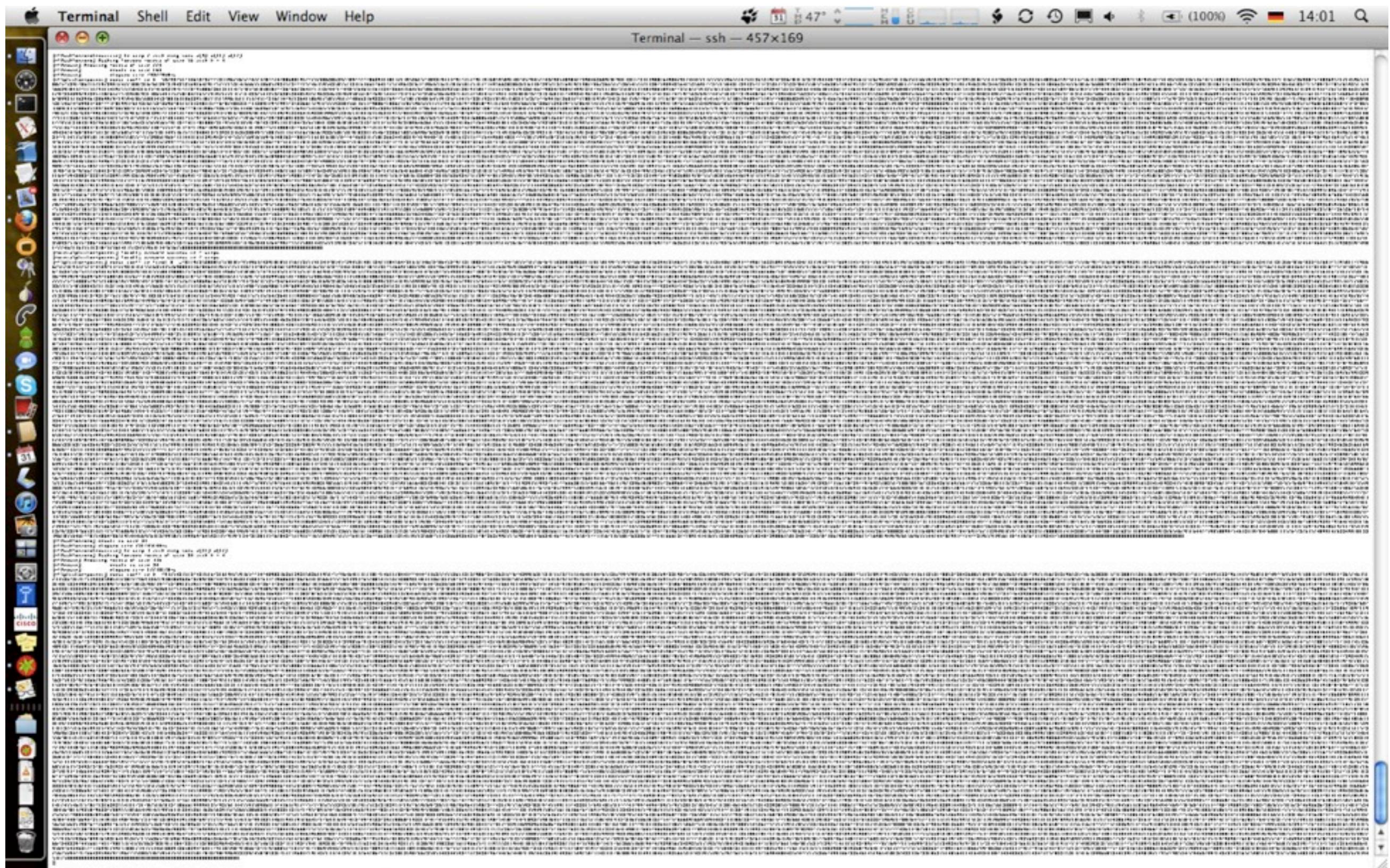
$$\begin{aligned} & q^{-10}t^{-3}(q^{20} + 2q^{18}t + q^{18} + 2q^{16}t + 2q^{14}t^3 + q^{14}t \\ & + 7q^{12}t^3 + 2q^{12}t^2 + q^{12}t + 2q^{10}t^4 + 9q^{10}t^3 + 2q^{10} \\ & + t^2 + q^8t^5 + 2q^8t^4 + 7q^8t^3 + q^6t^5 + 2q^6t^3 \\ & + 2q^4t^5 + q^2t^6 + 2q^2t^5 + t^6) \end{aligned}$$



N=3

```

Terminal — ssh — 228x60
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N = 4 Borromean rings

Terminal Shell Edit View Window Help

44° 13 12 44% (96%) 08:23

Terminal — ssh — 457x169



N = 4 L6n1