Ploog (May 7th, 2011, 11:20 – 11:50)

Kleinian and Fuchsian singularities

① Surface singularities with good $\mathbb{C}^*$-action

S is an affine, 2-dim, normal surface with good $\mathbb{C}^*$-action.

$\parallel$

$\mathrm{Spec}(R)$, $R = \bigoplus_{k \geq 0} R_k$, $R_0 = \mathbb{C}$

Hypersurface case: $S = \{f = 0\} \subseteq \mathbb{C}^3 \rightsquigarrow R = \mathbb{C}[x,y,z]/f$

f is weighted homogeneous

Ex $f_{A_1} = x^2 + y^2 + z^2 \in \mathbb{C}[x,y,z]$ (degrees $1\ 1\ 1$)

$f_{E_6} = x^3 + y^4 + z^2 \in \mathbb{C}[x,y,z]$ (degrees $4\ 3\ 6$)

Want to incorporate $\mathbb{C}^*$-action into singularity categories:

$D^{\mathbb{C}^*}(S) = \mathbb{C}^*$-linearised sheaves

$\parallel$

$$D^b([S/\mathbb{C}^*]) \twoheadrightarrow D_{sg}^{\mathbb{C}^*}(S) = D^{\mathbb{C}^*}(S) / \mathrm{Perf}^{\mathbb{C}^*}(S) \cong \underline{\mathrm{MCM}}^{gr}(R)$$

graded MCM modules

$$R = \mathbb{C}[x,y,z]/f \Rightarrow D_{sg}^{\mathbb{C}^*}(S) \cong \underline{\mathrm{MF}}^{gr}(f)$$

$$\ni \left(F_0 \xrightarrow{\alpha} F_1 \xrightarrow{\beta} F_0 \right)$$

graded, free

α, β homogeneous, $\alpha\beta = \beta\alpha = f$

Thm (Kajiura, Saito, Takahashi, Ueda)

The category $\underline{\mathrm{MF}}^{gr}(f_{ADE}) \cong D^b(\mathbb{C}\vec{Q}_{ADE})$ has a full strongly exceptional collection.

Ex. $f_{E_6} = x^3 + y^4 + z^2$ as above

$$
\begin{array}{c} R(-6) \\ \oplus \\ R(-11) \\ \oplus \\ R(-8) \\ \oplus \\ R(-9) \end{array}
\xrightarrow[\alpha]{\begin{pmatrix} -z & 0 & x^2 & y^3 \\ 0 & -z & y & -x \\ x & y^3 & z & 0 \\ y & x^2 & 0 & z \end{pmatrix}}
\begin{array}{c} R \\ \oplus \\ R(-5) \\ \oplus \\ R(-2) \\ \oplus \\ R(-3) \end{array}
$$

$\beta = \alpha$

Gorenstein Parameter: $a = \deg(f) - \deg(x) - \deg(y) - \deg(z)$

$a_{ADE} = 12 - 3 - 4 - 6 = -1$

② Fuchsian singularities

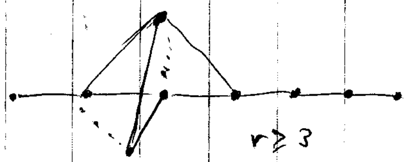
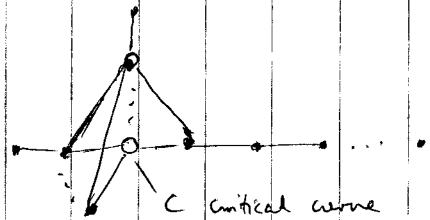
These are hyperbolic analogs of Kleinians:

$S_{ADE} = \mathbb{C}^2/G$, $G \subseteq SU(2)$ finite

$R_{ADE} = \bigoplus_{k=0}^{\infty} H^0(\mathbb{P}^1, T_{\mathbb{P}^1}^{\otimes k})^T$, $T \subseteq PSU(2)$

$R_{Fuchs} = \bigoplus_{k=0}^{\infty} H^0(\mathbb{H}, T_{\mathbb{H}}^{\times \otimes k})^T$, $T \subseteq PSU(1,1)$ (cocompact)

- not always complete intersection
- $a = 1$
- contain 14 exc. unimodular sing. of Arnold's strange duality
- genus $g = g(\mathbb{H}/T)$ ← Riemann surface

	Klein	Fuchs $g=0$	Fuchs $g>0$
a	-1	1	1
Coxeter-Dynkin	 ADE $r\leq 3$	 $r\geq 3$	 C critical curve $g(C)=g>0$, $C^2=2-2g\neq -2$
$\mathcal{D}^{\mathbb{C}^*}(S)$	$\mathcal{D}^b(\mathbb{C}\vec{Q}_{ADE})$	$\mathcal{D}^b(\mathbb{C}[P]) = \langle E_0, \mathcal{D}^b(\mathbb{X}_{g=0})\rangle$ full str. exc. coll.	$\langle E_0, \mathcal{D}^b(\mathbb{X}_{g>0})\rangle$
$\mathcal{T}_{res}$	$Y \to \mathbb{C}^2/G$ $\mathcal{D}^b_E(Y) \cap \mathcal{O}_Y^{\perp}$		