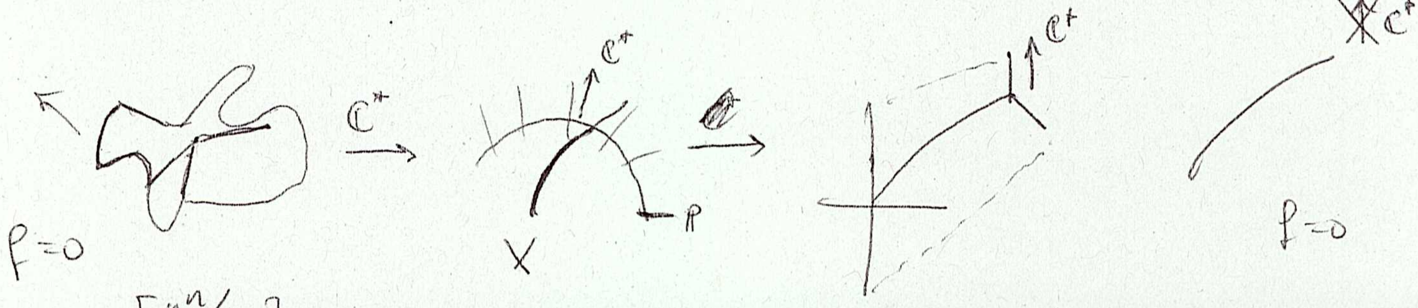
SHIPMAN

Geometric approach to Orlov's Thm

Goal: soft (geometric) pf, Fix $f \in \mathbb{C}[x_1, \ldots, x_n]$ homogeneous, $\deg f = d$, defining sm. hypersurface X

orbifold   resolution   normal bundle   hypersurface



$[\mathbb{A}^n/\mu_d]$ $\mathrm{tot}\,\mathcal{O}(-d)$ (f linear on fibres.)   General $f^{-1}a$

orbifold $\longleftrightarrow$ resolution $\longleftrightarrow$ normal bundle $\longleftrightarrow$ hypersurface

Segal (many others)   def. to normal cone   fibrewise Knörrer periodicity

data defines a dg cat of graded MF's.
- space
- scaling
- $\mathbb{C}^*$-action

Def'n (even) A Landau-Ginzburg model is a ~~pair~~ scheme or stack X + $\mathbb{C}^*$-action where $\mu_2 \subseteq \mathbb{C}^*$ acts trivially and a semi invariant fn $f \in \Gamma(\mathcal{O}_X)$ of weight 2.

LG pair $\rightsquigarrow$ $MF^{gr}(X, f)$

Ex (1) X w/ trivial action, $f=0$ $\rightsquigarrow$ perf(X)

2) Y scheme V/Y vector bundle $s \in \Gamma(Y, V^\vee)$

X = total space of V, w/ square scaling along fibres and f induced by s.

Def'n (X, f) LG-pair. A graded matrix fact. is a $\mathbb{C}^*$-equivariant vector bundle $\mathcal{E}$, with $d: \mathcal{E} \to \mathcal{E}$ degree 1 with $d^2 = f$.

They form a dg-category where

$\mathrm{Hom}((\mathcal{E}, d_\mathcal{E}), (\mathcal{F}, d_\mathcal{F})) = \check{C}(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$, the Čech complex.

To ~~get back~~ relate to usual matrix facts. work on orbifold

$P \xrightarrow{\alpha} Q \xrightarrow{\beta} P$, α, β homogeneous, P, Q graded free / $\mathbb{C}[\underline{x}]$.

$\tilde{X} = \mathrm{Spec}\, \mathbb{C}[x_1, \dots, x_n, p, p^{-1}] = \mathbb{A}^n \times \mathbb{G}_m$

$|x_i|_{hom} = 0$, $|p|_{hom} = 2$ $\qquad |fp|_{hom} = 2$

$T = \mathbb{C}^*$, $|x_i|_{str} = 1$, $|p|_{str} = -d$ $\qquad |fp|_{str} = 0$

$f_p \in \Gamma([\tilde{X}/T])$.

$$P \xrightarrow{\alpha} Q \xrightarrow{\beta} P(d) \xrightarrow{\alpha(d)} Q(d) \to \cdots$$

$$\updownarrow$$

$$((P\oplus Q)\otimes \mathbb{C}[p,p^{-1}],\ d = \alpha + \beta p)$$

Knörrer periodicity.

$$\begin{tikzcd} & V\oplus V^\vee = X \arrow[dl] \arrow[dr] \arrow[dd, "p"] & \\ V \arrow[dr, "p_1"'] & & V^\vee \arrow[dl, "p_2"] \\ & Y & \end{tikzcd}$$

Y scheme, V vector bundle

$\mathbb{C}^*$ action square scaling on $V^\vee$, trivial on V i.e. take

$$\begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \chi_2 & & \\ & & & & \ddots & \\ 0 & & & & & \chi_2 \end{pmatrix} \text{ on } \mathbb{C}[\underbrace{x_1,\dots,x_r}_{V},\underbrace{y_1,\dots,y_r}_{V^\vee}]$$

χ_2: character $p \mapsto p^2$

$$f = \mathrm{Id} \in V\otimes V^\vee \subseteq \mathrm{Sym}^2(V\oplus V^\vee) \subseteq p_*\mathcal{O}_X$$

get from $f : \mathcal{O}_Y \to V$ section on V, cosection on $V^\vee$

i.e. $p_1^* f$, $p_2^* f^\vee$

canonical object $\quad p^*V \underset{s_1}{\overset{s_2}{\rightleftarrows}} \mathcal{O}_X$

$S = \mathrm{Koszul}\left(p^*V(1) \underset{s_1}{\overset{s_2}{\rightleftarrows}} \mathcal{O}_X\right)$ locally $\simeq \bigotimes_{i=1}^{k} \{x_i, y_i\}$.

Thm

$$P^\bullet \mapsto S \otimes p^* P \text{ defines an equivalence}$$

$$\mathrm{Perf}(Y) \xrightarrow{\sim} MF(X, f).$$

Deformation to the normal cone.

As in (2) $\begin{array}{c} V = X \\ \downarrow \\ Y \end{array}$, $s \in \Gamma(Y, V) \rightsquigarrow f \in \Gamma(X, \mathcal{O}_X)$

$Z = Z(s)$ zero scheme of s embedded along zero section

Exists deformation

$$\begin{array}{ccc} \tilde{X} & \longleftarrow & N_{Z/X} \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longleftarrow & 0 \end{array}$$

generic fibre X,

+ $\exists\ \tilde{f}$ on $\tilde{X}$ st. $\tilde{f}$ restricts to f on X, f_{can} on $N_{Z/X}$.

$\Updownarrow$

get Koszul objects associated to these, ie. one from f_{can} deforms. and get equivalence by passing through the deformation.