# Michael Wemyss: MAXIMAL MODIFICATION ALGEBRA AND NON-ISOLATED AR DUALITY

Workshop on "Matrix factorisations"
Bielefeld, 2011/05/07

<u>Input:</u> $R$ 3d Gorenstein normal domain / $\mathbb{C}$
<u>not</u> local <u>not</u> isolated

$\rightsquigarrow$

look inside $\operatorname{refl} R \supseteq \operatorname{CM} R$ for certain modifying / max modifying modules

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pass to $\operatorname{End}_R(M)$

$\swarrow$ $\searrow$

possibly singular

$$Y \to \operatorname{Spec} R$$

study certain crepant modifications via moduli spaces

describe derived categories $D^b(Y) \simeq D^b(\operatorname{End}_R M)$

<u>Note</u> in this level of generality <u>$\operatorname{CM} R$</u> is Hom-infinite, not KS, not 2-CY

<u>Plan:</u> ① <u>CM</u> $\operatorname{End}_R(M)$ (in particular $\underline{\operatorname{CM} R}$) has some form of duality
② <u>CM</u> $\operatorname{End}_R(M)$ measures $\mathbb{Q}$-~~functoriality~~ factoriality

① $R$ $d$-dimensional CM ring with a canonical module $\omega_R$ such that $\dim R_m = \dim R$ $\forall m \in \operatorname{Max} R$ (e.g. normal domain $/ \mathbb{C}$) "equicodimensional"

If $R$ is isolated sing, then AR duality is

$$\underline{\operatorname{Hom}}_R(X,Y) \simeq D\operatorname{Ext}^1(Y, \tau X) \qquad \forall X,Y \in \operatorname{CM} R$$

$D =$ dual, $\tau = \operatorname{Hom}_R(\Omega^d \operatorname{Tr}(-), \omega_R)$

($\overset{R}{\underset{\text{Gorenstein}}{\Rightarrow}}$ $\underline{\operatorname{CM} R}$ is $(d-1)$-CY)

If $M \in \operatorname{mod} R$, denote $\operatorname{fl} M$ to be maximal finite length submodule of $M$

<u>Thm</u>: $R$ $d$-dim., CM, $\exists$ canonical module, $\dim(\operatorname{Sing} R) \leq 1$. Then

$$\operatorname{fl} \underline{\operatorname{Hom}}_R(X,Y) \simeq D(\operatorname{fl} \operatorname{Ext}^1(Y, \tau X))$$

<u>Corollary</u>: $R$ 3d Gorenstein normal domain $/ \mathbb{C}$

$$\operatorname{fl} \underline{\operatorname{Hom}}(X,Y) \simeq D(\operatorname{fl} \underline{\operatorname{Hom}}_R(Y, X[2]))$$

"almost 2-CY"

Rest of talk: $R$ is 3d Gorenstein normal domain $/ \mathbb{C}$

<u>Def</u> • $M \in \operatorname{refl} R$ is called modifying if $\operatorname{End}_R(M) \in \operatorname{CM} R$
($\Leftrightarrow \operatorname{fl} \operatorname{Ext}^1_R(M,M) = 0 \Leftrightarrow \operatorname{depth}_{R_m} \operatorname{Ext}^1_R(M,M) > 0$ $\forall m \in \operatorname{Max} R$)

• $M \in \operatorname{refl} R$ is called <u>max modifying</u> if it is modifying & maximal with respect to the property, i.e., $\operatorname{End}_R(M \oplus \underset{\in \operatorname{refl} R}{Y}) \in \operatorname{CM} R \Rightarrow Y \in \operatorname{add} M$

(equiv. $\operatorname{add} M = \{Y \in \operatorname{refl} R \mid \operatorname{End}_R(M \oplus Y) \in \operatorname{CM} R\}$)

We call $\operatorname{End}_R(M)$ a MMA.

<u>Note</u>: NCCR $\Rightarrow$ MMA, but converse in general is false

<u>Recall:</u> $X$ normal variety / $\mathbb{C}$, then

- $X$ is $\mathbb{Q}$-factorial if, for every Weil divisor $\exists\, n \in \mathbb{N}$ s.t. $nD$ is Cartier
- $X$ is complete locally $\mathbb{Q}$-factorial if $\hat{\mathcal{O}}_{X,x}$ is $\mathbb{Q}$-factorial $\forall x \in X$

<u>Note</u> $\mathbb{Q}$-factorial $\Rightarrow$ complete locally $\mathbb{Q}$-factorial
(if smooth $\Rightarrow$ complete local regular)

<u>Thm</u> Suppose $X$ is normal 3d / $\mathbb{C}$, with terminal Gorenstein sings. Then
(a) $X$ is $\mathbb{Q}$-factorial $\Rightarrow$ $D^b_{sg}(X)$ has no rigid <u>objects</u>
(b) $X$ is complete locally $\mathbb{Q}$-factorial $\Leftrightarrow$ $D^b_{sg}(X)$ has no rigid obj.

<u>Corollary</u>: If further $D^b(X) \simeq D^b(\overbrace{\mathrm{End}_R M}^{\Lambda})$, then
(a) $X$ $\mathbb{Q}$-factorial terminal $\Rightarrow$ $\Lambda$ is MMA
(b) $R$ complete local, $\Lambda$ is MMA $\Leftrightarrow$ $X$ is $\mathbb{Q}$-factorial