On the Milnor K-Groups of Complete Discrete Valuation Fields

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Abstract. For a discrete valuation field K, the unit group $K^\times$ of K has a natural decreasing filtration with respect to the valuation, and the graded quotients of this filtration are given in terms of the residue field. The Milnor K-group $K_q^M(K)$ is a generalization of the unit group, and it also has a natural decreasing filtration. However, if K is of mixed characteristics and has an absolute ramification index greater than one, the graded quotients of this filtration are not yet known except in some special cases.

The aim of this paper is to determine them when K is absolutely tamely ramified discrete valuation field of mixed characteristics $(0, p > 2)$ with possibly imperfect residue field.

Furthermore, we determine the kernel of the Kurihara's K_q^M-exponential homomorphism from the differential module to the Milnor K-group for such a field.

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1 Introduction

For a ring R, the Milnor K-group of R is defined as follows. We denote the unit group of R by $R^\times$. Let $J(R)$ be the subgroup of the q-fold tensor product of $R^\times$ over$\mathbb{Z}$ generated by the elements $a_1 \otimes \cdots \otimes a_q$, where $a_1, \ldots, a_q$ are elements of $R^\times$ such that $a_i + a_j = 0$ or 1 for some $i \neq j$. Define

$$K_q^M(R) = (R^\times \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^\times)/J(R).$$

We denotes the image of $a_1 \otimes \cdots \otimes a_q$ by $\{a_1, \ldots, a_q\}$.

Now we assume K is a discrete valuation field. Let v_K be the normalized valuation of K. Let $\mathcal{O}_K$, F and $\mathfrak{m}_K$ be the valuation ring, the residue field and the valuation ideal of K, respectively. There is a natural filtration on $K^\times$ defined by

$$U_K^i = \begin{cases} \mathcal{O}_K^\times & \text{for } i = 0 \\ 1 + \mathfrak{m}_K^i & \text{for } i \geq 1. \end{cases}$$

We know that the graded quotients U_K^i / U_K^{i+1} are isomorphic to $F^\times$ if $i = 0$ and F if $i \geq 1$. Similarly, there is a natural filtration on $K_q^M(K)$ defined by

$$U^i K_q^M(K) = \left\{ \{x_1, \ldots, x_q\} \in K_q^M(K) \;\middle|\; x_1 \in U_K^i,\ x_2, \ldots, x_q \in K^\times \right\}.$$

Let $\operatorname{gr}^i K_q^M(K) = U^i K_q^M(K) / U^{i+1} K_q^M(K)$ for $i \geq 0$. $\operatorname{gr}^i K_q^M(K)$ are determined in the case that the characteristics of K and F are both equal to 0 in [5], and in the case that they are both nonzero in [2] and [9]. If K is of mixed characteristics $(0, p)$, $\operatorname{gr}^i K_q^M(K)$ is determined in [3] in the range $0 \leq i \leq e_K p/(p-1)$, where $e_K = v_K(p)$. However, $\operatorname{gr}^i K_q^M(K)$ still remains mysterious for $i > ep/(p-1)$. In [16], Kurihara determined $\operatorname{gr}^i K_q^M(K)$ for all i if K is absolutely unramified, i.e., $v_K(p) = 1$. In [13] and [19], $\operatorname{gr}^i K_q^M(K)$ is determined for some K with absolute ramification index greater than one.

The purpose of this paper is to determine $\operatorname{gr}^i K_q^M(K)$ for all i and a discrete valuation field K of mixed characteristics $(0, p)$, where p is an odd prime and $p \nmid e_K$. We do not assume F to be perfect. Note that the graded quotient $\operatorname{gr}^i K_q^M(K)$ is equal to $\operatorname{gr}^i K_q^M(\hat{K})$, where $\hat{K}$ is the completion of K with respect to the valuation, thus we may assume that K is complete under the valuation.

Let F be a field of positive characteristic. Let $\Omega_F^1 = \Omega_{F/\mathbb{Z}}^1$ be the module of absolute differentials and Ω_F^q the q-th exterior power of Ω_F^1 over F. As in [7], we define the following subgroups of Ω_F^q. $Z_1^q = Z_1 \Omega_F^q$ denotes the kernel of $d : \Omega_F^q \to \Omega_F^{q+1}$ and $B_1^q = B_1 \Omega_F^q$ denotes the image of $d : \Omega_F^{q-1} \to \Omega_F^q$. Then there is an exact sequence

$$0 \longrightarrow B_1^q \longrightarrow Z_1^q \xrightarrow{\mathrm{C}} \Omega_F^q \longrightarrow 0,$$

where C is the Cartier operator defined by

$$\begin{aligned} & x^p \frac{dy_1}{y_1} \wedge \ldots \frac{dy_q}{y_q} \longmapsto x \frac{dy_1}{y_1} \wedge \ldots \frac{dy_q}{y_q}, \\ & B_1^q \to 0. \end{aligned}$$

The inverse of C induces the isomorphism

$$\begin{aligned} \mathrm{C}^{-1} : \Omega_F^q & \xrightarrow{\cong} Z_1^q / B_1^q \\ x \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_q}{y_q} & \longmapsto x^p \frac{dy_1}{y_1} \wedge \ldots \wedge \frac{dy_q}{y_q} \end{aligned} \tag{1}$$

for $x \in F$ and $y_1, \dots, y_q \in F^\times$. For $i \geq 2$, let $B_i^q = B_i\Omega_F^q$ (resp. $Z_i^q = Z_i\Omega_F^q$) be the subgroup of Ω_F^q defined inductively by

$$\begin{aligned} &B_i^q \supset B_{i-1}^q, \ \mathrm{C}^{-1} : B_{i-1}^q \xrightarrow{\cong} B_i^q/B_1^q \\ &(\text{resp. } Z_i^q \subset Z_{i-1}^q, \ \mathrm{C}^{-1} : Z_{i-1}^q \xrightarrow{\cong} Z_i^q/B_1^q). \end{aligned}$$

Let Z_∞^q be the intersection of all Z_i^q for $i \geq 1$. We denote $Z_i^q = \Omega_F^q$ for $i \leq 0$.

The main result of this paper is the following

Theorem 1.1. *Let K be a discrete valuation field of characteristic zero, and F the residue field of K. Assume that $p = \mathrm{char}(F)$ is an odd prime and $e = e_K = v_K(p)$ is prime to p. For $i > ep/(p-1)$, let n be the maximal integer which satisfies $i - ne \geq e/(p-1)$ and let $s = v_p(i - ne)$, where v_p is the p-adic order. Then*

$$\mathrm{gr}^i\, K_q^M(K) \cong \Omega_F^{q-1}/B_{s+n}^{q-1}.$$

Corollary 1.2. *Let $U^i(K_q^M(K)/p^m)$ be the image of $U^iK_q^M(K)$ in $K_q^M(K)/p^mK_q^M(K)$ for $m \geq 1$ and $\mathrm{gr}^i(K_q^M(K)/p^m) = U^i(K_q^M(K)/p^m)/U^{i+1}(K_q^M(K)/p^m)$. Then*

$$\mathrm{gr}^i(K_q^M(K)/p^m) \cong \begin{cases} \Omega_F^{q-1}/B_{s+n}^{q-1} & (\text{ if } m > s+n) \\ \Omega_F^{q-1}/Z_{m-n}^{q-1} & (\text{ if } m \leq s+n, i - en \neq \frac{e}{p-1}) \\ \Omega_F^{q-1}/(1+aC)Z_{m-n+1}^{q-1} & (\text{ if } m \leq s+n, i - en = \frac{e}{p-1}) \end{cases}$$

where a is the residue class of p/π^e for a fixed prime element π of K.

Remark 1.3. If $0 \leq i \leq ep/(p-1)$, $\mathrm{gr}^i\, K_q^M(K)$ is known by [3].

To show (1.1), we use the (truncated) syntomic complexes with respect to $\mathcal{O}_K$ and $\mathcal{O}_K/p\mathcal{O}_K$, which were introduced in [11]. In [12], it was proved that there exists an isomorphism between some subgroup of the q-th cohomology group of the syntomic complex with respect to $\mathcal{O}_K$ and some subgroup of $K_q^M(K)\hat{}$ which includes the image of $U^1K_q^M(K)$ (cf. (2.1)). On the other hand, the cohomology groups of the syntomic complex with respect to $\mathcal{O}_K/p\mathcal{O}_K$ can be calculated easily because $\mathcal{O}_K/p\mathcal{O}_K$ depends only on F and e. Comparing these two complexes, we have the exact sequence (2.4)

$$H^1(\mathbb{S}_q) \longrightarrow \hat{\Omega}_{A/\mathbb{Z}}^{q-1}/pd\hat{\Omega}_{A/\mathbb{Z}}^{q-2} \xrightarrow{\exp_p} K_q^M(K)\hat{}$$

as an long exact sequence of syntomic complexes, where $\mathbb{S}_q$ is the truncated translated syntomic complex with respect to $\mathcal{O}_K/p\mathcal{O}_K$, hat means the p-adic completion, and $\exp_p$ is the Kurihara's K_q^M-exponential homomorphism with respect to p. For more details, see Section 2. The left hand side of this exact sequence is determined in (2.6), and we have (1.1) by calculating these groups and the relations explicitly.

In Section 2, we see the relations between the syntomic complexes mentioned above, the Milnor K-groups, and the differential modules. The method of the proof of (1.1) is mentioned here. Note that we do not assume $p \nmid e$ in this section and we get the explicit description of the cohomology group of the syntomic complex with respect to $\mathcal{O}_K/p\mathcal{O}_K$ which was used in the proof of (1.1) without the assumption $p \nmid e$. In Section 3, we calculate differential module of $\mathcal{O}_K$. We calculate the kernel of the K_q^M-exponential homomorphism (4) explicitly in Section 4, 5, 6 and 7. In Section 8, we show Theorem 1.1 and Corollary 1.2. In Section 9, we have an application related to higher local class field theory.

Notations and Definitions. All rings are commutative with 1. For an element x of a discrete valuation ring, $\bar{x}$ means the residue class of x in the residue field. For an abelian group M and positive integer n, we denote $M/p^n = M/p^nM$ and $\hat{M} = \varprojlim_n M/p^n$. For a subset N of M, $\langle N \rangle$ means the subgroup of M generated by N. For a ring R, let $\Omega^1_R = \Omega^1_{R/\mathbb{Z}}$ be the absolute differentials of R and Ω^q_R the q-th exterior power of Ω^1_R over R for $q \geq 2$. We denote $\Omega^0_R = R$ and $\Omega^q_R = 0$ for negative q. If R is of characteristic zero, let

$$\mathfrak{Z}_n\hat{\Omega}^q_R = \operatorname{Ker}\left(\hat{\Omega}^q_R \xrightarrow{d} \hat{\Omega}^{q+1}_R/p^n\right)$$

for positive n. For an element $\omega \in \hat{\Omega}^q_R$, let $v_p(\omega)$ be the maximal n which satisfies $\omega \in \mathfrak{Z}_n\hat{\Omega}^q_R$. For $n \leq 0$, let $\mathfrak{Z}_n\hat{\Omega}^q_R = \hat{\Omega}^q_R$. Let $\mathfrak{Z}_\infty\hat{\Omega}^q_R$ be the intersection of $\mathfrak{Z}_n\hat{\Omega}^q_R$ of all $n \geq 0$. All complexes are cochain complexes. For a morphism of non-negative complexes $f\colon C^\cdot \to D^\cdot$, $[f\colon C^\cdot \to D^\cdot]$ and

$$\begin{bmatrix} C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & C^2 & \xrightarrow{d} & \dots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ D^0 & \xrightarrow{d} & D^1 & \xrightarrow{d} & D^2 & \xrightarrow{d} & \dots \end{bmatrix}$$

both denote the mapping fiber complex with respect to the morphism f, namely, the complex

$$(C^0 \xrightarrow{d} C^1 \oplus D^0 \xrightarrow{d} C^2 \oplus D^1 \xrightarrow{d} \dots),$$

where the leftmost term is the degree-0 part and where the differentials are defined by

$$\begin{aligned} C^i \oplus D^{i-1} &\longrightarrow C^{i+1} \oplus D^i \\ (a,b) &\longmapsto (da, f(a) - db). \end{aligned}$$

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In [20], I.Zhukov calculated the Milnor K-groups of multidimensional complete fields in a different way. He gives an explicit description by using topological generators. In [8], B.Kahn also calculated $K_2(K)$ of local fields with perfect residue fields without an assumption $p \nmid e_K$.

2 Exponential homomorphism and syntomic cohomology

Let K be a complete discrete valuation field of mixed characteristics $(0,p)$. Assume that p is an odd prime. Let $A=\mathcal{O}_K$ be the ring of integers of K and F the residue field of K. Let A_0 be the Cohen subring of A with respect to F, namely, A_0 is a complete discrete valuation ring under the restriction of the valuation of A with the residue field F and p is a prime element of A_0 (cf. [4], IX, Section 2). Let K_0 be the fraction field of A_0. Then K/K_0 is finite and totally ramified extension of extension degree $e=e_K$. We denote $e'=ep/(p-1)$. Let π be a prime element of K and fix it. We further assume that F has a finite p-base and fix their liftings $\mathbb{T}\subset A_0$. We can take the frobenius endomorphism f of A_0 such that $f(T)=T^p$ for $T\in\mathbb{T}$ (cf. [12] or [17]). Let $U^iK_q^M(A)$ be the subgroup defined by the same way of $U^iK_q^M(K)$, namely,

$$U^iK_q^M(A)=\left\langle \{x_1,\dots,x_q\}\in K_q^M(A)\ \middle|\ x_1\in U_K^i,\ x_2,\dots,x_q\in A^\times\right\rangle.$$

Let $U^iK_q^M(K)\hat{}$ (resp. $U^iK_q^M(A)\hat{}$) be the closure of the image of $U^iK_q^M(K)$ (resp. $U^iK_q^M(A)$) in $K_q^M(K)\hat{}$ (resp. $K_q^M(A)\hat{}$). Note that $\operatorname{gr}^i K_q^M(K)\cong \operatorname{gr}^i K_q^M(K)\hat{}$ for $i>0$.

At first, we introduce an isomorphism between $U^1K_q^M(K)\hat{}$ and a subgroup of the cohomology group of the syntomic complex with respect to A. For further details, see [12]. Let $B=A_0[[X]]$, where X is an indeterminate. We extend the operation of the frobenius f on B by $f(X)=X^p$. We define $\mathcal{I}$ and $\mathcal{J}$ as follows.

$$\begin{aligned}\mathcal{J}&=\operatorname{Ker}\left(B\xrightarrow{X\mapsto\pi}A\right)\\ \mathcal{I}&=\operatorname{Ker}\left(B\xrightarrow{X\mapsto\pi}A\xrightarrow{\bmod p}A/p\right)=\mathcal{J}+pB.\end{aligned}$$

Let D and $J\subset D$ be the PD-envelope and the PD-ideal with respect to $B\to A$, respectively ([1],Section 3). Let $I\subset D$ be the PD-ideal with respect to $B\to A/p$. D is also the PD-envelope with respect to $B\to A/p$. Let $J^{[q]}$ and $I^{[q]}$ be their q-th divided powers. Notice that $I^{[1]}=I$, $J^{[1]}=J$ and $I^{[0]}=J^{[0]}=D$. If q is an negative integer, we denote $J^{[q]}=I^{[q]}=D$. We define the complexes $\mathbb{J}^{[q]}$ and $\mathbb{I}^{[q]}$ as

$$\begin{aligned}\mathbb{J}^{[q]}&=\left(J^{[q]}\xrightarrow{d}J^{[q-1]}\underset{B}{\otimes}\hat{\Omega}_B^1\xrightarrow{d}J^{[q-2]}\underset{B}{\otimes}\hat{\Omega}_B^2\longrightarrow\cdots\right)\\ \mathbb{I}^{[q]}&=\left(I^{[q]}\xrightarrow{d}I^{[q-1]}\underset{B}{\otimes}\hat{\Omega}_B^1\xrightarrow{d}I^{[q-2]}\underset{B}{\otimes}\hat{\Omega}_B^2\longrightarrow\cdots\right),\end{aligned}$$

where $\hat{\Omega}_B^q$ is the p-adic completion of Ω_B^q. The leftmost term of each complex is the degree 0 part. We define $\mathbb{D}=\mathbb{I}^{[0]}=\mathbb{J}^{[0]}$. For $1\le q<p$, let $\mathcal{S}(A,B)(q)$ and $\mathcal{S}'(A,B)(q)$ be the mapping fibers of

$$\begin{aligned}&\mathbb{J}^{[q]}\xrightarrow{1-f_q}\mathbb{D}\\ &\mathbb{I}^{[q]}\xrightarrow{1-f_q}\mathbb{D},\end{aligned}$$

respectively, where $f_q = f/p^q$. $\mathcal{S}(A,B)(q)$ is called the syntomic complex of A with respect to B, and $\mathcal{S}'(A,B)(q)$ is also called the syntomic complex of A/p with respect to B (cf. [11]). We notice that

$$H^q(\mathcal{S}(A,B)(q)) = \frac{\operatorname{Ker}\Big((D\otimes\hat\Omega_B^q)\oplus(D\otimes\hat\Omega_B^{q-1})\to(D\otimes\hat\Omega_B^{q+1})\oplus(D\otimes\hat\Omega_B^q)\Big)}{\operatorname{Im}\Big((J\otimes\hat\Omega_B^{q-1})\oplus(D\otimes\hat\Omega_B^{q-2})\to(D\otimes\hat\Omega_B^q)\oplus(D\otimes\hat\Omega_B^{q-1})\Big)}, \tag{2}$$

where the maps are the differentials of the mapping fiber. If $q \geq p$, we cannot define the map $1 - f_q$ on $\mathbb{J}^{[q]}$ and $\mathbb{I}^{[q]}$, but we define $H^q(\mathcal{S}(A,B)(q))$ by using (2) in this case. This is equal to the cohomology of the mapping fiber of

$$\sigma_{>q-3}\mathbb{J}^{[q]} \xrightarrow{1-f_q} \sigma_{>q-3}\mathbb{D},$$

where $\sigma_{>n}C^{\cdot}$ means the brutal truncation for a complex $C^{\cdot}$, i.e., $(\sigma_{>n}C^{\cdot})^i$ is C^i if $i > n$ and 0 if $i \leq n$. Let $U^1(D \otimes \hat\Omega_B^{q-1})$ be the subgroup of $D \otimes \hat\Omega_B^{q-1}$ generated by $XD \otimes \hat\Omega_B^{q-1}$, $(X^e)^{[m]}D \otimes \hat\Omega_B^{q-1}$ for all $m \geq 1$ and $D \otimes \hat\Omega_B^{q-2} \wedge dX$. Let $U^1H^q(\mathcal{S}(A,B)(q))$ be the subgroup of $H^q(\mathcal{S}(A,B)(q))$ generated by the image of $(D \otimes \hat\Omega_B^q) \oplus U^1(D \otimes \hat\Omega_B^{q-1})$. Then there is a result of Kurihara:

THEOREM 2.1 (KURIHARA, [12]). *A and B are as above. Then*

$$U^1H^q(\mathcal{S}(A,B)(q)) \cong U^1K_q^M(A)\hat{}.$$

Furthermore, we have the following

LEMMA 2.2. *A and K are as above. Assume that A has the primitive p-th roots of unity. Then*

(i) *The natural map $K_q^M(A)\hat{} \to K_q^M(K)\hat{}$ is an injection.*

(ii) $U^1H^q(\mathcal{S}(A,B)(q)) \cong U^1K_q^M(A)\hat{} \cong U^1K_q^M(K)\hat{}$.

Remark 2.3. When F is separably closed, this lemma is also the consequence of the result of Kurihara [14]. But even if F is not separably closed, calculation goes similarly to [14].

Proof of Lemma 2.2. The first isomorphism of (ii) is (2.1). The natural map

$$U^1K_q^M(A)\hat{} \to U^1K_q^M(K)\hat{}$$

is a surjection by the definition of the filtrations and the fact that we can define an element $\{1+\pi^i a_1, a_2, \ldots, a_{q-1}, \pi\}$ as an element of $K_q^M(A)\hat{}$ by using Dennis-Stain Symbols, see [17]. Thus we only have to show (i). Let ζ_p be a primitive

p-th root of unity and fix it. Let μ_p be the subgroup of $A^\times$ generated by ζ_p. For $n \geq 2$, see the following commutative diagram.

$$\begin{array}{ccccc}
K^M_{q-1}(K)/p & \xrightarrow{\{*,\zeta_p\}} & K^M_q(A)/p^{n-1} & \xrightarrow{p} & K^M_q(A)/p^n \\
{\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\
K^M_{q-1}(K)/p & \xrightarrow{\{*,\zeta_p\}} & K^M_q(K)/p^{n-1} & \xrightarrow{p} & K^M_q(K)/p^n \\
& & \longrightarrow & K^M_q(A)/p & \longrightarrow 0 \\
& & & \downarrow & \\
& & \longrightarrow & K^M_q(K)/p & \longrightarrow 0.
\end{array} \tag{3}$$

The bottom row are exact by using Galois cohomology long exact sequence with respect to the Bockstein

$$\cdots \to H^{q-1}(K, \mathbb{Z}/p(q)) \to H^q(K, \mathbb{Z}/p^{n-1}(q)) \to H^q(K, \mathbb{Z}/p^n(q)) \to \dots$$

and

$$K^M_q(K)/p^n \cong H^q(K, \mathbb{Z}/p^n(q))$$

by [3]. The map $\{*, \zeta_p\}$ in the top row is well-defined if $K^M_q(A)/p^{n-1} \to K^M_q(K)/p^{n-1}$ is injective, and the top row are exact except at $K^M_q(A)/p^{n-1}$. Using the induction on n, we only have to show the injectivity of $K^M_q(A)/p \to K^M_q(K)/p$. We know the subquotients of the filtration of $K^M_q(K)/p$ by [3] and we also know the subquotients of the filtration of $K^M_q(A)/p$ using the isomorphism $U^1H^q(\mathcal{S}(A,B)(q)) \cong U^1K^M_q(A)\hat{}$ in [12] and the explicit calculation of $H^q(\mathcal{S}(A,B)(q))$ by [14] except $\mathrm{gr}^0(K^M_q(A)/p)$. Natural map preserves filtrations and induces isomorphisms of subquotients. Thus $U^1(K^M_q(A)/p) \to U^1(K^M_q(K)/p)$ is an injection. Lastly, the composite map of the natural maps

$$K^M_q(F)/p \twoheadrightarrow \mathrm{gr}^0(K^M_q(A)/p) \to \mathrm{gr}^0(K^M_q(K)/p) \xrightarrow{\cong} K^M_q(F)/p \oplus K^M_{q-1}(F)/p$$

is also an injection. Hence $K^M_q(A)/p \to K^M_q(K)/p$ is injective. □

Next, we introduce K^M_q-exponential homomorphism and consider the kernel. By [17], there is the K^M_q-exponential homomorphism with respect to η for $q \geq 2$ and $\eta \in K$ such that $v_K(\eta) \geq 2e/(p-1)$ defined by

$$\begin{aligned} \exp_\eta \ : \ \hat{\Omega}^{q-1}_A &\longrightarrow K^M_q(K)\hat{} \\ a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}} &\longmapsto \{\exp(\eta a), b_1, \dots, b_{q-1}\} \end{aligned} \tag{4}$$

for $a \in A$, $b_1, \dots, b_{q-1} \in A^\times$. Here exp is

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

We use this K_q^M-exponential homomorphism only in the case $\eta = p$ in this paper. On the other hand, there exists an exact sequence of complexes

$$0 \to \begin{bmatrix} \sigma_{>q-3}\mathbb{J}^{[q]} \\ \downarrow 1-f_q \\ \sigma_{>q-3}\mathbb{D} \end{bmatrix} \to \begin{bmatrix} \sigma_{>q-3}\mathbb{I}^{[q]} \\ \downarrow 1-f_q \\ \sigma_{>q-3}\mathbb{D} \end{bmatrix} \to \begin{bmatrix} \sigma_{>q-3}\mathbb{I}^{[q]}/\sigma_{>q-3}\mathbb{J}^{[q]} \\ \downarrow \\ 0 \end{bmatrix} \to 0. \qquad (5)$$

$[\sigma_{>q-3}\mathbb{I}^{[q]}/\sigma_{>q-3}\mathbb{J}^{[q]} \to 0]$ is none other than the complex $\sigma_{>q-3}\mathbb{I}^{[q]}/\sigma_{>q-3}\mathbb{J}^{[q]}$.

We denote the complex $[\sigma_{>q-3}\mathbb{I}^{[q]} \overset{1-f_q}{\to} \sigma_{>q-3}\mathbb{D}][q-2]$ by $\mathbb{S}_q$. It is the mapping fiber complex

$$\begin{bmatrix} I^{[2]} \otimes \hat{\Omega}_B^{q-2} & \xrightarrow{d} & I \otimes \hat{\Omega}_B^{q-1} & \xrightarrow{d} & D \otimes \hat{\Omega}_B^{q} & \xrightarrow{d} & \dots \\ \downarrow 1-f_q & & \downarrow 1-f_q & & \downarrow 1-f_q & & \\ D \otimes \hat{\Omega}_B^{q-2} & \xrightarrow{d} & D \otimes \hat{\Omega}_B^{q-1} & \xrightarrow{d} & D \otimes \hat{\Omega}_B^{q} & \xrightarrow{d} & \dots \end{bmatrix}. \qquad (6)$$

Taking cohomology, we have the following

PROPOSITION 2.4. *A, B and K are as above. Then K_q^M-exponential homomorphism with respect to p factors through $\hat{\Omega}_A^{q-1}/pd\Omega_A^{q-2}$ and there is an exact sequence*

$$H^1(\mathbb{S}_q) \xrightarrow{\psi} \hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2} \xrightarrow{\exp_p} K_q^M(K)\hat{}.$$

Proof. See the cohomological long exact sequence with respect to the exact sequence (5). The q-th cohomology group of the left complex of (5) is equal to $H^q(\mathcal{S}(A,B)(q))$, thus the sequence

$$H^1(\mathbb{S}_q) \xrightarrow{\psi} H^1((\sigma_{>q-3}\mathbb{I}^{[q]}/\sigma_{>q-3}\mathbb{J}^{[q]})[q-2]) \to H^q(\mathcal{S}(A,B)(q))$$

is exact. Here we denote the first map by ψ. The complex $(\sigma_{>q-3}\mathbb{I}^{[2]}/\sigma_{>q-3}\mathbb{J}^{[2]})[q-2]$ is

$$\left((I^{[2]} \otimes \hat{\Omega}_B^{q-2})/(J^{[2]} \otimes \hat{\Omega}_B^{q-2}) \to (I \otimes \hat{\Omega}_B^{q-1})/(J \otimes \hat{\Omega}_B^{q-1}) \to 0 \to \cdots\right).$$

$(I \otimes \hat{\Omega}_B^{q-1})/(J \otimes \hat{\Omega}_B^{q-1})$ is the subgroup of $(D \otimes \hat{\Omega}_B^{q-1})/(J \otimes \hat{\Omega}_B^{q-1}) = A \otimes \hat{\Omega}_B^{q-1}$. The image of $I \otimes \hat{\Omega}_B^{q-1}$ in $A \otimes \hat{\Omega}_B^{q-1}$ is equal to $pA \otimes \hat{\Omega}_B^{q-1}$. Thus $(I \otimes \hat{\Omega}_B^{q-1})/(J \otimes \hat{\Omega}_B^{q-1}) = pA \otimes \hat{\Omega}_B^{q-1}$. The image of

$$(I^{[2]} \otimes \hat{\Omega}_B^{q-2})/(J^{[2]} \otimes \hat{\Omega}_B^{q-2}) \xrightarrow{d} pA \otimes \hat{\Omega}_B^{q-1}$$

is equal to the image of $\mathcal{I}^2 \otimes \hat{\Omega}_B^{q-2}$. By $\mathcal{I} = (p) + \mathcal{J}$, $d(\mathcal{I}^2 \otimes \hat{\Omega}_B^{q-2})$ is equal to $d(\mathcal{J}^2 \otimes \hat{\Omega}_B^{q-2}) + pd(\mathcal{J} \otimes \hat{\Omega}_B^{q-2}) + p^2 d(\hat{\Omega}_B^{q-2})$. By the exact sequence

$$0 \longrightarrow \mathcal{J} \longrightarrow B \longrightarrow A \longrightarrow 0,$$

we have an exact sequence

$$(\mathcal{J}/\mathcal{J}^2)\otimes\hat{\Omega}_B^{q-2}\xrightarrow{d}A\otimes\hat{\Omega}_B^{q-1}\longrightarrow\hat{\Omega}_A^{q-1}\longrightarrow 0. \tag{7}$$

Thus the image of $d(\mathcal{J}^2\otimes\hat{\Omega}_B^{q-2})$ in $A\otimes\hat{\Omega}_B^{q-1}$ is zero. $A\otimes\hat{\Omega}_B^{q-1}$ is torsion free, thus

$$\frac{pA\otimes\hat{\Omega}_B^{q-1}}{pd(\mathcal{J}\otimes\hat{\Omega}_B^{q-2})+p^2d\hat{\Omega}_B^{q-2}}\overset{p^{-1}}{\cong}\frac{A\otimes\hat{\Omega}_B^{q-1}}{d(\mathcal{J}\otimes\hat{\Omega}_B^{q-2})+pd\hat{\Omega}_B^{q-2}}\cong\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}. \tag{8}$$

Hence we have $H^1((\sigma_{>q-3}\mathbb{I}^{[2]}/\sigma_{>q-3}\mathbb{J}^{[2]})[q-2])\cong\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}$. By chasing the connecting homomorphism $\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}\to H^q(\mathcal{S}(A,B)(q))$, we can show that the image is contained by $U^1H^q(\mathcal{S}(A,B)(q))$ and the composite map

$$\hat{\Omega}_A^{q-1}\to\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}\to U^1H^q(\mathcal{S}(A,B)(q))\overset{\cong}{\to}U^1K_q^M(K)\hat{}$$

is equal to $\exp_p$. We got the desired exact sequence. ☐

Remark 2.5. By [3], there exist surjections

$$\begin{aligned}\Omega_F^{q-2}\oplus\Omega_F^{q-1}&\longrightarrow\operatorname{gr}^i K_q^M(K)\\ \left(x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-2}}{f_{q-2}},0\right)&\longmapsto\{1+\pi^i\tilde{x},\tilde{y}_1,\ldots,\tilde{y}_{q-2},\pi\}\\ \left(0,x\frac{dy_1}{y_1}\wedge\cdots\wedge\frac{dy_{q-1}}{f_{q-1}}\right)&\longmapsto\{1+\pi^i\tilde{x},\tilde{y}_1,\ldots,\tilde{y}_{q-1}\}\end{aligned} \tag{9}$$

for $i\geq 1$, where $x\in F$, $y_1,\ldots,y_{q-1}\in F^\times$ and where $\tilde{x},\tilde{y}_1,\ldots,\tilde{y}_{q-1}$ are their liftings to A. If $i\geq e+1$, then we can construct all elements of $\operatorname{gr}^i K_q^M(K)$ as the image of $\exp_p$, namely,

$$\begin{aligned}\left\{\omega\in\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}\;\middle|\;\exp_p(\omega)\in U^iK_q^M(K)\hat{}\right\}&\xrightarrow{\exp_p}\operatorname{gr}^i K_q^M(K)\\ \frac{\pi^{i-1}}{p}a\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_{q-2}}{b_{q-2}}\wedge d\pi&\longmapsto\{\exp(\pi^ia),b_1,\ldots,b_{q-2},\pi\}\\ &\qquad=\{1+\pi^ia,b_1,\ldots,b_{q-2},\pi\}\\ \frac{\pi^i}{p}a\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_{q-1}}{b_{q-1}}&\longmapsto\{\exp(\pi^ia),b_1,\ldots,b_{q-1}\}\\ &\qquad=\{1+\pi^ia,b_1,\ldots,b_{q-1}\}.\end{aligned}$$

Thus $U^{e+1}K_q^M(K)\hat{}$ is contained by the image of $\exp_p$. On the other hand, (2.4) says the kernel of the K_q^M-exponential homomorphism is $\psi(H^1(\mathbb{S}_q))$. Recall that the aim of this paper is to determine $\operatorname{gr}^i K_q^M(K)$ for all i, but we already know them in the range $0\leq i\leq e'$ in [3]. Thus if we want to know $\operatorname{gr}^i K_q^M(K)$ for all i, we only have to know $\psi(H^1(\mathbb{S}_q))$. We determine $H^1(\mathbb{S}_q)$ in the rest of this section, and $\psi(H^1(\mathbb{S}_q))$ in Section 4, 5, 6 and 7.

To determine $H^1(\mathbb{S}_q)$, we introduce a filtration into it. Let $0 \le r < p$ and $s \ge 0$ be integers. Recall that $B = A_0[[X]]$. For $i \ge 0$ and $s \ge 0$, let $\mathrm{fil}^i(I^{[r]} \otimes \hat{\Omega}^s_B)$ be the subgroup of $I^{[r]} \otimes \hat{\Omega}^s_B$ generated by the elements

$$\begin{aligned}&\left\{X^n(X^e)^{[j]}\omega \;\middle|\; n+ej \ge i, n \ge 0, j \ge r, \omega \in D \otimes \hat{\Omega}^s_B\right\}\\ \cup &\left\{X^{n-1}(X^e)^{[j]}\omega \wedge dX \;\middle|\; n+ej \ge i, n \ge 1, j \ge r, \omega \in D \otimes \hat{\Omega}^{s-1}_B\right\}.\end{aligned}$$

The homomorphism $1 - f_{r+s} : I^{[r]} \otimes \hat{\Omega}^s_B \to D \otimes \hat{\Omega}^s_B$ preserves filtrations. Thus we can define the following complexes

$$\begin{aligned}
\mathrm{fil}^i(\sigma_{>q-3}\mathbb{I}^{[q]})[q-2]&\\
&= \left(\mathrm{fil}^i\ (I^{[2]} \otimes \hat{\Omega}^{q-2}_B) \to \mathrm{fil}^i(I \otimes \hat{\Omega}^{q-1}_B) \to \mathrm{fil}^i(D \otimes \hat{\Omega}^q_B) \to \dots\right)\\
\mathrm{fil}^i(\sigma_{>q-3}\mathbb{D})[q-2]&\\
&= \left(\mathrm{fil}^i\ (D \otimes \hat{\Omega}^{q-2}_B) \to \mathrm{fil}^i(D \otimes \hat{\Omega}^{q-1}_B) \to \mathrm{fil}^i(D \otimes \hat{\Omega}^q_B) \to \dots\right)\\
\mathrm{fil}^i\,\mathbb{S}_q &= \left[\mathrm{fil}^i(\sigma_{>q-3}\mathbb{I}^{[q]})[q-2] \xrightarrow{1-f_q} \mathrm{fil}^i(\sigma_{>q-3}\mathbb{D})[q-2]\right]\\
\mathrm{gr}^i(\sigma_{>q-3}\mathbb{I}^{[r]})[q-2] &= \frac{\mathrm{fil}^i(\sigma_{>q-3}\mathbb{I}^{[r]})[q-2]}{\mathrm{fil}^{i+1}(\sigma_{>q-3}\mathbb{I}^{[r]})[q-2]} \quad \text{for } r = 0, q\\
\mathrm{gr}^i\,\mathbb{S}_q &= \left[\mathrm{gr}^i(\sigma_{>q-3}\mathbb{I}^{[q]})[q-2] \xrightarrow{1-f_q} \mathrm{gr}^i(\sigma_{>q-3}\mathbb{D})[q-2]\right].
\end{aligned}$$

Note that if $i \ge 1$, $1 - f_q$: $\mathrm{gr}^i(\sigma_{>q-3}\mathbb{I}^{[q]})[q-2] \to \mathrm{gr}^i(\sigma_{>q-3}\mathbb{D})[q-2]$ is none other than 1 because f_q takes the elements to the higher filters. $\mathrm{fil}^i\,\mathbb{S}_q$ forms the filtration of $\mathbb{S}_q$ and we have the exact sequences

$$0 \longrightarrow \mathrm{fil}^{i+1}\,\mathbb{S}_q \longrightarrow \mathrm{fil}^i\,\mathbb{S}_q \longrightarrow \mathrm{gr}^i\,\mathbb{S}_q \longrightarrow 0$$

for $i \ge 0$. This exact sequence of complexes give a long exact sequence

$$\dots \to H^n(\mathrm{fil}^{i+1}\,\mathbb{S}_q) \to H^n(\mathrm{fil}^i\,\mathbb{S}_q) \to H^n(\mathrm{gr}^i\,\mathbb{S}_q) \to H^{n+1}(\mathrm{fil}^{i+1}\,\mathbb{S}_q) \to \dots \quad (10)$$

Furthermore, we have the following

Proposition 2.6. *$\{H^1(\mathrm{fil}^i\,\mathbb{S}_q)\}_i$ forms the finite decreasing filtration of $H^1(\mathbb{S}_q)$. Denote $\mathrm{fil}^i H^1(\mathbb{S}_q) = H^1(\mathrm{fil}^i\,\mathbb{S}_q)$ and $\mathrm{gr}^i H^1(\mathbb{S}_q) = \mathrm{fil}^i H^1(\mathbb{S}_q)/\,\mathrm{fil}^{i+1} H^1(\mathbb{S}_q)$. Then*

$$\mathrm{gr}^i H^1(\mathbb{S}_q) = \begin{cases} 0 & (\text{ if } i > 2e) \\ X^{2e-1}dX \wedge \left(\hat{\Omega}^{q-3}_{A_0}/p\right) & (\text{ if } i = 2e) \\ X^i \left(\hat{\Omega}^{q-2}_{A_0}/p\right) \oplus X^{i-1}dX \wedge (\hat{\Omega}^{q-3}_{A_0}/p) & (\text{ if } e < i < 2e) \\ X^e \left(\hat{\Omega}^{q-2}_{A_0}/p\right) \oplus X^{e-1}dX \wedge \left(\mathfrak{z}_1\hat{\Omega}^{q-3}_{A_0}/p^2\hat{\Omega}^{q-3}_{A_0}\right) & (\text{ if } i = e, p \mid e) \\ X^{e-1}dX \wedge \left(\mathfrak{z}_1\hat{\Omega}^{q-3}_{A_0}/p^2\hat{\Omega}^{q-3}_{A_0}\right) & (\text{ if } i = e, p \nmid e) \\ \left(X^i \dfrac{\left(p^{\mathrm{Max}(\eta'_i - v_p(i),0)}\hat{\Omega}^{q-2}_{A_0} \cap \mathfrak{z}_{\eta_i}\hat{\Omega}^{q-2}_{A_0}\right) + p^2\hat{\Omega}^{q-2}_{A_0}}{p^2\hat{\Omega}^{q-2}_{A_0}}\right) & \\ \quad \oplus \left(X^{i-1}dX \wedge \dfrac{\mathfrak{z}_{\eta_i}\hat{\Omega}^{q-3}_{A_0} + p^2\hat{\Omega}^{q-3}_{A_0}}{p^2\hat{\Omega}^{q-3}_{A_0}}\right) & (\text{ if } 1 \le i < e) \\ 0 & (\text{ if } i = 0), \end{cases}$$

where η_i and η'_i be the integers which satisfy $p^{\eta_i - 1}i < e \le p^{\eta_i}i$ and $p^{\eta'_i - 1}i - 1 < e \le p^{\eta'_i}i - 1$ for each i.

To prove (2.6), we need the following lemmas.

LEMMA 2.7. *For $\omega \in D \otimes \hat{\Omega}^q_B$ and $n \ge 0$,*

$$v_p(f^n(\omega)) \ge v_p(\omega) + nq. \tag{11}$$

In particular, if $\omega \in \hat{\Omega}^q_{A_0}$, then

$$v_p(f^n(\omega)) = v_p(\omega) + nq. \tag{12}$$

Proof. $\omega \in D \otimes \hat{\Omega}^q_B$ can be rewrite as $\omega = \sum_i a_i\omega_i$, where $a_i \in D$ and ω_i are the canonical generators of $\hat{\Omega}^q_B$, which are

$$\omega_i = \frac{dT_1}{T_1} \wedge \cdots \wedge \frac{dT_q}{T_q}$$

for $T_1, \ldots, T_q \in \mathbb{T} \cup \{X\}$. Canonical generators have the property $f(\omega_i) = p^q\omega_i$, thus we have (11). Furthermore, if $\omega \in \hat{\Omega}^q_{A_0}$, then $a_i \in A_0$ and we have $v_p(f(a_i)) = v_p(a_i)$. Thus (12) follows. □

LEMMA 2.8. *If $1 \le r < p$, $s \ge 0$ and $i > er$, then there exists a homomorphism*

$$\sum_{m=0}^{\infty} f^m_{r+s} \ : \ \mathrm{fil}^i(D \otimes \hat{\Omega}^s_B) \longrightarrow \mathrm{fil}^i(I^{[r]} \otimes_B \hat{\Omega}^s_B)$$

This is the inverse map of $1 - f_{r+s}$, hence $1 - f_{r+s} : \mathrm{fil}^i(I^{[r]} \otimes \hat{\Omega}^s_B) \to \mathrm{fil}^i(D \otimes \hat{\Omega}^s_B)$ is an isomorphism.

Proof. By $i > er$, $\mathrm{fil}^i(I^{[r]} \otimes \hat{\Omega}^s_B) = \mathrm{fil}^i(D \otimes \hat{\Omega}^s_B)$ because $X^i = r!X^{i-er}(X^e)^{[r]}$. All elements of $\mathrm{fil}^i D \otimes \hat{\Omega}^s_B$ can be written as the sum of the elements of the form $X^n(X^e)^{[j]}\omega$, where $\omega \in D \otimes \hat{\Omega}^s_B$ and $n + ej \geq i$. Now $r < p$, thus $(X^e)^{[r]} = X^{er}/r!$ in D, hence we may assume $j \geq r$. The image of $X^n(X^e)^{[j]}\omega$ is

$$\sum_{m=0}^{\infty} f^m_{r+s}(X^n(X^e)^{[j]}\omega) = \sum_{m=0}^{\infty} \frac{(p^m j)!}{p^{rm}(j!)} X^{np^m}(X^e)^{[p^m j]} \frac{f^m(\omega)}{p^{sm}}.$$

Here, $f^m(\omega)$ is divisible by p^{sm} by (11). The coefficients $(p^m j)!/p^{rm}(j!)$ are p-integers for all m and if $j \geq 1$ then the sum converges p-adically. If $j = r = 0$, $n \geq 1$ says that the order of the power of X is increasing. This also means the sum converges p-adically in $D \otimes_B \hat{\Omega}^s_B$. The image is in $\mathrm{fil}^i(I^{[r]} \otimes_B \hat{\Omega}^s_B)$ because $p^m j \geq r$ for all m, thus the map is well-defined. Obviously, $\sum_{m=0}^{\infty} f^m_{r+s}$ is the inverse map of $1 - f_{r+s}$. □

Lemma 2.9. *Let $i \geq 1$ and $e \geq 1$ be integers. For each $n \geq 0$, let m_n (resp. m'_n) be the maximal integer which satisfies $ip^n \geq m_n e$ (resp. $ip^n - 1 \geq m'_n e$). Then*

$$\begin{aligned} &\mathrm{Min}\,\{v_p(m_n!) + m_n - n\}_n \\ &\qquad = \begin{cases} 1 - \eta_i \leq 0 & (\textit{ when } n = \eta_i - 1,\ \textit{if } \eta_i \geq 1) \\ v_p(m_0!) + m_0 \geq 1 & (\textit{ when } n = 0,\ \textit{if } \eta_i = 0) \end{cases} \\ &\mathrm{Min}\,\{v_p(m'_n!) + m'_n - n\}_n \\ &\qquad = \begin{cases} 1 - \eta'_i \leq 0 & (\textit{ when } n = \eta'_i - 1,\ \textit{if } \eta'_i \geq 1) \\ v_p(m'_0!) + m'_0 \geq 1 & (\textit{ when } n = 0,\ \textit{if } \eta'_i = 0), \end{cases} \end{aligned}$$

where η_i and η'_i are as in (2.6).

Proof. By the definition of $\{m_n\}_n$, m_{n+1} is greater than or equal to pm_n. Thus $v_p(m'_{n+1}!) \geq v_p(pm'_n!)$ and

$$\begin{aligned} &v_p(m_{n+1}!) + m_{n+1} - (n+1) - (v_p(m_n!) + m_n - n) \\ &\quad = v_p(m_{n+1}!) - v_p(m_n!) + m_{n+1} - m_n - 1 \end{aligned} \tag{13}$$

is greater than zero if $m_n > 0$. On the other hand, η_i is the number which has the property that if $n < \eta_i$, then $m_n = 0$ and $m_{\eta_i} \geq 1$. Thus the value of (13) is less than zero if and only if $n < \eta_i$. Hence the minimum of $v_p(m_n!) + m_n - n$ is the value when $n = \eta_i - 1$ if $\eta > 0$ and $n = 0$ if $\eta_i = 0$. The rest of the desired equation comes from the same way. □

Proof of Proposition 2.6. At first, we show that $\{H^1(\mathrm{fil}^i \mathbb{S}_q)\}_i$ forms the finite decreasing filtration of $H^1(\mathbb{S}_q)$. See

$$\mathrm{gr}^i \mathbb{S}_q = \left[\begin{array}{ccccccc} \mathrm{gr}^i I^{[2]} \otimes \hat{\Omega}_B^{q-2} & \xrightarrow{d} & \mathrm{gr}^i I \otimes \hat{\Omega}_B^{q-1} & \xrightarrow{d} & \mathrm{gr}^i D \otimes \hat{\Omega}_B^{q} & \xrightarrow{d} & \cdots \\ \downarrow{\scriptstyle 1-f_q} & & \downarrow{\scriptstyle 1-f_q} & & \downarrow{\scriptstyle 1-f_q} & & \\ \mathrm{gr}^i D \otimes \hat{\Omega}_B^{q-2} & \xrightarrow{d} & \mathrm{gr}^i D \otimes \hat{\Omega}_B^{q-1} & \xrightarrow{d} & \mathrm{gr}^i D \otimes \hat{\Omega}_B^{q} & \xrightarrow{d} & \cdots \end{array}\right]. \tag{14}$$

If $i \geq 1$, all vertical arrows of (14) are equal to 1. Thus they are injections by the definition of the filtration. Especially, the injectivity of the first vertical arrow gives $H^0(\mathrm{gr}^i \mathbb{S}_q) = 0$, this means

$$0 \longrightarrow H^1(\mathrm{fil}^{i+1} \mathbb{S}_q) \longrightarrow H^1(\mathrm{fil}^i \mathbb{S}_q) \longrightarrow H^1(\mathrm{gr}^i \mathbb{S}_q) \tag{15}$$

is exact. If $i = 0$, the first vertical arrow of (14) is $1 - f_q$: $p^2\hat{\Omega}_{A_0}^{q-2} \to \hat{\Omega}_{A_0}^{q-2}$. This is also injective because of the invariance of the valuation of A_0 by the action of f. Thus the exact sequence (15) also follows when $i = 0$. Hence $\{H^1(\mathrm{fil}^i \mathbb{S}_q)\}_i$ forms a decreasing filtration of $H^1(\mathbb{S}_q)$.

Next we calculate $H^1(\mathrm{gr}^i \mathbb{S}_q)$. If $i > 2e$, $\mathrm{fil}^i \mathbb{S}_q$ is acyclic by (2.8). Thus we only consider the case $i \leq 2e$. Furthermore, if $i \geq 1$, we may consider that $H^1(\mathrm{gr}^i \mathbb{S}_q)$ is the subgroup of $\mathrm{gr}^i D \otimes \hat{\Omega}_B^{q-2}$ because of the injectivity of the vertical arrows of (14).

Let $i = 2e$. Then $\mathrm{gr}^{2e} \mathbb{S}_q$ is

$$\left[\begin{array}{ccc} X^{2e}\hat{\Omega}_{A_0}^{q-2} \oplus pX^{2e-1}dX \wedge \hat{\Omega}_{A_0}^{q-3} & \xrightarrow{d} X^{2e}\hat{\Omega}_{A_0}^{q-1} \oplus X^{2e-1}dX \wedge \hat{\Omega}_{A_0}^{q-2} & \xrightarrow{d} \cdots \\ \downarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \\ X^{2e}\hat{\Omega}_{A_0}^{q-2} \oplus X^{2e-1}dX \wedge \hat{\Omega}_{A_0}^{q-3} & \xrightarrow{d} X^{2e}\hat{\Omega}_{A_0}^{q-1} \oplus X^{2e-1}dX \wedge \hat{\Omega}_{A_0}^{q-2} & \xrightarrow{d} \cdots \end{array}\right].$$

The second vertical arrow is a surjection, thus

$$H^1(\mathrm{gr}^{2e} \mathbb{S}_q) \cong X^{2e-1}dX \wedge (\hat{\Omega}_{A_0}^{q-3}/p). \tag{16}$$

Let $e < i < 2e$. Then $\mathrm{gr}^{2e} \mathbb{S}_q$ is

$$\left[\begin{array}{ccc} pX^{i}\hat{\Omega}_{A_0}^{q-2} \oplus pX^{i-1}dX \wedge \hat{\Omega}_{A_0}^{q-3} & \xrightarrow{d} X^{i}\hat{\Omega}_{A_0}^{q-1} \oplus X^{i-1}dX \wedge \hat{\Omega}_{A_0}^{q-2} & \xrightarrow{d} \cdots \\ \downarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \\ X^{i}\hat{\Omega}_{A_0}^{q-2} \oplus X^{i-1}dX \wedge \hat{\Omega}_{A_0}^{q-3} & \xrightarrow{d} X^{i}\hat{\Omega}_{A_0}^{q-1} \oplus X^{i-1}dX \wedge \hat{\Omega}_{A_0}^{q-2} & \xrightarrow{d} \cdots \end{array}\right].$$

The second vertical arrow is also a surjection, thus

$$H^1(\mathrm{gr}^i \mathbb{S}_q) \cong X^i(\hat{\Omega}_{A_0}^{q-2}/p) \oplus X^{i-1}dX \wedge (\hat{\Omega}_{A_0}^{q-3}/p). \tag{17}$$

Let $i = e$. Then $\mathrm{gr}^e\,\mathbb{S}_q$ is

$$\begin{bmatrix} pX^e\hat{\Omega}^{q-2}_{A_0} \oplus p^2X^{e-1}dX \wedge \hat{\Omega}^{q-3}_{A_0} & \xrightarrow{d} & X^e\hat{\Omega}^{q-1}_{A_0} \oplus pX^{e-1}dX \wedge \hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} \cdots \\ \downarrow 1 & & \downarrow 1 & \\ X^e\hat{\Omega}^{q-2}_{A_0} \oplus X^{e-1}dX \wedge \hat{\Omega}^{q-3}_{A_0} & \xrightarrow{d} & X^e\hat{\Omega}^{q-1}_{A_0} \oplus X^{e-1}dX \wedge \hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} \cdots \end{bmatrix}.$$

The second vertical arrow is not a surjection. For an element $X^e\omega \in X^e\hat{\Omega}^{q-2}_{A_0}$, $d(X^e\omega)$ is included in $X^e\hat{\Omega}^{q-1}_{A_0} \oplus pX^{e-1}dX \wedge \hat{\Omega}^{q-2}_{A_0}$ if and only if $p \mid e$ or $p \mid \omega$. For an element $X^{e-1}\omega \wedge dX \in X^{e-1}dX \wedge \hat{\Omega}^{q-3}_{A_0}$, $d(X^{e-1}\omega \wedge dX)$ is included in $X^e\hat{\Omega}^{q-1}_{A_0} \oplus pX^{e-1}dX \wedge \hat{\Omega}^{q-2}_{A_0}$ if and only if $p \mid d\omega$. Thus we have

$$H^1(\mathrm{gr}^e\,\mathbb{S}_q) \cong \begin{cases} X^e(\hat{\Omega}^{q-2}_{A_0}/p) \oplus X^{e-1}dX \wedge \left(\mathfrak{Z}_1\hat{\Omega}^{q-3}_{A_0}/p^2\hat{\Omega}^{q-3}_{A_0}\right) & (\text{ if } p \mid e) \\ X^{e-1}dX \wedge \left(\mathfrak{Z}_1\hat{\Omega}^{q-3}_{A_0}/p^2\hat{\Omega}^{q-3}_{A_0}\right) & (\text{ if } p \nmid e). \end{cases} \tag{18}$$

Let $1 \le i < e$. Then $\mathrm{gr}^i\,\mathbb{S}_q$ is

$$\begin{bmatrix} p^2X^i\hat{\Omega}^{q-2}_{A_0} \oplus p^2X^{i-1}dX \wedge \hat{\Omega}^{q-3}_{A_0} & \xrightarrow{d} & pX^i\hat{\Omega}^{q-1}_{A_0} \oplus pX^{i-1}dX \wedge \hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} \cdots \\ \downarrow 1 & & \downarrow 1 & \\ X^i\hat{\Omega}^{q-2}_{A_0} \oplus X^{i-1}dX \wedge \hat{\Omega}^{q-3}_{A_0} & \xrightarrow{d} & X^i\hat{\Omega}^{q-1}_{A_0} \oplus X^{i-1}dX \wedge \hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} \cdots \end{bmatrix}.$$

The image of $X^i\omega \in X^i\hat{\Omega}^{q-2}_{A_0}$ is included in $pX^i\hat{\Omega}^{q-1}_{A_0} \oplus pX^{i-1}dX \wedge \hat{\Omega}^{q-2}_{A_0}$ if and only if $p \mid i\omega$ and $p \mid d\omega$, and the image of $X^{i-1}dX \wedge \omega \in X^{i-1}dX \wedge \hat{\Omega}^{q-3}_{A_0}$ is included in $pX^i\hat{\Omega}^{q-1}_{A_0} \oplus pX^{i-1}dX \wedge \hat{\Omega}^{q-2}_{A_0}$ if and only if $p \mid d\omega$. Thus

$$H^1(\mathrm{gr}^i\,\mathbb{S}_q) \cong$$
$$\begin{cases} X^i\left(\mathfrak{Z}_1\hat{\Omega}^{q-2}_{A_0}/p^2\hat{\Omega}^{q-2}_{A_0}\right) \oplus X^{i-1}dX \wedge \left(\mathfrak{Z}_1\hat{\Omega}^{q-3}_{A_0}/p^2\hat{\Omega}^{q-3}_{A_0}\right) & (\text{ if } p \mid i) \\ X^i(p\hat{\Omega}^{q-2}_{A_0}/p^2\hat{\Omega}^{q-2}_{A_0}) \oplus X^{i-1}dX \wedge \left(\mathfrak{Z}_1\hat{\Omega}^{q-3}_{A_0}/p^2\hat{\Omega}^{q-3}_{A_0}\right) & (\text{ if } p \nmid i). \end{cases} \tag{19}$$

If $i = 0$, we need more calculation. The complex $\mathrm{gr}^0\,\mathbb{S}_q$ is

$$\begin{bmatrix} p^2\hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} & p\hat{\Omega}^{q-1}_{A_0} & \xrightarrow{d} & \hat{\Omega}^{q}_{A_0} & \xrightarrow{d} & \cdots \\ \downarrow 1-f_q & & \downarrow 1-f_q & & \downarrow 1-f_q & & \\ \hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} & \hat{\Omega}^{q-1}_{A_0} & \xrightarrow{d} & \hat{\Omega}^{q}_{A_0} & \xrightarrow{d} & \cdots \end{bmatrix}.$$

We introduce a p-adic filtration to $\mathrm{gr}^0\,\mathbb{S}_q$ as follows.

$$\mathrm{fil}^m_p(\mathrm{gr}^0\,\mathbb{S}_q) = \begin{bmatrix} p^{2+m}\hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} & p^{1+m}\hat{\Omega}^{q-1}_{A_0} & \xrightarrow{d} & p^m\hat{\Omega}^{q}_{A_0} & \xrightarrow{d} & \cdots \\ \downarrow 1-f_q & & \downarrow 1-f_q & & \downarrow 1-f_q & & \\ p^m\hat{\Omega}^{q-2}_{A_0} & \xrightarrow{d} & p^m\hat{\Omega}^{q-1}_{A_0} & \xrightarrow{d} & p^m\hat{\Omega}^{q}_{A_0} & \xrightarrow{d} & \cdots \end{bmatrix}.$$

Then, for all $m \geq 0$,

$$\mathrm{gr}_p^m(\mathrm{gr}^0 \mathbb{S}_q) = \begin{bmatrix} \Omega_F^{q-2} & \xrightarrow{0} & \Omega_F^{q-1} & \xrightarrow{0} & \Omega_F^q & \longrightarrow & \cdots \\ \big\downarrow{\scriptstyle -\mathrm{C}^{-1}} & & \big\downarrow{\scriptstyle -\mathrm{C}^{-1}} & & \big\downarrow{\scriptstyle 1-\mathrm{C}^{-1}} & & \\ \Omega_F^{q-2} & \xrightarrow{d} & \Omega_F^{q-1} & \xrightarrow{d} & \Omega_F^q & \longrightarrow & \cdots \end{bmatrix}. \tag{20}$$

The injectivity of the leftmost vertical arrow of (20) says that

$$H^0(\mathrm{gr}_p^m(\mathrm{gr}^0 \mathbb{S}_q)) = 0$$

for all $m \geq 0$. Thus $\{H^1(\mathrm{fil}_p^m(\mathrm{gr}^0 \mathbb{S}_q))\}_m$ is a decreasing filtration of $H^1(\mathrm{gr}^0 \mathbb{S}_q)$. On the other hand, the intersection of the image of $-\mathrm{C}^{-1} : \Omega_F^{q-1} \to \Omega_F^{q-1}$ and the image of d: $\Omega_F^{q-2} \to \Omega_F^{q-1} = B_1^{q-1}$ is $\{0\}$ by (1). Thus we also have $H^1(\mathrm{gr}_p^m(\mathrm{gr}^0 \mathbb{S}_q)) = 0$ for all $m \geq 0$. Hence we have $H^1(\mathrm{gr}^0 \mathbb{S}_q) = 0$.

We already have known $H^1(\mathrm{gr}^i \mathbb{S}_q)$ for all $i \geq 0$, but the third arrow of (15) is not surjective in general. So we must know the image of $H^1(\mathrm{fil}^i \mathbb{S}_q) \to H^1(\mathrm{gr}^i \mathbb{S}_q)$. Let $i \geq 1$ and let x be an element of $\mathrm{fil}^i D \otimes \hat{\Omega}_B^{q-2}$ which represents an element of $H^1(\mathrm{gr}^i \mathbb{S}_q)$. $H^1(\mathrm{fil}^i \mathbb{S}_q)$ is

$$H^1 \begin{bmatrix} \mathrm{fil}^i I^{[2]} \otimes \hat{\Omega}_B^{q-2} & \xrightarrow{d} & \mathrm{fil}^i I \otimes \hat{\Omega}_B^{q-1} & \xrightarrow{d} & \mathrm{fil}^i D \otimes \hat{\Omega}_B^q & \xrightarrow{d} & \cdots \\ \big\downarrow{\scriptstyle 1-f_q} & & \big\downarrow{\scriptstyle 1-f_q} & & \big\downarrow{\scriptstyle 1-f_q} & & \\ \mathrm{fil}^i D \otimes \hat{\Omega}_B^{q-2} & \xrightarrow{d} & \mathrm{fil}^i D \otimes \hat{\Omega}_B^{q-1} & \xrightarrow{d} & \mathrm{fil}^i D \otimes \hat{\Omega}_B^q & \xrightarrow{d} & \cdots \end{bmatrix}.$$

Now the second vertical arrow is an injection. Thus x also represents the element of $H^1(\mathrm{fil}^i \mathbb{S}_q)$ if and only if

$$\sum_{n=0}^{\infty} f_q^n(dx) \in \mathrm{fil}^i I \otimes \hat{\Omega}_B^{q-1}. \tag{21}$$

The elements of $H^1(\mathrm{gr}^i \mathbb{S}_q)$ are represented by two types of the elements of $D \otimes \hat{\Omega}_B^{q-2}$, these are $X^i\omega$ for $\omega \in \hat{\Omega}_{A_0}^{q-2}$ and $X^{i-1}dX \wedge \omega$ for $\omega \in \hat{\Omega}_{A_0}^{q-3}$. Thus we must know the condition when (21) follows for these elements.

At first, we calculate $X^i\omega$ for $\omega \in \hat{\Omega}_{A_0}^{q-2}$.

$$\begin{aligned} &\sum_{n=0}^{\infty} f_q^n(dX^i\omega) \\ &\quad = \sum_{n=0}^{\infty} f_q^n(X^i d\omega + iX^{i-1}dX \wedge \omega) \\ &\quad = \sum_{n=0}^{\infty} \left(\frac{1}{p^{nq}} X^{ip^n} f^n(d\omega) + \frac{ip^n}{p^{nq}} X^{ip^n-1} dX \wedge f^n(\omega) \right) \\ &\quad = \sum_{n=0}^{\infty} \left(\frac{m_n!}{p^n} X^{ip^n - m_n e} (X^e)^{[m_n]} \frac{f^n(d\omega)}{p^{n(q-1)}} \right. \\ &\qquad\qquad \left. + \frac{i(m'_n!)}{p^n} X^{ip^n-1-m'_n e} (X^e)^{[m'_n]} dX \wedge \frac{f^n(\omega)}{p^{n(q-2)}} \right). \end{aligned}$$

Here m_n and m'_n be the maximal integers which satisfy $ip^n - m_n e \geq 0$ and $ip^n - 1 - m'_n e \geq 0$. Note that $f^n(d\omega)$ and $f^n(\omega)$ can be divided by $p^{n(q-1)}$ and $p^{n(q-2)}$, respectively, by (11). Furthermore, $v_p(f^n(d\omega)/p^{n(q-1)}) = v_p(d\omega)$ and $v_p(f^n(\omega)/p^{n(q-2)}) = v_p(\omega)$ by (12). To be included in $I \otimes \hat{\Omega}_B^{q-1}$, the sum of the p-adic order and divided power degree must be greater than or equal to 1, i.e., $v_p(m_n!) - n + m_n + v_p(d\omega) \geq 1$ and $v_p(i) + v_p(m'_n!) - n + m'_n + v_p(\omega) \geq 1$ must be satisfied for all n. We already know the minimal of $v_p(m_n!) - n + m_n$ and $v_p(m'_n!) - n + m'_n$ by (2.9), thus $\sum_{n=0}^{\infty} f_q^n(dX^i\omega)$ belongs to $I \otimes_B^{q-1}$ if and only if

$$\begin{cases} \text{no condition} & (\text{ if } e+1 \leq i) \\ v_p(i) + v_p(\omega) \geq 1 & (\text{ if } e = i) \\ v_p(d\omega) \geq \eta_i \text{ and } v_p(i) + v_p(\omega) \geq \eta'_i & (\text{ if } 1 \leq i < e). \end{cases} \tag{22}$$

Next, we calculate $X^{i-1}dX \wedge \omega$ for $\omega \in \hat{\Omega}_{A_0}^{q-3}$.

$$\begin{aligned} &\sum_{n=0}^{\infty} f_q^n(dX^{i-1}dX \wedge \omega) \\ &= \sum_{n=0}^{\infty} f_q^n(X^{i-1}dX \wedge d\omega) \\ &= \sum_{n=0}^{\infty} \left(\frac{p^n}{p^{nq}} X^{ip^n-1} dX \wedge f^n(d\omega) \right) \\ &= \sum_{n=0}^{\infty} \left(\frac{(m'_n!)}{p^n} X^{ip^n-1-m'_n e} (X^e)^{[m'_n]} dX \wedge \frac{f^n(d\omega)}{p^{n(q-2)}} \right). \end{aligned}$$

To be included in $I \otimes \hat{\Omega}_B^{q-1}$, $v_p(m'_n!) - n + m'_n + v_p(d\omega) \geq 1$ must be satisfied for all n. As the same way as above, $\sum_{n=0}^{\infty} f_q^n(X^{i-1}dX \wedge \omega)$ belongs to $I \otimes_B^{q-1}$ if and only if

$$\begin{cases} \text{no condition} & (\text{ if } e+1 \leq i) \\ v_p(d\omega) \geq \eta'_i & (\text{ if } 1 \leq i \leq e). \end{cases} \tag{23}$$

For $\omega \in \hat{\Omega}_{A_0}^{q-1}$, the condition $v_p(\omega) \geq n$ means $\omega \in p^n \hat{\Omega}_{A_0}^{q-1}$ and $v_p(d\omega) \geq n$ means $\omega \in \mathfrak{Z}_n \hat{\Omega}_{A_0}^{q-1}$. Thus, by (16), (17), (18), (19), (22) and (23), we get (2.6). □

3 Differential modules and filtrations

Let K, A, A_0, K_0 and B are as in Section 2. We assume that $p \nmid e = e_K$, i.e., K/K_0 is tamely totally ramified extension from here. Let k be the constant field of K (cf. [18]), i.e., k is the complete discrete valuation subfield of K with the restriction of the valuation of K, algebraically closed in K, and the residue field of k is the maximal perfect subfield of F. Then there exists a prime element of K such that π is the element of k. Let $k_0 = K_0 \cap k$. Then π is algebraic over k_0 and we get $\hat{\Omega}^1_{\mathcal{O}_{k_0}} = 0$, where $\mathcal{O}_{k_0}$ is the ring of integers of k_0. Thus $\pi^{e-1}d\pi = 0$ in $\hat{\Omega}^1_A$ by taking the differential of the minimal equation of π over k_0.

By the equation $\pi^{e-1}d\pi = 0$ in $\hat{\Omega}^1_A$, we have

$$\begin{aligned}\hat{\Omega}^q_A \cong & \left(\bigoplus_{i_1<i_2<\cdots<i_q} A\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_q}}{T_{i_q}} \right) \\ & \oplus \left(\bigoplus_{i_1<i_2<\cdots<i_{q-1}} A/(\pi^{e-1})\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_{q-1}}}{T_{i_{q-1}}} \wedge d\pi \right),\end{aligned} \tag{24}$$

where $\{T_i\} = \mathbb{T}$. We introduce a filtration on $\hat{\Omega}^q_A$ as follows. Let

$$\operatorname{fil}^i \hat{\Omega}^q_A = \begin{cases} \hat{\Omega}^q_A & (\text{ if } i = 0) \\ \pi^i\hat{\Omega}^q_A + \pi^{i-1}d\pi \wedge \hat{\Omega}^{q-1}_A & (\text{ if } i \geq 1). \end{cases}$$

The subquotients are

$$\begin{aligned}\operatorname{gr}^i \hat{\Omega}^q_A &= \operatorname{fil}^i \hat{\Omega}^q_A / \operatorname{fil}^{i+1} \hat{\Omega}^q_A \\ &= \begin{cases} \Omega^q_F & (\text{ if } i = 0 \text{ or } i \geq e) \\ \Omega^q_F \oplus \Omega^{q-1}_F & (\text{ if } 1 \leq i < e), \end{cases}\end{aligned}$$

where the map is

$$\begin{aligned}&\pi^i\hat{\Omega}^q_A \ni \pi^i\omega \longmapsto \bar{\omega} \in \Omega^q_F \\ &\pi^{i-1}d\pi \wedge \hat{\Omega}^{q-1}_A \ni \pi^{i-1}d\pi \wedge \omega \longmapsto \bar{\omega} \in \Omega^{q-1}_F.\end{aligned}$$

Let $\operatorname{fil}^i(\hat{\Omega}^q_A/pd\hat{\Omega}^{q-1}_A)$ be the image of $\operatorname{fil}^i \hat{\Omega}^q_A$ in $\hat{\Omega}^q_A/pd\hat{\Omega}^{q-1}_A$. Then we have the following

PROPOSITION 3.1. *For $j \geq 0$,*

$$\operatorname{gr}^j \left(\hat{\Omega}^q_A/pd\hat{\Omega}^{q-1}_A\right) = \begin{cases} \Omega^q_F & (j = 0) \\ \Omega^q_F \oplus \Omega^{q-1}_F & (1 \leq j < e) \\ \Omega^q_F/B^q_l & (e \leq j), \end{cases}$$

where l be the maximal integer which satisfies $j - le \geq 0$.

Proof. If $1 \leq j < e$, $\operatorname{gr}^j \hat{\Omega}^q_A = \operatorname{gr}^j(\hat{\Omega}^q_A/pd\hat{\Omega}^{q-1}_A)$ because $pd\hat{\Omega}^{q-1}_A \subset \operatorname{fil}^e \hat{\Omega}^q_A$. Assume that $j \geq e$ and let l as above. By (24) and $pi^e d\pi = 0$, $\hat{\Omega}^{q-1}_A$ is generated by the elements $p\pi^i d\omega$ for $0 \leq i < e$ and $\omega \in \hat{\Omega}^{q-1}_{A_0}$. By [7] (Cor. 2.3.14), $p\pi^i d\omega \in \operatorname{fil}^{e(1+n)+i} \hat{\Omega}^q_A$ if and only if the residue class of $p^{-n}d\omega$ belongs to B_{n+1}. Thus $\operatorname{gr}^j(\hat{\Omega}^q_A/pd\hat{\Omega}^{q-1}_A) \cong \Omega^q_F/B^q_l$. ∎

We need the lemma in the following sections.

Lemma 3.2. (i) *For $n \geq 0$, there exist maps*

$$f_q^n = \frac{f^n}{p^{nq}} : \hat{\Omega}^q_{A_0} \longrightarrow \mathfrak{Z}_n \hat{\Omega}^q_{A_0}.$$

(ii) *For $n \geq 1$,*

$$\mathfrak{Z}_n \hat{\Omega}^q_{A_0} = \left(\sum_{l=0}^{n-1} p^l f_q^{n-l} \hat{\Omega}^q_{A_0} \right) + p^n \hat{\Omega}^q_{A_0} + \mathfrak{Z}_\infty \hat{\Omega}^q_{A_0}.$$

(iii) *For $n \geq 1$,*

$$\bigoplus_{i=0}^{n-1} \frac{d}{p^i} \circ f_{q-1}^i \colon \left(\hat{\Omega}^{q-1}_{A_0} / \mathfrak{Z}_1 \hat{\Omega}^{q-1}_{A_0} \right)^{\oplus n} \longrightarrow \hat{\Omega}^q_{A_0}/p \cong \Omega^q_F$$

is injective and the image is $B_n \Omega^q_F$.

(iv) *For any $n \geq 0$,*

$$\hat{\Omega}^q_{A_0} / \mathfrak{Z}_1 \hat{\Omega}^q_{A_0} \xrightarrow{f_q^n} \mathfrak{Z}_n \hat{\Omega}^q_{A_0} / (\mathfrak{Z}_{n+1} \hat{\Omega}^q_{A_0} + \mathfrak{Z}_n \hat{\Omega}^q_{A_0} \cap p \hat{\Omega}^q_{A_0})$$

is an isomorphism.

(v) *For any $n \geq 0$,*

$$\left(\hat{\Omega}^q_{A_0} / p \right) \oplus \left(\hat{\Omega}^{q-1}_{A_0} / \mathfrak{Z}_1 \hat{\Omega}^{q-1}_{A_0} \right)^{\oplus n} \xrightarrow{f_q^n \oplus \bigoplus_{i=0}^{n-1} \frac{d}{p^i} \circ f_{q-1}^i} \frac{\mathfrak{Z}_n \hat{\Omega}^q_{A_0}}{\mathfrak{Z}_n \hat{\Omega}^q_{A_0} \cap p \hat{\Omega}^q_{A_0}}$$

is an isomorphism.

Proof. (i) By (12), $f^n(\omega)$ belongs to $p^{nq} \hat{\Omega}^q_{A_0}$. $\hat{\Omega}^q_{A_0}$ is p-torsion free, thus f_q^n is well-defined as the map to $\hat{\Omega}^q_{A_0}$. Furthermore,

$$d(f_q^n(\omega)) = \frac{1}{p^{nq}} f^n(d\omega) = p^n f_{q+1}^n(d\omega),$$

thus $f_q^n(\omega) \in \mathfrak{Z}_n \hat{\Omega}^q_{A_0}$.

(ii) For $0 \leq l \leq n-1$, the image of the natural injection

$$\frac{\mathfrak{Z}_n \hat{\Omega}^q_{A_0} \cap p^l \hat{\Omega}^q_{A_0}}{(\mathfrak{Z}_n \hat{\Omega}^q_{A_0} \cap p^{l+1} \hat{\Omega}^q_{A_0}) + (\mathfrak{Z}_\infty \hat{\Omega}^q_{A_0} \cap p^l \hat{\Omega}^q_{A_0})}$$
$$\longrightarrow \frac{p^l \hat{\Omega}^q_{A_0}}{p^{l+1} \hat{\Omega}^q_{A_0} + (\mathfrak{Z}_\infty \hat{\Omega}^q_{A_0} \cap p^l \hat{\Omega}^q_{A_0})} \cong \Omega^q_F / Z_\infty \Omega^q_F$$

is coincide with $Z_{n-l}\Omega_F^q/Z_\infty\Omega_F^q$ by [7] (Cor. 3.2.14). The image of $p^l f_q^{n-l}\hat{\Omega}_{A_0}^q$ is also $Z_{n-l}\Omega_F^q/Z_\infty\Omega_F^q$ for all l, thus the natural projection

$$\left(\sum_{l=0}^{n-1} p^l f_q^{n-l}\hat{\Omega}_{A_0}^q\right) \longrightarrow \frac{\mathfrak{Z}_n\hat{\Omega}_{A_0}^q}{p^n\hat{\Omega}_{A_0}^q + \mathfrak{Z}_\infty\hat{\Omega}_{A_0}^q}$$

is surjective. Hence we have (ii).

(iii) The following diagram commute

$$\begin{array}{ccc}
\left(\hat{\Omega}_{A_0}^{q-1}/\mathfrak{Z}_1\hat{\Omega}_{A_0}^{q-1}\right)^{\oplus n} & \xrightarrow{\bigoplus_{i=0}^{n-1}\frac{d}{p^i}\circ f_{q-1}^i} & \hat{\Omega}_{A_0}^q/p \\
\cong\downarrow & & \cong\downarrow \\
\left(\Omega_F^{q-1}/Z_1^{q-1}\right)^{\oplus n} & \xrightarrow{\bigoplus_{i=0}^{n-1}\mathrm{C}^{-i}\,d} & \Omega_F^q.
\end{array}$$

The image of the bottom arrow is B_n^q.

(iv) The image of $\mathfrak{Z}_n\hat{\Omega}_{A_0}^q/(\mathfrak{Z}_n\hat{\Omega}_{A_0}^q \cap p\hat{\Omega}_{A_0}^q)$ under the isomorphism $\hat{\Omega}_{A_0}^q/p \to \Omega_F^q$ is Z_n^q by [7] (Cor. 3.2.14). (iv) follows from the diagram

$$\begin{array}{ccc}
\hat{\Omega}_{A_0}^q/\mathfrak{Z}_1\hat{\Omega}_{A_0}^q & \xrightarrow{f_q^n} & \mathfrak{Z}_l\hat{\Omega}_{A_0}^q/(\mathfrak{Z}_{n+1}\hat{\Omega}_{A_0}^q + \mathfrak{Z}_n\hat{\Omega}_{A_0}^q \cap p\hat{\Omega}_{A_0}^q) \\
\cong\downarrow & & \cong\downarrow \\
\Omega_F^q/Z_1^q & \xrightarrow{\mathrm{C}^{-n}} & Z_n^q/Z_{n+1}^q.
\end{array}$$

(v) The image of

$$\left(\hat{\Omega}_{A_0}^{q-1}/\mathfrak{Z}_1\hat{\Omega}_{A_0}^{q-1}\right)^{\oplus n} \xrightarrow{\bigoplus_{i=0}^{n-1}\frac{d}{p^i}\circ f_{q-1}^i} \hat{\Omega}_{A_0}^q/p \cong \Omega_F^q$$

is B_n^q by (iii), and the image of the composite

$$\hat{\Omega}_{A_0}^q/p \xrightarrow{f_q^n} \hat{\Omega}_{A_0}^q/p \cong \Omega_F^q \to \Omega_F^q/B_n^q$$

is Z_n^q/B_n^q. Hence we get (v). □

4 The image of $H^1(\mathbb{S}_q)$

We assume $p \nmid e$. We further assume that there exists the prime element π of K such that $\pi^e = p$. If there does not exist such π, we replace K by $K(p^{\frac{1}{e}})$. Note that the extension $K(p^{\frac{1}{e}})/K$ is unramified of degree prime to p. In this section, we calculate $\psi(H^1(\mathbb{S}_q))$ explicitly. We need some preparations.

Let N_0^q be the subset of $\hat{\Omega}_{A_0}^q$ such that the canonical map $N_0^q \to \Omega_F^q \setminus Z_1^q$ is an injection, the image generates Ω_F^q/Z_1^q and have the property

$$\text{If } \bar{\omega} + C^{-1}\bar{\omega} = 0, \text{ then } d\omega = 0. \tag{25}$$

We can take such N_0^q because of the following

Lemma 4.1. *Take $x \in \hat{\Omega}^q_F$. If $x + \mathrm{C}^{-1} x = 0$ then there exists $\omega \in \hat{\Omega}^q_{A_0}$ such that $\bar{\omega} = x$ and $d\omega = 0$.*

Proof. x can be written as

$$x = \sum_{\tau} x_\tau \tau,$$

where τ runs through the canonical generators (cf. in the proof of (2.7)) and $x_\tau \in F$. The assumption $x + \mathrm{C}^{-1} x = 0$ means that $x_\tau + x_\tau^p = 0$ for all τ, thus $x_\tau \in E$ for all τ, where E is the maximal perfect subfield of F. The canonical generators have the fixed lifts denoted by $\tilde{\tau}$, and we can take lifts of x_τ, denoted by $\tilde{x}_\tau$, in the ring of Witt vectors with coefficients in E, denotes $W(E)$. Fix an inclusion $W(E) \to A_0$. Let

$$\omega = \sum_{\tau} \tilde{x}_\tau \tilde{\tau}.$$

Then $d\omega = 0$ in $\hat{\Omega}^q_{A_0}$ because $d\tilde{x}_\tau = 0$ in $\hat{\Omega}^q_{A_0}$ and $\bar{\omega} = x$. This ω is the desired one. □

For any $q, l \geq 0$, let $N^q_l = f^l_q(N^q_0)$ as a subset of $\hat{\Omega}^q_{A_0}$ and let

$$N^q_\infty = \mathfrak{Z}_\infty \hat{\Omega}^q_{A_0} \setminus (\mathfrak{Z}_\infty \hat{\Omega}^q_{A_0} \cap p\hat{\Omega}^q_{A_0}),$$
$$N^q_f = \bigcup_{l \geq 0} N^q_l, \;\; N^q = N^q_f \cup N^q_\infty.$$

Then, by (3.2,iv), N^q generates $\hat{\Omega}^q_{A_0}/p$ and $\omega \neq 0$ in $\hat{\Omega}^q_{A_0}/p$ for all $\omega \in N^q$. Furthermore, by using (3.2,v) and the isomorphism

$$\frac{\mathfrak{Z}_{n-1}\hat{\Omega}^q_{A_0}}{\mathfrak{Z}_{n-1}\hat{\Omega}^q_{A_0} \cap p\hat{\Omega}^q_{A_0}} \xrightarrow{p} \frac{\mathfrak{Z}_n\hat{\Omega}^q_{A_0} \cap p\hat{\Omega}^q_{A_0}}{\mathfrak{Z}_n\hat{\Omega}^q_{A_0} \cap p^2\hat{\Omega}^q_{A_0}},$$

we have

$$\begin{aligned} \left\langle f^n_q N^q \cup \bigcup_{m=0}^{n-1} \frac{d}{p^m} f^m_{q-1} N^{q-1}_0 \right\rangle &= \frac{\mathfrak{Z}_n\hat{\Omega}^q_{A_0}}{\mathfrak{Z}_n\hat{\Omega}^q_{A_0} \cap p\hat{\Omega}^q_{A_0}}, \\ \left\langle p f^{n-1}_q N^q \cup \bigcup_{m=0}^{n-2} p\frac{d}{p^m} f^m_{q-1} N^{q-1}_0 \right\rangle &= \frac{\mathfrak{Z}_n\hat{\Omega}^q_{A_0} \cap p\hat{\Omega}^q_{A_0}}{\mathfrak{Z}_n\hat{\Omega}^q_{A_0} \cap p^2\hat{\Omega}^q_{A_0}}. \end{aligned} \tag{26}$$

Thus the union of the sets of the left hand side of (26) generates $\mathfrak{Z}_n\hat{\Omega}^q_{A_0}/(\mathfrak{Z}_n\hat{\Omega}^q_{A_0} \cap p^2\hat{\Omega}^q_{A_0})$. If $q < 0$ then let $N^q_l = \emptyset$.

Let $S^0_{i,1}$, $S^1_{i,1}$, $S^0_{i,2}$ and $S^1_{i,2}$ be the subsets of $D\otimes\hat\Omega^{q-2}_B$ defined as follows.

$$S^0_{i,1}=\begin{cases}\emptyset & (i=0,e \text{ or } i\geq 2e)\\ X^iN^{q-2} & (\text{if } e<i<2e)\\ \emptyset & (\text{if } 1\leq i<e, \eta_i-v_p(i)\geq 1)\\ X^i\left(f^{\eta_i}_{q-2}N^{q-2}\cup\bigcup_{m=0}^{\eta_i-1}\frac{d}{p^m}f^m_{q-3}N_0^{q-3}\right) & (\text{if } 1\leq i<e,\eta_i-v_p(i)\leq 0),\end{cases}$$

$$S^1_{i,1}=\begin{cases}\emptyset & (i=0 \text{ or } i\geq e\ )\\ \emptyset & (1\leq i<e,\eta_i-v_p(i)\geq 2)\\ X^i\left(pf^{\eta_i-1}_{q-2}N^{q-2}\cup\bigcup_{m=0}^{\eta_i-2}p\frac{d}{p^m}f^m_{q-3}N_0^{q-3}\right) & (1\leq i<e,\eta_i-v_p(i)\leq 1),\end{cases}$$

$$S^0_{i,2}=\begin{cases}\emptyset & (i=0 \text{ or } i>2e)\\ X^{i-1}dX\wedge N^{q-3} & (e<i\leq 2e)\\ X^{e-1}dX\wedge\left(f^1_{q-3}N^{q-3}\cup dN_0^{q-4}\right) & (i=e)\\ X^{i-1}dX\wedge\left(f^{\eta_i}_{q-3}N^{q-3}\cup\bigcup_{m=0}^{\eta_i-1}\frac{d}{p^m}f^m_{q-4}N_0^{q-4}\right) & (1\leq i<e),\end{cases}$$

$$S^1_{i,2}=\begin{cases}\emptyset & (i=0 \text{ or } i>e)\\ X^{e-1}dX\wedge pN^q & (\text{if } i=e)\\ X^{i-1}dX\wedge\left(pf^{\eta_i-1}_{q-3}N^{q-3}\cup\bigcup_{m=0}^{\eta_i-2}p\frac{d}{p^m}f^m_{q-4}N_0^{q-4}\right) & (\text{if } 1\leq i<e).\end{cases}$$

Let $S_{i,1}=S^0_{i,1}\cup S^1_{i,1}$, $S_{i,2}=S^0_{i,2}\cup S^1_{i,2}$, $S_i=S_{i,1}\cup S_{i,2}$ and S the union of all S_i. By the above definitions, S_i generates $\mathrm{gr}^i\,H^1(\mathbb{S}_q)$, hence S generates $H^1(\mathbb{S}_q)$.

The following lemma is useful to calculate ψ.

LEMMA 4.2. *If $1\leq i<e$ then the minimal value of $v_K(\pi^{ip^n}/p^{n+1})=ip^n-e(n+1)$ is*

$$\begin{cases}ip^{\eta_i-1}-e\eta_i & (\text{ when } n=\eta_i-1;\quad \text{if } e'<ip^{\eta_i}<ep)\\ ip^{\eta_i}-e(\eta_i+1) & (\text{ when } n=\eta_i;\quad \text{if } e<ip^{\eta_i}<e')\\ ip^{\eta_i-1}-e\eta_i & (\text{ when } n=\eta_i-1,\eta_i;\quad \text{if } <ip^{\eta_i}=e')\end{cases}$$

and if $e<i$ then the minimal value of $v_K(\pi^{ip^n}/p^{n+1})$ is $i-e$.

Proof. Lemma follows from the definition of η_i. ☐

Remark 4.3. Method of calculation of ψ. In (2.6) and in the definition of S, we use elements of $D\otimes\hat\Omega^{q-2}_B$, which is the degree zero part of the complex $\sigma_{>q-3}\mathbb{D}[q-2]$, to represent elements of $H^1(\mathbb{S}_q)$. Chasing the complex (6) and the map (8), ψ is the composite of

$$D\otimes\hat\Omega^{q-2}_B\xrightarrow{d}D\otimes\hat\Omega^{q-1}_B\xrightarrow{\sum_{n\geq 0}f^n_q}I\otimes\hat\Omega^{q-1}_B\xrightarrow{I\to pA}pA\otimes\hat\Omega^{q-1}_B$$
$$\xrightarrow{p^{-1}}A\otimes\hat\Omega^{q-1}_B\xrightarrow{dX=d\pi}\hat\Omega^{q-1}_A/pd\hat\Omega^{q-2}_A.$$

Thus, for $\omega \in \hat{\Omega}^{q-2}_{A_0}$ (resp. $\omega \in \hat{\Omega}^{q-3}_{A_0}$) and $i \geq 1$,

$$\psi(X^i\omega) = \sum_{n\geq 0}\left(\frac{i}{p^{n+1}}\pi^{ip^n}\frac{d\pi}{\pi}\wedge f^n_{q-2}(\omega) + \frac{1}{p^{n+1}}\pi^{ip^n}f^n_{q-1}(d\omega)\right)$$
$$\left(\text{resp. } \psi\left(X^i\frac{dX}{X}\wedge\omega\right) = \sum_{n\geq 0}\frac{1}{p^{n+1}}\pi^{ip^n}\frac{d\pi}{\pi}\wedge f^n_{q-2}(d\omega)\right). \tag{27}$$

Here, to avoid the complication of notations, we use

$$X^i\frac{dX}{X} \quad \left(\text{resp. } \pi^i\frac{d\pi}{\pi}\right)$$

which only denotes the meaning of $X^{i-1}dX$ (resp. $\pi^{i-1}d\pi$) when $i \geq 1$. By using (4.2), $n = \eta_i - 1$ or $n = \eta_i$ is the number at which the value v_K of the coefficients of $d\pi$ in (27) is the minimal. If $X^i\omega \in S$ (resp. $X^{i-1}dX\wedge\omega \in S$) for $\omega \in \hat{\Omega}^{q-2}_{A_0}$ (resp. $\omega \in \hat{\Omega}^{q-3}_{A_0}$), then ω has the property (22) (resp. (23)). Under this condition, the right hand side of (27) belongs to $\hat{\Omega}^{q-1}_A$. Furthermore, by $\pi^{e-1}\pi = 0$, if $\eta_i \geq 1$ then

$$\begin{aligned}\psi(X^i\omega) &= \frac{i}{p^{\eta_i}}\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f^{\eta_i-1}_{q-2}(\omega) + \frac{i}{p^{\eta_i+1}}\pi^{ip^{\eta_i}}\frac{d\pi}{\pi}\wedge f^{\eta_i}_{q-2}(\omega)\\ &\quad + \sum_{n\geq 0}\left(\frac{1}{p^{n+1}}\pi^{ip^n}f^n_{q-1}(d\omega)\right),\\ \psi(X^i\frac{dX}{X}\wedge\omega) &= \frac{1}{p^{\eta_i}}\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f^{\eta_i-1}_{q-2}(d\omega)\frac{1}{p^{\eta_i+1}}\pi^{ip^{\eta_i}}\frac{d\pi}{\pi}\wedge f^{\eta_i}_{q-2}(d\omega),\end{aligned}$$

and if $\eta_i = 0$ then

$$\begin{aligned}\psi(X^i\omega) &= \frac{i}{p}\pi^i\frac{d\pi}{\pi}\wedge\omega + \sum_{n\geq 0}\left(\frac{1}{p^{n+1}}\pi^{ip^n}f^n_{q-1}(d\omega)\right),\\ \psi(X^i\frac{dX}{X}\wedge\omega) &= \frac{1}{p}\pi^i\frac{d\pi}{\pi}\wedge d\omega.\end{aligned}$$

Note that if $\eta_i \geq 1$,

$$v_K\left(\frac{1}{p^{\eta_i}}\pi^{ip^{\eta_i-1}}\right) - v_K\left(\frac{1}{p^{\eta_i+1}}\pi^{ip^{\eta_i}}\right)\begin{cases} < 0 & (\text{ if } e' < ip^{\eta_i} < ep)\\ > 0 & (\text{ if } e < ip^{\eta_i} < e')\\ = 0 & (\text{ if } ip^{\eta_i} = e').\end{cases} \tag{28}$$

By the definition, S generates $H^1(\mathbb{S}_q)$. But $\psi\colon H^1(\mathbb{S}_q) \to \hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A$ has the kernel in general. The following lemma compute some subset of this kernel.

Lemma 4.4. (i) *$S_{2e}, S_e \subset \operatorname{Ker}\psi$.*

(ii) *If $e < i < 2e$ then $\psi(S^0_{i,2} \setminus (X^{i-1}dX \wedge N_0^{q-3})) = 0$. If $1 \leq i < e$ then $\psi(S^0_{i,2} \setminus (X^{i-1}dX \wedge f^{\eta_i}_{q-3}N_0^{q-3})) = 0$ and $\psi(S^1_{i,2} \setminus (X^{i-1}dX \wedge pf^{\eta_i-1}_{q-3}N_0^{q-3})) = 0$.*

(iii) *If $e < i < 2e$ and $p \nmid i$, then $\psi(S_{i,2}) \subset \langle \psi(S_{i,1}) \rangle$.*

(iv) *If $e' < i < 2e$ and $p \mid i$, then*

$$\psi(S_{i,2}) \subset \left\langle \psi \left(\bigcup_{1 \leq j < e} S^1_{j,1} \right) \right\rangle.$$

(v) *Let $1 \leq i < e$, $s = \eta_i + v_p(i)$ and $i_0 = i/p^{v_p(i)}$. If $e' < ip^{\eta_i} < ep$ and $s \geq 2$, then*

$$\psi(S_{i,2}) \subset \left\langle \left(\bigcup_{1 \leq j < e} S_{j,1} \right) \cup S_{ip^{\eta_i}-e,1} \right\rangle.$$

Furthermore, let

$$j = \begin{cases} i_0 p^{\frac{s}{2}} & (\text{ if } s \text{ is even }), \\ i_0 p^{\frac{s-1}{2}} & (\text{ if } s \text{ is odd }). \end{cases}$$

Then

$$\begin{aligned} \psi(S^0_{i,2}) &\subset \langle \psi(S_{j,1}) \rangle \quad \text{if } 3\eta_i \geq v_p(i), \\ \psi(S^1_{i,2}) &\subset \langle \psi(S_{j,1}) \rangle \quad \text{if } 3\eta_i \geq v_p(i) + 2. \end{aligned}$$

Proof. (i) Take $X^{2e-1}dX \wedge \omega \in S_{2e,2}$. Then

$$\psi\left(X^{2e}\frac{dX}{X} \wedge \omega\right) = \frac{1}{p}\pi^{2e}\frac{d\pi}{\pi} \wedge d\omega = 0$$

by (4.3). Next, take $X^{e-1}dX \wedge \omega \in S_{e,2}$. By the definition of $S_{e,2}$, such an ω can be divided by p. Thus, by using (4.3), we get

$$\psi\left(X^{e}\frac{dX}{X} \wedge \omega\right) = \frac{1}{p}\pi^{e}\frac{d\pi}{\pi} \wedge p\frac{d\omega}{p} = 0.$$

(ii) At first, let $e < i < 2e$. When we take $X^{i-1}dX \wedge \omega$ from $S^0_{i,2} \setminus (X^{i-1}dX \wedge N_0^{q-3})$, then ω has the property $v_p(d\omega) \geq 1$. Thus by using (4.3),

$$\psi\left(X^{i}\frac{dX}{X} \wedge \omega\right) = \frac{1}{p}\pi^{i}\frac{d\pi}{\pi} \wedge p\frac{d\omega}{p} = 0.$$

Next let $1 \leq i < e$. When we take $X^{i-1}dX \wedge \omega$ from

$$\left(S^0_{i,2} \setminus \left(X^i \frac{dX}{X} \wedge f^{\eta_i}_{q-3} N^{q-3}_0\right)\right) \cup \left(S^1_{i,2} \setminus \left(X^i \frac{dX}{X} \wedge p f^{\eta_i - 1}_{q-3} N^{q-3}_0\right)\right),$$

ω has the property $v_p(d\omega) \geq \eta_i + 1$. Thus $\psi(X^{i-1}dX \wedge \omega) = 0$ by using (4.3).

(iii) In this case, $S_{i,2} = X^{i-1}dX \wedge N^{q-3}$ and $S_{i,1} = X^i N^{q-2}$. For an element $X^{i-1}dX \wedge \omega \in S_{i,2}$ with $\omega \in N^{q-3}$, there exists $X^i d\omega \in S_{i,1}$ because $d\omega \in N^{q-2}_\infty$, and

$$d\left(X^i \frac{dX}{X} \wedge \omega\right) = d\left(\frac{X^i d\omega}{i}\right).$$

This means $\psi(X^{i-1}dX \wedge \omega) = \psi(X^i d\omega)/i$. Thus $\psi(S_{i,2}) \subset \langle \psi(S_{i,1}) \rangle$.

(iv) Take an element $X^{i-1}dX \wedge \omega \in S_{i,2}$ with $\omega \in N^{q-3}$. Let $j = j_0 = i - e$ and $j_l = j_{l-1}p - e$ for $j \geq 1$. Then, $\{j_l\}_l$ have the property

$$p \nmid j_l,$$
$$\frac{e}{p-1} < j_0 < j_1 < j_2 < \dots$$

by $p \mid i$ and $i > e'$. Let L be the minimal integer such that $j_L \geq 2e/p$. Then $\eta_{j_l} = 1$ for all $0 \leq l \leq L$. There exist the elements $X^{j_l} p f^l_{q-2}(d\omega) \in S^1_{j_l,1}$ because $S^1_{j_l,1} = X^{j_l} N^{q-2}$ and $p f^l_{q-2}(d\omega) \in N^{q-2}_\infty$. Thus the element, denoted by Y,

$$Y = \sum_{l=0}^{L} \frac{(-1)^l}{j_l} X^{j_l} p f^l_{q-2}(d\omega)$$

exists in $\langle \bigcup_{k=1}^{e-1} S^1_{k,1} \rangle$. By (4.3), $\psi(X^{i-1}dX \wedge \omega) = \pi^{i-e-1} d\pi \wedge d\omega$. On the other hand,

$$\begin{aligned}
\psi(Y) &= \sum_{l=0}^{L} \left((-1)^l \pi^{j_l} \frac{d\pi}{\pi} \wedge f^l_{q-2}(d\omega) + (-1)^l \frac{1}{p} \pi^{j_l p} \frac{d\pi}{\pi} \wedge f^{l+1}_{q-2}(d\omega) \right) \\
&= \sum_{l=0}^{L} \left((-1)^l \pi^{j_l} \frac{d\pi}{\pi} \wedge f^l_{q-2}(d\omega) + (-1)^l \pi^{j_{l+1}} \frac{d\pi}{\pi} \wedge f^{l+1}_{q-2}(d\omega) \right) \\
&= \pi^{j_0} \frac{d\pi}{\pi} \wedge d\omega.
\end{aligned}$$

The third equation follows from $j_{L+1} - 1 \geq e - 1$. Hence $\psi(X^{i-1}dX \wedge \omega) = \psi(Y)$ because $j_0 = i - e$, and we get (iv).

(v) Now $S^0_{i,2}$ and $S^1_{i,2}$ are

$$\begin{aligned}
S^0_{i,2} &= X^i \frac{dX}{X} \wedge \left(f^{\eta_i}_{q-3} N^{q-3} \cup \bigcup_{m=0}^{\eta_i - 1} \frac{d}{p^m} f^m_{q-4} N^{q-4}_0 \right), \\
S^1_{i,2} &= X^i \frac{dX}{X} \wedge \left(p f^{\eta_i - 1}_{q-3} N^{q-3} \cup \bigcup_{m=0}^{\eta_i - 2} p \frac{d}{p^m} f^m_{q-4} N^{q-4}_0 \right).
\end{aligned}$$

By (ii), we only have to calculate the element of

$$X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}N_0^{q-3},\ X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}N_0^{q-3}$$

to show (v). If $e'<ip^{\eta_i}<ep$ and $s\geq 2$, then

$$\begin{aligned}X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}\omega &= \pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)\\ &\quad+\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i}(d\omega),\\ X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}\omega &= \pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-2}(d\omega)\\ &\quad+\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega).\end{aligned}$$

The first terms of the right hand side come from

$$\begin{aligned}S_{ip^{\eta_i-1}+e,1}\supset & X^{ip^{\eta_i-1}+e}N_\infty^{q-2}\\ &\ni X^{ip^{\eta_i-1}+e}f_{q-2}^{2\eta_i-1}(d\omega)\xrightarrow{\psi}e\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega),\\ S_{ip^{\eta_i-1}+e,1}\supset & X^{ip^{\eta_i-1}+e}N_\infty^{q-2}\\ &\ni X^{ip^{\eta_i-1}+e}f_{q-2}^{2\eta_i-2}(d\omega)\xrightarrow{\psi}e\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-2}(d\omega).\end{aligned}$$

On the other hand, the second terms of the right hand side are, if $ip^{\eta_i}\geq 2e$ then vanished. If $ip^{\eta_i}<2e$, then by using the same argument of (iv) with $j_0=ip^{\eta_i}-e$ and

$$Y=\begin{cases}\sum_{l=0}^{L}\frac{(-1)^l}{j_l}X^{j_l}pf_{q-2}^l(f_{q-2}^{2\eta_i}(d\omega)) & (\text{ the first case })\\ \sum_{l=0}^{L}\frac{(-1)^l}{j_l}X^{j_l}pf_{q-2}^l(f_{q-2}^{2\eta_i-1}(d\omega)) & (\text{ the second case }),\end{cases}$$

we get

$$\psi(S_{i,2})\subset\left\langle\left(\bigcup_{1\leq j<e}S_{j,1}\right)\cup S_{ip^{\eta_i}-e,1}\right\rangle.$$

Next, we do not assume $e<ip^{\eta_i}<e'$ and $s\geq 2$. In this case, we have to show

$$\begin{aligned}\psi\left(X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}N_0^{q-3}\right)&\subset\psi(S_{j,1})\quad\text{if } 3\eta_i\geq v_p(i),\\ \psi\left(X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}N_0^{q-3}\right)&\subset\psi(S_{j,1})\quad\text{if } 3\eta_i\geq v_p(i)+2.\end{aligned}$$

Take $\omega \in N_0^{q-3}$. Then, by (4.3),

$$\begin{aligned}
&\psi\left(X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}(\omega)\right)\\
&\quad =\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)+\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i}(d\omega)\,,\\
&\psi\left(X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}(\omega)\right)\\
&\quad =\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-2}(d\omega)+\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)\,.
\end{aligned}$$

On the other hand, there exist elements

$$\begin{aligned}
\omega_1' &= f_{q-2}^{2\eta_i-\frac{s}{2}}(d\omega) && (\text{ if } s \text{ is even and } 3\eta_i\geq v_p(i)),\\
\omega_2' &= pf_{q-2}^{2\eta_i-\frac{s+1}{2}}(d\omega) && (\text{ if } s \text{ is odd and } 3\eta_i\geq v_p(i)),\\
\omega_3' &= f_{q-2}^{2\eta_i-\frac{s}{2}-1}(d\omega) && (\text{ if } s \text{ is even and } 3\eta_i\geq v_p(i)+2),\\
\omega_4' &= pf_{q-2}^{2\eta_i-\frac{s+1}{2}-1}(d\omega) && (\text{ if } s \text{ is odd and } 3\eta_i\geq v_p(i)+2)
\end{aligned}$$

in $\hat{\Omega}_{A_0}^{q-2}$ because the conditions are, $2\eta_i \geq s/2$ if and only if $3\eta_i \geq v_p(i)$ when s is even, $2\eta_i \geq (s+1)/2$ if and only if $3\eta_i \geq v_p(i)$ when s is odd, $2\eta_i \geq (s/2)+1$ if and only if $3\eta_i \geq v_p(i)+2$ when s is even, and $2\eta_i \geq ((s+1)/2)+1$ if and only if $3\eta_i \geq v_p(i)+2$ when s is odd. The image of ψ of an each element is

$$\begin{aligned}
\psi(X^j\omega_1') &= i_0\pi^{i_0p^{s-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s}{2}-1}(\omega_1')+\frac{i_0}{p}\pi^{i_0p^s}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s}{2}}(\omega_1')\\
&= i_0\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)+i_0\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i}(d\omega)\\
&= i_0\psi\left(X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}(\omega)\right),
\end{aligned}$$

$$\begin{aligned}
\psi(X^j\omega_2') &= \frac{i_0}{p}\pi^{i_0p^{s-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s-1}{2}}(\omega_2')+\frac{i_0}{p^2}\pi^{i_0p^s}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s+1}{2}}(\omega_2')\\
&= i_0\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)+i_0\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i}(d\omega)\\
&= i_0\psi\left(X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}(\omega)\right),
\end{aligned}$$

$$\begin{aligned}
\psi(X^j\omega_3') &= i_0\pi^{i_0p^{s-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s}{2}-1}(\omega_3')+\frac{i_0}{p}\pi^{i_0p^s}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s}{2}}(\omega_3')\\
&= i_0\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-2}(d\omega)+i_0\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)\\
&= i_0\psi\left(X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}(\omega)\right),
\end{aligned}$$

$$\begin{aligned}\psi(X^j\omega_4') &= \frac{i_0}{p}\pi^{i_0p^{s-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s-1}{2}}(\omega_4') + \frac{i_0}{p^2}\pi^{i_0p^s}\frac{d\pi}{\pi}\wedge f_{q-2}^{\frac{s+1}{2}}(\omega_4')\\ &= i_0\pi^{ip^{\eta_i-1}}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-2}(d\omega) + i_0\pi^{ip^{\eta_i}-e}\frac{d\pi}{\pi}\wedge f_{q-2}^{2\eta_i-1}(d\omega)\\ &= i_0\psi\left(X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}(\omega)\right).\end{aligned}$$

Compare the definition of $S_{j,1}$ with the condition of $\omega_1',\dots,\omega_4'$. If s is even and $3\eta_i \geq v_p(i)$ then

$$X^j\omega_1' = \begin{cases} X^j\frac{d}{p^{2\eta_i-\frac{s}{2}}}f_{q-2}^{2\eta_i-\frac{s}{2}}(d\omega) \in X^j\frac{d}{p^{2\eta_i-\frac{s}{2}}}f_{q-2}^{2\eta_i-\frac{s}{2}}N_0^{q-3} & \left(\text{if } \eta_i-\frac{s}{2}\leq \eta_j-1\right),\\ X^jf_{q-2}^{\eta_j}f_{q-2}^{2\eta_i-\frac{s}{2}-\eta_j}(d\omega) \in X^jf_{q-2}^{\eta_j}N_\infty^{q-2} & \left(\text{if } \eta_i-\frac{s}{2}\geq \eta_j\right).\end{cases}$$

Thus $X^j\omega_1' \in S_{i,1}^0$. By the similar way, we have

$$\begin{array}{ll} X^j\omega_2' \in S_{j,1}^1 & (\text{ if } s \text{ is odd and } 3\eta_i \geq v_p(i)),\\ X^j\omega_3' \in S_{j,1}^0 & (\text{ if } s \text{ is even and } 3\eta_i \geq v_p(i)+2),\\ X^j\omega_4' \in S_{j,1}^1 & (\text{ if } s \text{ is odd and } 3\eta_i \geq v_p(i)+2).\end{array}$$

The claim (v) was proved.

□

Remark 4.5. Let $S_{i,2}'^{0}$ (resp. $S_{i,2}'^{1}$) be the subset of $S_{i,2}^0$ (resp. $S_{i,2}^1$) defined as follows.

$$S_{i,2}'^{0} = \begin{cases} X^i\frac{dX}{X}\wedge N_0^{q-3} & (e < i \leq e', p \mid i)\\ X^i\frac{dX}{X}\wedge f_{q-3}^{\eta_i}N_0^{q-3} & 1\leq i<e,\ e<ip^{\eta_i}\leq e',\ 3\eta_i<v_p(i))\\ \emptyset & (\text{ otherwise }),\end{cases}$$

$$S_{i,2}'^{1} = \begin{cases} X^i\frac{dX}{X}\wedge pf_{q-3}^{\eta_i-1}N_0^{q-3} & (1\leq i<e,\ e<ip^{\eta_i}\leq e',\ 3\eta_i<v_p(i)+2)\\ \emptyset & (\text{ otherwise }).\end{cases}$$

Remark that if $1 \leq i < e$ satisfies $v_p(i)+\eta_i = 1$ and $e' < ip^{\eta_i} < ep$, then $v_p(i) = 0$, $e/(p-1) < i < e$ and $\eta_i = 1$. Thus this i satisfies neither $3\eta_i < v_p(i)+2$ nor $3\eta_i < v_p(i)$. Let $S_{i,2}' = S_{i,2}'^{0}\cup S_{i,2}'^{1}$. Then by (4.4), $\psi(H^1(\mathbb{S}_q))$ is generated by

$$\left(\bigcup_{1\leq i<2e} S_{i,1}\right)\cup\left(\bigcup_{1\leq i<2e} S_{i,2}'\right).$$

We need some modification of generators of $\psi(H^1(\mathbb{S}_q))$ as follows.

Let the index sets Λ_0 and Λ_1 be

$$\Lambda_0 = \{i\ ;\ 1\leq i<e,\ e'<ip^{\eta_i}<2e,\ \eta_i = v_p(i)\}, \tag{29}$$

$$\Lambda_1 = \{ i \;;\; 1 \le i < e,\; e' < ip^{\eta_i} < 2e,\; \eta_i = v_p(i)+1,\; p \nmid (i+e) \}. \tag{30}$$

Let $\Lambda = \Lambda_0 \cup \Lambda_1$. For $i \in \Lambda$, let

$$\begin{aligned} s &= \eta_i + v_p(i), \\ i' &= i/p^{v_p(i)}, \\ i_0 &= i' p^{s-1}, \\ i_l &= i_{l-1}p - e \text{ for } l \ge 1, \\ L &= \mathrm{Min}\{ l \;;\; i_l \ge 2e/p \}. \end{aligned} \tag{31}$$

$\{i_l\}_l$ are monotonely increasing, thus we can take such L. Note that $p \nmid i'$, $\eta_{i_l} = 1$ for $0 \le l \le L$ and $p \nmid i_l$ for $l \ge 1$. If $i \in \Lambda_0$ then let $g_{i,0}$ be

$$g_{i,0}(X^i\omega) = \frac{1}{i'}X^i\omega - \frac{1}{i_0+e}X^{i_0+e}f_{q-2}^{\eta_i-1}(\omega) + \sum_{l=1}^{L-1}\frac{(-1)^l}{i_l}pX^{i_l}f_{q-2}^{\eta_i+l-1}(\omega)$$

for $\omega \in \mathfrak{Z}_{\eta_i}\hat{\Omega}_{A_0}^{q-2}$. This function satisfies $g_{i,0}(\omega) \equiv (1/i')X^i\omega$ modulo $\mathrm{fil}^{i+1} H^1(\mathbb{S}_q)$, thus we can replace $S_{i,1}^0$ by $g_{i,0}(S_{i,1}^0)$ to generate $\psi(H^1(\mathbb{S}_q))$. When $i \in \Lambda_1$, then let $g_{i,1}$ be

$$g_{i,1}(X^ip\omega) = \frac{1}{i'}X^ip\omega - \frac{1}{i_0+e}X^{i_0+e}f_{q-2}^{\eta_i-1}(\omega) + \sum_{l=1}^{L-1}\frac{(-1)^l}{i_l}pX^{i_l}f_{q-2}^{\eta_i+l-1}(\omega)$$

for $p\omega \in p\hat{\Omega}_{A_0}^{q-2} \cap \mathfrak{Z}_{\eta_i}\hat{\Omega}_{A_0}^{q-2}$. This function satisfies $g_{i,1}(p\omega) \equiv (1/i')X^ip\omega$ modulo $\mathrm{fil}^{i+1} H^1(\mathbb{S}_q)$, thus we can replace $S_{i,1}^1$ by $g_{i,1}(S_{i,1}^1)$ to generate $\psi(H^1(\mathbb{S}_q))$.

5 Explicit Calculation, Case (a)

We compute $\psi(S_{i,1})$, $\psi(S'_{i,2})$, $\psi(g_{i,0}S_{i,1}^0)$ and $\psi(g_{i,1}S_{i,1}^1)$ explicitly in Section 5, 6 and 7.

Define the index sets as

$$\begin{aligned} \Gamma_a &= \left\{ i \;\middle|\; 1 \le i < e, \frac{e}{p-1} < ip^{\eta_i-1} < e \right\} \cup \left\{ i \;\middle|\; e' < i < 2e \right\}, \\ \Gamma_b &= \left\{ i \;\middle|\; 1 \le i < e, e < ip^{\eta_i} < e' \right\} \cup \left\{ i \;\middle|\; e < i < e' \right\}, \\ \Gamma_c &= \left\{ \frac{e}{p-1}, e' \right\}. \end{aligned}$$

These sets are disjoint to each other, and $\Gamma_a \cup \Gamma_b \cup \Gamma_c$ is coincide with $\{i;\ 1 \le i < 2e, i \ne e\}$. Λ is the subset of Γ_a. In this section, we compute $\psi(S_{i,1} \cup S'_{i,2})$ for $i \in \Gamma_a \setminus \Lambda$, $\psi(g_{i,0}(S_{i,1}^0) \cup S_{i,1}^1 \cup S'_{i,2})$ for $i \in \Lambda_0$ and $\psi(g_{i,1}(S_{i,1}^1) \cup S'_{i,2})$ for $i \in \Lambda_1$. We compute $\psi(S_{i,1} \cup S'_{i,2})$ when $i \in \Gamma_b$ in Section 6 and when $i \in \Gamma_c$ in Section 7.

At first, we compute ψ when $i \in \Gamma_a$ and $1 \leq i < e$. Let $e/(p-1) < j < e$, $s = v_p(j)+1$ and $j_0 = j/p^{s-1}$. Then the integers i which satisfy $ip^{\eta_i - 1} = j$ are

$$(i, \eta_i) = (j_0, s), (j_0 p, s-1), \ldots, (j_0 p^{s-1}, 1).$$

Let $i = j_0 p^t$. Then $i \in \Gamma_a$ for all t. Notice that if $i \in \Gamma_a$ and $i < e$ then there exists $e/(p-1) < j < e$ such that $ip^{\eta_i - 1} = j$.

If $t < \frac{s-1}{2}$ then $S_{i,1} = \emptyset$.

If $t = (s-1)/2$ and $p \mid (j+e)$, then $i \in \Lambda_1$, $S^0_{i,1} = \emptyset$ and

$$S^1_{i,1} = X^i \left(p f^{\eta_i - 1}_{q-2} N^{q-2} \cup \bigcup_{m=0}^{\eta_i - 2} p \frac{d}{p^m} f^m_{q-3} N^{q-3}_0 \right).$$

For $X^i p\omega \in S^1_{i,1}$, $\psi(g_{i,1}(X^i p\omega))$ is

$$\begin{aligned} \psi(g_{i,1}(X^i p\omega)) = & \sum_{n \geq 0} \frac{p\pi^{ip^n}}{i' p^{n+1}} f^n_{q-1}(d\omega) \\ & - \sum_{n \geq 0} \frac{p^{\eta_i - 1}}{(i_0 + e)p^{n+1}} \pi^{(i_0+e)p^n} f^{\eta_i + n - 1}_{q-1}(d\omega) \\ & + \sum_{l=1}^{L-1} \sum_{n \geq 0} \frac{(-1)^l p^{\eta_i + l - 1}}{i_l p^n} \pi^{i_l p^n} f^{\eta_i + l + n - 1}_{q-1}(d\omega) \end{aligned} \tag{32}$$

by using (4.3) and the same kind of calculation in (4.4,iv) with the notation (31). If $p\omega \in p f^{\eta_i - 1}_{q-2} N^{q-2}_\infty$ or $p\omega \in p\frac{d}{p^m} f^m_{q-3} N^{q-3}_0$ for some m, then $\psi(g'_i(X^i p\omega)) = 0$ by (32). If $p\omega \in p f^{\eta_i - 1}_{q-2} N^{q-2}_f$, then take $l \geq 0$ and $f^l_{q-2}\omega' \in N^{q-2}_l$ for $\omega' \in N^{q-2}_0$ such that $\omega = f^{\eta_i + l - 1}_{q-2}(\omega')$. For this ω', we have

$$\psi(g_{i,1}(X^i p f^{\eta_i + l - 1}_{q-2}(\omega'))) \equiv \begin{cases} \frac{p^l}{i'} \pi^{ip^{\eta_i - 1}} f^{2\eta_i + l - 2}_{q-1}(d\omega') & (\text{ if } \eta_i \neq 1) \\ \frac{ep^l}{i(i+e)} \pi^i f^l_{q-1}(d\omega') & (\text{ if } \eta_i = 1) \end{cases}$$
$$\bmod \operatorname{fil}^{ip^{\eta_i - 1} + el + 1}(\hat{\Omega}^{q-1}_A / pd\hat{\Omega}^{q-2}_A).$$

Now $ip^{\eta_i - 1} = j$ and $\eta_i = s - t = (s+1)/2$,

$$\psi(g_{i,1}(X^i p f^{\eta_i + l - 1}_{q-2}(\omega'))) \equiv \begin{cases} \frac{p^l}{i'} \pi^j f^{s+l-1}_{q-1}(d\omega') & (\text{ if } s - t > 1) \\ \frac{ep^l}{i(i+e)} \pi^j f^l_{q-1}(d\omega') & (\text{ if } t = 0, s = 1) \end{cases} \tag{33}$$
$$\bmod \operatorname{fil}^{j + el + 1}(\hat{\Omega}^{q-1}_A / pd\hat{\Omega}^{q-2}_A).$$

If $s = 1$ and $p \mid (j+e)$, then t can be taken only 0. $S^0_{i,1} = \emptyset$. This i is not in Λ_1, hence we compute $S^1_{i,1}$ without $g_{i,1}$. Now $S^1_{i,1} = X^i p N^{q-2}$. For

$X^ip\omega \in X^ipN^{q-2}$,

$$\begin{aligned} \psi(X^ip\omega) &= i\pi^{i-1}d\pi \wedge \omega + i\pi^{ip-e-1}d\pi \wedge f_{q-2}(\omega) \\ &\quad + \sum_{n\geq 0} \frac{1}{p^n}\pi^{ip^n} \wedge f_{q-1}^n(d\omega) \\ &\equiv i\pi^{i-1}d\pi \wedge \omega + \pi^i \wedge d\omega \\ &\qquad \operatorname{mod}\ \operatorname{fil}^{i+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}). \end{aligned} \tag{34}$$

If $t = s/2$, then $i \in \Lambda_0$ and

$$S_{i,1}^0 = X^i\left(f_{q-2}^{\eta_i}N^{q-2} \cup \bigcup_{m=0}^{\eta_i-1} p\frac{d}{p^m}f_{q-3}^m N_0^{q-3}\right).$$

For $X^i\omega \in S_{i,1}^0$, $\psi(g_{i,0}(X^i\omega))$ is

$$\begin{aligned} \psi(g_{i,0}(X^i\omega)) &= \sum_{n\geq 0} \frac{\pi^{ip^n}}{i'p^{n+1}} f_{q-1}^n(d\omega) \\ &\quad - \sum_{n\geq 0} \frac{1}{(i_0+e)p^{n+1}}\pi^{(i_0+e)p^n} f_{q-1}^n(df_{q-2}^{\eta_i-1}(\omega)) \\ &\quad + \sum_{l=1}^{L-1}\sum_{n\geq 0} \frac{(-1)^l p^{\eta_i+l-1}}{i_l p^n}\pi^{i_l p^n} f_{q-1}^{\eta_i+l+n-1}(d\omega). \end{aligned} \tag{35}$$

If $\omega \in f_{q-2}^{\eta_i}N_\infty^{q-2}$ or $\omega \in \frac{d}{p^m}f_{q-3}^m N_0^{q-3}$ for some m, then $\psi(g_{i,0}(X^i\omega)) = 0$ by (35). If $\omega \in f_{q-2}^{\eta_i}N_f^{q-2}$, then take $l \geq 0$ and $f_{q-2}^l\omega' \in N_l^{q-2}$ for $\omega' \in N_0^{q-2}$ such that $\omega = f_{q-2}^{\eta_i+l}(\omega')$. For this ω', we have

$$\begin{aligned} \psi(g_{i,0}(X^if_{q-2}^{\eta_i+l}(\omega'))) &\equiv \frac{p^l}{i'}\pi^{ip^{\eta_i-1}} f_{q-1}^{2\eta_i+l-1}(d\omega') \\ &\equiv \frac{p^l}{i'}\pi^j f_{q-1}^{s+l-1}(d\omega') \\ &\qquad \operatorname{mod}\ \operatorname{fil}^{j+el+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}). \end{aligned} \tag{36}$$

$S_{i,1}^1$ is

$$S_{i,1}^1 = X^i\left(pf_{q-2}^{\eta_i-1}N^{q-2} \cup \bigcup_{m=0}^{\eta_i-2} p\frac{d}{p^m}f_{q-3}^m N_0^{q-3}\right).$$

For $X^ip\omega \in S_{i,1}^1$, $\psi(X^ip\omega)$ is

$$\psi(X^ip\omega) = \sum_{n\geq 0} \frac{p^{\eta_i-1}}{p^n}\pi^{ip^n} f_{q-1}^n\left(\frac{d\omega}{p^{\eta_i-1}}\right). \tag{37}$$

Thus if $\omega \in f_{q-2}^{\eta_i-1}N_\infty^{q-2}$ or $\omega \in \frac{d}{p^m}f_{q-3}^m N_0^{q-3}$ for some m, then $\psi(X^i p\omega) = 0$. If $\omega \in f_{q-2}^{\eta_i-1}N_f^{q-2}$, let $l \geq 0$ and $f_{q-2}^l\omega' \in N_l^{q-2}$ for $\omega' \in N_0^{q-2}$ such that $\omega = f_{q-2}^{\eta_i+l}(\omega')$. For this ω', we have

$$\begin{aligned}\psi(X^i p f_{q-2}^{\eta_i+l-1}(\omega')) &\equiv p^l \pi^{ip^{\eta_i-1}} f_{q-1}^{2\eta_i+l-2}(d\omega') \\ &\equiv p^l\pi^j f_{q-1}^{s+l-2}(d\omega') \\ &\quad \bmod \operatorname{fil}^{j+el+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}).\end{aligned} \tag{38}$$

If $t > s/2$, then $i \notin \Lambda$ and

$$S_{i,1}^0 = X^i\left(f_{q-2}^{\eta_i}N^{q-2} \cup \bigcup_{m=0}^{\eta_i-1} p\frac{d}{p^m}f_{q-3}^m N_0^{q-3}\right).$$

For $X^i\omega \in S_{i,1}^0$, $\psi(X^i\omega)$ is

$$\psi(X^i\omega) = \sum_{n\geq 0}\frac{p^{\eta_i}}{p^{n+1}}\pi^{ip^n}f_{q-1}^n\left(\frac{d\omega}{p^{\eta_i}}\right). \tag{39}$$

If $\omega \in f_{q-2}^{\eta_i}N_\infty^{q-2}$ or $\omega \in \frac{d}{p^m}f_{q-3}^m N_0^{q-3}$ for some m, then $\psi(X^i\omega) = 0$. If $\omega \in f_{q-2}^{\eta_i}N_f^{q-2}$, then take $l \geq 0$ and $f_{q-2}^l\omega' \in N_l^{q-2}$ for $\omega' \in N_0^{q-2}$ such that $\omega = f_{q-2}^{\eta_i+l}(\omega')$. For this ω', we have

$$\begin{aligned}\psi(X^i f_{q-2}^{\eta_i+l}(\omega')) &\equiv p^l\pi^{ip^{\eta_i-1}} f_{q-1}^{2\eta_i+l-1}(d\omega') \\ &\equiv p^l\pi^j f_{q-1}^{2s-2t+l-1}(d\omega') \\ &\quad \bmod \operatorname{fil}^{j+el+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}).\end{aligned} \tag{40}$$

When we take $X^ip\omega \in S_{i,1}^1$, by the same calculation of the case $t = s/2$, we have

$$\begin{aligned}\psi(X^i p f_{q-2}^{\eta_i+l-1}(\omega')) &\equiv p^l\pi^{ip^{\eta_i-1}} f_{q-1}^{2\eta_i+l-2}(d\omega') \\ &\equiv p^l\pi^j f_{q-1}^{2s-2t+l-2}(d\omega') \\ &\quad \bmod \operatorname{fil}^{j+el+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}).\end{aligned} \tag{41}$$

When $i \in \Gamma_a$ and $i < e$ then $S'_{i,2} = \emptyset$.

Next, we compute $i \in \Gamma_a$ and $e < i$. In this case $S'_{i,2} = \emptyset$, thus we only have to compute $S_{i,1}$. Let $e/(p-1) < j < e$ and $i = j + e$. Then $S_{i,1}^1 = \emptyset$ and $S_{i,1}^0 = X^iN^{q-2}$. For an element $X^i\omega \in S_{i,1}^0$,

$$\psi(X^i\omega) = i\pi^{i-e}\frac{d\pi}{\pi}\wedge\omega + \sum_{n\geq 0}\frac{1}{p^{n+1}}\pi^{ip^n}f_{q-1}^n(d\omega).$$

If $p \mid i$, then the first term of the right hand side is zero. Hence if $\omega = f^l_{q-2}(\omega')$ then

$$\psi(X^i\omega) \equiv p^l \pi^{i-e} f^l_{q-1}(d\omega') \mod \mathrm{fil}^{j+el+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A) \tag{42}$$

and if $\omega \in N^{q-2}_\infty$ then $\psi(X^i\omega) = 0$. If $p \nmid i$, then

$$\psi(X^i\omega) \equiv i\pi^{i-e}\frac{d\pi}{\pi} \wedge \omega + \pi^{i-e} d\omega \mod \mathrm{fil}^{j+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A). \tag{43}$$

We have computed all S_i or substitutes of S_i for $i \in \Gamma_a$ as above. Next, we construct the sets $\{M^j\}_{j\geq 0}$ which are rearrangements of the generators of $\psi(H^1(\mathbb{S}_q))$. The law of rearrangement is, for example, as follows. See (33). For an element $g_{i,1}(X^i p f^{\eta_i+l-1}_{q-2}(\omega'))$, the image of ψ is

$$\psi(g_{i,1}(X^i p f^{\eta_i+l-1}_{q-2}(\omega'))) \equiv \frac{p^l}{i'}\pi^j f^{s+l-1}_{q-1}(d\omega') \mod \mathrm{fil}^{j+el+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A)$$

when $s - t > 1$. Thus this element goes to $\mathrm{gr}^{j+el}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A)$ and it seems non-zero. So we put $g_{i,1}(X^i p f^{\eta_i+l-1}_{q-2}(\omega'))$ into M^{j+el}. We will know its image is really non-zero in Section 8 but now we do not know it is true or not. We construct the set M^{j+el} by, roughly speaking, the set of the elements which come to $\mathrm{gr}^{j+el}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A)$ and seem non-zero. The real definition of M^* is as follows.

Use (33), (34), (36), (38), (40), (41), (42) and (43) to define M^{j+el} for $e/(p-1) < j < e$ and $l \geq 0$. Let $e/(p-1) < j < e$, $s = v_p(j) + 1$ and $l \geq 0$. If $s = 1$ then let

$$M^{j+el} = \begin{cases} g_{j,1}(X^j p N^{q-2}_0) \cup X^{j+e} N^{q-2} & (p \nmid (j+e) \text{ and } l = 0) \ \ldots\ (33),(43) \\ g_{j,1}(X^j p f^l_{q-2} N^{q-2}_0) & (p \nmid (j+e) \text{ and } l \geq 1) \ \ldots\ (33) \\ X^j p N^{q-2} \cup X^{j+e} N^{q-2}_0 & (p \mid (j+e) \text{ and } l = 0) \ \ldots\ (34),(42) \\ X^{j+e} f^l_{q-2} N^{q-2}_0 & (p \mid (j+e) \text{ and } l \geq 1) \ \ldots\ (42). \end{cases} \tag{44}$$

By (3.1), $\mathrm{gr}^j(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A) \cong \Omega^{q-1}_F \oplus \Omega^{q-2}_F$ and $\mathrm{gr}^{j+el}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A) \cong \Omega^{q-1}_F/B^{q-1}_l$ for $l \geq 1$. The image of M^j is, if $p \nmid (j+e)$,

$$\begin{array}{l} \Omega^{q-2}_F/Z^{q-2}_1 \xrightarrow{\psi\circ g_{j,1}X^j p} \mathrm{gr}^j(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A) \xrightarrow{\cong} \Omega^{q-1}_F \oplus \Omega^{q-2}_F \\ \omega \longmapsto \psi \circ g_{j,1} X^j p\omega = \dfrac{e}{j+e}\pi^j d\omega \longmapsto \left(\dfrac{e}{j+e}d\bar{\omega}, 0\right) \end{array}$$

and

$$\Omega_F^{q-2} \xrightarrow{\psi X^{j+e}} \mathrm{gr}^j(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}) \xrightarrow{\cong} \Omega_F^{q-1} \oplus \Omega_F^{q-2}$$
$$\omega \longmapsto \psi \circ X^{j+e}\omega = \pi^j d\omega + (j+e)\pi^j \frac{d\pi}{\pi} \wedge \omega \longmapsto (d\bar{\omega}, i\bar{\omega}).$$

Thus we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^j \text{ in } \mathrm{gr}^j(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^j(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle \psi(M^j)\rangle \cong \Omega_F^{q-1}/B_1^{q-1}. \end{aligned} \tag{45}$$

The case $s = 1$ and $p \mid (j+e)$ goes similarly to the case above. If $l \geq 1$, the image of M^{l+el} in $\mathrm{gr}^{j+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}) \cong \Omega_F^{q-1}/B_l^{q-1}$ is

$$\Omega_F^{q-2}/Z_1^{q-2} \ni x \longmapsto \mathrm{C}^{-l}\, dx \in \Omega_F^{q-1}/B_l^{q-1}$$

and hence non-zero. Thus

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{j+el} \text{ in } \mathrm{gr}^{j+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^{j+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle \psi(M^{j+el})\rangle \cong \Omega_F^{q-1}/B_{l+1}^{q-1}. \end{aligned} \tag{46}$$

If s is even and $s \geq 2$, let

$$\begin{aligned} M^j = & X^{j+e}N^{q-2} \quad \ldots\ (43) \\ & \cup g_{j_0p^{\frac{s}{2}},0}\left(X^{j_0p^{\frac{s}{2}}} f_{q-2}^{\frac{s}{2}} N_0^{q-2}\right) \quad \ldots\ (36) \\ & \cup X^{j_0p^{\frac{s}{2}}} p f_{q-2}^{\frac{s}{2}-1} N_0^{q-2} \quad \ldots\ (38) \\ & \cup \left(\bigcup_{s/2<t\leq s-1} X^{j_0p^t} f_{q-2}^{s-t} N_0^{q-2} \cup X^{j_0p^t} p f_{q-2}^{s-t-1} N_0^{q-2} \right) \quad \ldots\ (40),(41), \end{aligned} \tag{47}$$

and

$$\begin{aligned} M^{j+el} = & \\ & g_{j_0p^{\frac{s}{2}},0}\left(X^{j_0p^{\frac{s}{2}}} f_{q-2}^{\frac{s}{2}+l} N_0^{q-2}\right) \quad \ldots\ (36) \\ & \cup X^{j_0p^{\frac{s}{2}}} p f_{q-2}^{\frac{s}{2}+l-1} N_0^{q-2} \quad \ldots\ (38) \\ & \cup \left(\bigcup_{s/2<t\leq s-1} X^{j_0p^t} f_{q-2}^{s-t+l} N_0^{q-2} \cup X^{j_0p^t} p f_{q-2}^{s-t+l-1} N_0^{q-2} \right) \quad \ldots\ (40),(41) \end{aligned} \tag{48}$$

for $l \geq 1$. The image of (43) is the image of

$$\Omega_F^{q-1} \ni x \overset{(d,i)}{\longmapsto} (dx, ix) \in \Omega_F^{q-1} \oplus \Omega_F^{q-2}$$

and the image of (36), (38) and (40) is

$$\Omega_F^{q-1}/Z_1^{q-1} \oplus \Omega_F^{q-1}/Z_1^{q-1} \oplus \bigoplus_{s/2<t\leq s-1} (\Omega_F^{q-1}/Z_1^{q-1} \oplus \Omega_F^{q-1}/Z_1^{q-1})$$
$$\xrightarrow{\mathrm{C}^{-(s-1)d} \oplus \mathrm{C}^{-(s-2)d} \oplus \bigoplus_{s/2<t\leq s-1}(\mathrm{C}^{-(2s-2t-1)}\, d \oplus \mathrm{C}^{-(2s-2t-2)})}$$
$$B_s^{q-1} \oplus 0 \subset \Omega_F^{q-1} \oplus \Omega_F^{q-2}.$$

Thus we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^j \text{ in } \mathrm{gr}^j(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^j(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle\psi(M^j)\rangle \cong \Omega_F^{q-1}/B_s^{q-1}. \end{aligned} \tag{49}$$

If $l \geq 1$, the image of M^{j+el} is

$$\Omega_F^{q-1}/Z_1^{q-1} \oplus \Omega_F^{q-1}/Z_1^{q-1} \oplus \bigoplus_{s/2<t\leq s-1} (\Omega_F^{q-1}/Z_1^{q-1} \oplus \Omega_F^{q-1}/Z_1^{q-1})$$
$$\xrightarrow{\mathrm{C}^{-(s+l-1)}\, d \oplus \mathrm{C}^{-(s+l-2)} \oplus \bigoplus_{s/2<t\leq s-1}(\mathrm{C}^{-(2s-2t+l-1)} \oplus \mathrm{C}^{-(2s-2t+l-2)})}$$
$$B_{s+l}^{q-1}/B_l^{q-1} \subset \Omega_F^{q-1}/B_l^{q-1}.$$

Thus we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{j+el} \text{ in } \mathrm{gr}^{j+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^{j+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{j+el})\rangle \cong \Omega_F^{q-1}/B_{s+l}^{q-1}. \end{aligned} \tag{50}$$

If s is odd and $s \geq 3$, let

$$\begin{aligned} M^j =& X^{j+e} N^{q-2} \quad \ldots\ (43) \\ &\cup g_{j_0 p^{\frac{s-1}{2}},1}\left(X^{j_0 p^{\frac{s-1}{2}}} p f_{q-2}^{\frac{s-1}{2}-1} N_0^{q-2}\right) \quad \ldots\ (33) \\ &\cup \left(\bigcup_{s/2<t\leq s-1} X^{j_0 p^t} f_{q-2}^{s-t} N_0^{q-2} \cup X^{j_0 p^t} p f_{q-2}^{s-t-1} N_0^{q-2}\right) \quad \ldots\ (40),(41), \end{aligned} \tag{51}$$

and

$$\begin{aligned} M^{j+el} =& \\ &g_{j_0 p^{\frac{s-1}{2}},1}\left(X^{j_0 p^{\frac{s-1}{2}}} p f_{q-2}^{\frac{s-1}{2}+l-1} N_0^{q-2}\right) \quad \ldots\ (33) \\ &\cup \left(\bigcup_{s/2<t\leq s-1} X^{j_0 p^t} f_{q-2}^{s-t+l} N_0^{q-2} \cup X^{j_0 p^t} p f_{q-2}^{s-t+l-1} N_0^{q-2}\right) \quad \ldots\ (40),(41) \end{aligned} \tag{52}$$

for $l \geq 1$. By the similar calculation as the case s is even, we get the same results (49) and (50).

By the definition of M^{j+el},

$$\left(\bigcup_{i\in\Gamma_a\setminus\Lambda} S^0_{i,1}\cup S^1_{i,1}\right)\cup\left(\bigcup_{i\in\Lambda_0} g_{i,0}S^0_{i,1}\cup S^1_{i,1}\right)\cup\left(\bigcup_{i\in\Lambda_1} g_{i,1}S^1_{i,1}\right)\cup\left(\bigcup_{i\in\Gamma_a} S'_{i,2}\right)$$

is equal to the union of M^{j+el} for all $e/(p-1) < j < e$ and all $l \geq 0$.

6 Explicit Calculation, Case (b)

In this section, we compute $\psi(S_{i,1})$ and $\psi(S'_{i,2})$ for $i \in \Gamma_b$.

At first, we compute ψ when $i \in \Gamma_b$ and $1 \leq i < e$. Let $e < j < e'$, $s = v_p(j)$ and $j_0 = j/p^s$. Then the integers i which satisfy $ip^{\eta_i} = j$ are

$$(i,\eta_i) = (j_0, s), (j_0p, s-1), \ldots, (j_0p^{s-1}, 1).$$

Let $i = j_0p^t$. Then $i \in \Gamma_b$ for all t. Notice that if $i \in \Gamma_b$ and $i < e$ then there exists $e < j < e'$ such that $ip^{\eta_i} = j$. But if $s = 0$ then there is no $i \in \Gamma_b$ such that $i < e$ and $ip^{\eta_i} = j$. Thus we assume $s \geq 1$ to calculate when $i < e$.

If $t < \frac{s-1}{2}$ then $S_{i,1} = \emptyset$.

If $t = (s-1)/2$, then $S^0_{i,1} = \emptyset$ and

$$S^1_{i,1} = X^i\left(pf^{\eta_i-1}_{q-2}N^{q-2}\cup\bigcup_{m=0}^{\eta_i-2} p\frac{d}{p^m}f^m_{q-3}N^{q-3}_0\right).$$

For $X^ip\omega \in S^1_{i,1}$, $\psi(X^ip\omega)$ is

$$\begin{aligned}\psi(X^ip\omega) &= j_0\pi^{j_0p^{s-1}}\frac{d\pi}{\pi}\wedge f^{\frac{s-1}{2}}_{q-2}(\omega) + j_0\pi^{j-e}\frac{d\pi}{\pi}\wedge f^{\frac{s+1}{2}}_{q-2}(\omega)\\ &\quad+\sum_{n\geq 0}\frac{1}{p^n}\pi^{ip^n}f^n_{q-1}(d\omega)\\ &\equiv j_0\pi^{j-e}\frac{d\pi}{\pi}\wedge f^{\frac{s+1}{2}}_{q-2}(\omega) + \pi^{j-e}f^{\frac{s+1}{2}}_{q-2}\left(\frac{d\omega}{p^{\frac{s-1}{2}}}\right)\\ &\qquad \bmod \operatorname{fil}^{j-e+1}(\hat\Omega^{q-1}_A/pd\hat\Omega^{q-2}_A).\end{aligned} \tag{53}$$

Note that if $X^ip\omega \in S^1_{i,1}$ then $\omega \in \mathfrak{Z}_{\eta_i-1}\hat\Omega^{q-2}_{A_0}$.

If $t = s/2$, then

$$S^0_{i,1} = X^i\left(f^{\eta_i}_{q-2}N^{q-2}\cup\bigcup_{m=0}^{\eta_i-1}\frac{d}{p^m}f^m_{q-3}N^{q-3}_0\right)$$

and

$$S^1_{i,1} = X^i\left(pf^{\eta_i-1}_{q-2}N^{q-2}\cup\bigcup_{m=0}^{\eta_i-2} p\frac{d}{p^m}f^m_{q-3}N^{q-3}_0\right).$$

For $X^i\omega \in S^0_{i,1}$, $\psi(X^i\omega)$ is

$$\psi(X^i\omega) \equiv j_0\pi^{j-e}\frac{d\pi}{\pi} \wedge f^{\frac{s}{2}}_{q-2}(\omega) + \pi^{j-e} f^{\frac{s}{2}}_{q-1}\left(\frac{d\omega}{p^{\frac{s}{2}}}\right) \mod \mathrm{fil}^{j-e+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A). \tag{54}$$

For $X^ip\omega \in S^1_{i,1}$, $\psi(X^ip\omega)$ is

$$\psi(X^ip\omega) = \sum_{n\geq 0} \frac{1}{p^n}\pi^{ip^n} f^n_{q-2}(d\omega).$$

Thus if $X^ip\omega \in S^1_{i,1} \setminus X^ipf^{\eta_i-1}_{q-2}N^{q-2}_f$ then $\psi(X^ip\omega) = 0$. Take $\omega' \in N^{q-2}_0$ such that $\omega = f^{\eta_i-1}_{q-2}f^l_{q-2}\omega'$. Then

$$\psi(X^ip\omega) \equiv p^l\pi^{j-e}f^{s-1+l}_{q-1}d\omega' \mod \mathrm{fil}^{j-e+el+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A). \tag{55}$$

If $t > s/2$, then

$$S^0_{i,1} = X^i\left(f^{\eta_i}_{q-2}N^{q-2} \cup \bigcup_{m=0}^{\eta_i-1} p\frac{d}{p^m}f^m_{q-3}N^{q-3}_0\right)$$

and

$$S^1_{i,1} = X^i\left(pf^{\eta_i-1}_{q-2}N^{q-2} \cup \bigcup_{m=0}^{\eta_i-2} p\frac{d}{p^m}f^m_{q-3}N^{q-3}_0\right).$$

For $X^i\omega \in S^0_{i,1}$, $\psi(X^i\omega)$ is

$$\psi(X^i\omega) = \sum_{n\geq 0} \frac{1}{p^{n+1}}\pi^{ip^n} f^n_{q-2}(d\omega).$$

Thus if $X^i\omega \in S^0_{i,1} \setminus X^if^{\eta_i}_{q-2}N^{q-2}_f$ then $\psi(X^i\omega) = 0$. Take $\omega' \in N^{q-2}_0$ such that $\omega = f^{\eta_i}_{q-2}f^l_{q-2}\omega'$. Then

$$\psi(X^i\omega) \equiv p^l\pi^{j-e}f^{2s-2t+l}_{q-1}d\omega' \mod \mathrm{fil}^{j-e+el+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A). \tag{56}$$

For $X^ip\omega \in S^1_{i,1}$, by the same calculation as in the case $t = s/2$,

$$\psi(X^ip\omega) \equiv p^l\pi^{j-e}f^{2s-2t+l-1}_{q-1}d\omega' \mod \mathrm{fil}^{j-e+el+1}(\hat{\Omega}^{q-1}_A/pd\hat{\Omega}^{q-2}_A). \tag{57}$$

Next, we compute $S'_{i,2}$ in the case $i \in \Gamma_b$ and $i < e$. By (4.5), $S'^{0}_{i,2}$ (resp. $S'^{1}_{i,2}$) exists when $3s/4 < t$ (resp. $(3s-2)/4 < t$). If $3s/4 < t$,

$$\begin{aligned} S'^{0}_{i,2} &= X^i \frac{dX}{X} \wedge f^{\eta_i}_{q-3} N_0^{q-3} \ni X^i \frac{dX}{X} \wedge f^{\eta_i}_{q-3}(\omega) \\ &\longmapsto \psi(X^i \frac{dX}{X} \wedge f^{\eta_i}_{q-3}(\omega)) \\ &= \pi^{ip^{\eta_i -1}} \frac{d\pi}{\pi} \wedge f^{2\eta_i -1}_{q-2}(d\omega) + \frac{1}{p} \pi^{ip^{\eta_i}} \frac{d\pi}{\pi} \wedge f^{2\eta_i}_{q-2}(d\omega) \\ &\equiv \pi^{j-e} \frac{d\pi}{\pi} \wedge f^{2s-2t}_{q-2}(d\omega) \\ &\qquad \mod \operatorname{fil}^{j-e+1}(\hat{\Omega}^{q-1}_A / pd\hat{\Omega}^{q-2}_A). \end{aligned} \tag{58}$$

If $(3s-2)/4 < t$,

$$\begin{aligned} S'^{1}_{i,2} &= X^i \frac{dX}{X} \wedge p f^{\eta_i -1}_{q-3} N_0^{q-3} \ni X^i \frac{dX}{X} \wedge p f^{\eta_i -1}_{q-3}(\omega) \\ &\longmapsto \psi(X^i \frac{dX}{X} \wedge f^{\eta_i -1}_{q-3}(\omega)) \\ &= \pi^{ip^{\eta_i -1}} \frac{d\pi}{\pi} \wedge f^{2\eta_i -2}_{q-2}(d\omega) + \frac{1}{p} \pi^{ip^{\eta_i}} \frac{d\pi}{\pi} \wedge f^{2\eta_i -1}_{q-2}(d\omega) \\ &\equiv \pi^{j-e} \frac{d\pi}{\pi} \wedge f^{2s-2t-1}_{q-2}(d\omega) \\ &\qquad \mod \operatorname{fil}^{j-e+1}(\hat{\Omega}^{q-1}_A / pd\hat{\Omega}^{q-2}_A). \end{aligned} \tag{59}$$

Next, we compute $S_{i,1}$ for $i \in \Gamma_b$ and $i > e$. Let $j = i$. Now $S^1_{i,1} = \emptyset$ and $S^0_{i,1} = X^i N^{q-2}$. For an element $X^i \omega \in S^0_{i,1}$,

$$\psi(X^i \omega) = i \pi^{i-e-1} d\pi \wedge \omega + \sum_{n \geq 0} \frac{1}{p^{n+1}} \pi^{ip^n} f^n_{q-1}(d\omega).$$

If $p \mid i$, then the first term of the right hand side is zero. Hence if $\omega = f^l_{q-2}(\omega')$ then

$$\begin{aligned} \psi(X^i \omega) &\equiv p^l \pi^{i-e} f^l_{q-1}(d\omega') \\ &\qquad \mod \operatorname{fil}^{j-e+el+1}(\hat{\Omega}^{q-1}_A / pd\hat{\Omega}^{q-2}_A) \end{aligned} \tag{60}$$

and if $\omega \in N^{q-2}_\infty$ then $\psi(X^i \omega) = 0$. If $p \nmid i$, then

$$\begin{aligned} \psi(X^i \omega) &\equiv i \pi^{i-e} \frac{d\pi}{\pi} \wedge \omega + \pi^{i-e} d\omega \\ &\qquad \mod \operatorname{fil}^{j-e+1}(\hat{\Omega}^{q-1}_A / pd\hat{\Omega}^{q-2}_A). \end{aligned} \tag{61}$$

For $i \in \Gamma_b$ and $i > e$, $S'_{i,2}$ is empty if $p \nmid i$. So assume $p \mid i$. Then, for

$X^{i-1}dX \wedge \omega \in X^{i-1}dX \wedge N_0^{q-3} = S'_{i,2} = S'^{0}_{i,2}$,

$$\begin{aligned} X^i \frac{dX}{X} \wedge \omega &= \pi^{i-e} \frac{d\pi}{\pi} \wedge d\omega \\ &\equiv \pi^{j-e} \frac{d\pi}{\pi} \wedge d\omega \quad \bmod \ \mathrm{fil}^{j-e+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}). \end{aligned} \tag{62}$$

Use (53), (54), (55), (56), (57), (58), (59), (60), (61) and (62) to define M^{j+el} for $e < j < e'$ and $l \geq 0$. Let $e < j < e'$, $s = v_p(j)$. By (3.1),

$$\mathrm{gr}^{j-e+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}) \cong \begin{cases} \hat{\Omega}_F^{q-1} \oplus \Omega_F^{q-2} & (l = 0) \\ \hat{\Omega}_F^{q-1}/B_l^{q-1} & (l \geq 1). \end{cases}$$

If $s = 0$ then let

$$\begin{aligned} &M^{j-e} = X^j N^{q-2}, \quad \dots \ (61) \\ &M^{j-e+el} = \emptyset \end{aligned} \tag{63}$$

for all $l \geq 0$. The image of M^{j-e} is the image of

$$\Omega_F^{q-2} \ni x \longmapsto (dx, jx) \in \Omega_F^{q-1} \oplus \Omega_F^{q-2},$$

hence we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{j-e} \text{ in } \mathrm{gr}^{j-e}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^{j-e}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle \psi(M^{j-e}) \rangle \\ &\quad \cong \mathrm{Coker}\left(\Omega_F^{q-2} \ni x \longmapsto (dx, jx) \in \Omega_F^{q-1} \oplus \Omega_F^{q-2}\right) \end{aligned} \tag{64}$$

and

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{j-e+el} \text{ in } \mathrm{gr}^{j-e+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^{j-e+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle \psi(M^{j-e+el}) \rangle \cong \Omega_F^{q-1}/B_l^{q-1} \end{aligned} \tag{65}$$

for $l \geq 1$ because $M^{j-e+el} = \emptyset$.

If s is even and $s \geq 2$, let

$$\begin{aligned}
&M^{j-e} = \\
&X^{j_0 p^{\frac{s}{2}}} \left(f_{q-2}^{\frac{s}{2}} N^{q-2} \cup \bigcup_{m=0}^{(s/2)-1} \frac{d}{p^m} f_{q-3}^m N_0^{q-3} \right) \quad \ldots \ (54) \\
&\cup X^{j_0 p^{\frac{s}{2}}} p f_{q-2}^{(s/2)-1} N_0^{q-2} \quad \ldots \ (55) \\
&\cup \bigcup_{\frac{s}{2} < t \leq s-1} \left(X^{j_0 p^t} f_{q-2}^{s-t} N_0^{q-2} \cup X^{j_0 p^t} f_{q-2}^{s-t-1} N_0^{q-2} \right) \quad \ldots \ (56),(57) \\
&\cup \bigcup_{\frac{3s}{4} < t \leq s-1} \left(X^{j_0 p^t} \frac{dX}{X} f_{q-3}^{s-t} N_0^{q-3} \right) \quad \ldots \ (58) \\
&\cup \bigcup_{\frac{3s-2}{4} < t \leq s-1} \left(X^{j_0 p^t} \frac{dX}{X} p f_{q-3}^{s-t-1} N_0^{q-3} \right) \quad \ldots \ (59) \\
&\cup X^j N_0^{q-2} \quad \ldots \ (60) \\
&\cup X^j \frac{dX}{X} N_0^{q-3} \quad \ldots \ (62)
\end{aligned} \tag{66}$$

and

$$\begin{aligned}
&M^{j-e+el} = \\
&X^{j_0 p^{\frac{s}{2}}} p f_{q-2}^{(s/2)-1} N_l^{q-2} \quad \ldots \ (55) \\
&\cup \bigcup_{\frac{s}{2} < t \leq s-1} \left(X^{j_0 p^t} f_{q-2}^{s-t} N_l^{q-2} \cup X^{j_0 p^t} f_{q-2}^{s-t-1} N_l^{q-2} \right) \quad \ldots \ (56),(57) \\
&\cup X^j N_l^{q-2} \quad \ldots \ (60)
\end{aligned} \tag{67}$$

for $l \geq 1$. When $l = 0$, the image of (58), (59) and (62) is

$$\left(\bigoplus_{3s/4 < t \leq s-1} \Omega_F^{q-3}/Z_1^{q-3} \right) \oplus \left(\bigoplus_{3s/4 < t \leq s-1} \Omega_F^{q-3}/Z_1^{q-3} \right) \oplus \Omega_F^{q-3}/Z_1^{q-3}$$
$$\xrightarrow{(\bigoplus_{3s/4 < t \leq s-1} \mathrm{C}^{-(2s-2t)}\, d) \oplus (\bigoplus_{(3s-2)/4 < t \leq s-1} \mathrm{C}^{-(2s-2t-1)}\, d) \oplus d}$$
$$0 \oplus B_{\frac{s}{2}}^{q-2} \subset \Omega_F^{q-1} \oplus \Omega_F^{q-2},$$

the image of (55), (56), (57) and (60) is

$$\Omega_F^{q-2}/Z_1^{q-2} \oplus \left(\bigoplus_{\frac{s}{2} < t \leq s-1} \Omega_F^{q-2}/Z_1^{q-2} \oplus \Omega_F^{q-2}/Z_1^{q-2} \right) \oplus \Omega_F^{q-2}/Z_1^{q-2}$$
$$\xrightarrow{\mathrm{C}^{-(s-1)}\, d \oplus \left(\bigoplus_{\frac{s}{2} < t \leq s-1} \mathrm{C}^{-(2s-2t)}\, d \oplus \mathrm{C}^{-(2s-2t-1)} \right) \oplus d}$$
$$B_s^{q-1} \oplus 0 \subset \Omega_F^{q-1} \oplus \Omega_F^{q-2}.$$

Furthermore, the image of (54) modulo the image of the group generated by the other generators of M^j is

$$\Omega_F^{q-2} \oplus \left(\bigoplus_{0 \le m < \frac{s}{2}} \Omega_F^{q-3}/Z_1^{q-3} \right) \xrightarrow{(\mathrm{C}^{-s}\, d, j_0\, \mathrm{C}^{-s}) \oplus \left(\bigoplus_{0\le m<\frac{s}{2}} \mathrm{C}^{-(\frac{s}{2}+m)}\, d\right)} (\mathrm{C}^{-s}\, d, j_0\, \mathrm{C}^{-s})\Omega_F^{q-2} + B_s^{q-2}/B_{\frac{s}{2}}^{q-2} \subset \Omega_F^{q-1}/B_s^{q-1} \oplus \Omega_F^{q-2}/B_{\frac{s}{2}}^{q-2}.$$

Hence we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{j-e} \text{ in } \operatorname{gr}^{j-e}(\hat\Omega_A^{q-1}/pd\hat\Omega_A^{q-2}), \\ &\operatorname{gr}^{j-e}(\hat\Omega_A^{q-1}/pd\hat\Omega_A^{q-2})/\langle \psi(M^{j-e})\rangle \\ &\cong \operatorname{Coker}\left(\Omega_F^{q-2} \ni x \longmapsto (\mathrm{C}^{-s}\, dx, j_0\, \mathrm{C}^{-s}\, x) \in \Omega_F^{q-1}/B_s^{q-1} \oplus \Omega_F^{q-2}/B_s^{q-2}\right). \end{aligned} \tag{68}$$

When $l \geq 1$, the image of M^{j-e+el} is

$$\Omega_F^{q-2}/Z_1^{q-2} \oplus \left(\bigoplus_{\frac{s}{2}<t\le s-1} \Omega_F^{q-2}/Z_1^{q-2} \oplus \Omega_F^{q-2}/Z_1^{q-2} \right) \oplus \Omega_F^{q-2}/Z_1^{q-2} \xrightarrow{\mathrm{C}^{-(s+l-1)}\, d \oplus \left(\bigoplus_{\frac{s}{2}<t\le s-1} \mathrm{C}^{-(2s-2t+l)}\, d \oplus \mathrm{C}^{-(2s-2t+l-1)}\right) \oplus \mathrm{C}^{-l}\, d} B_{s+l}^{q-1}/B_l^{q-1} \subset \Omega_F^{q-1}/B_l^{q-1}.$$

Hence we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{j-e+el} \text{ in } \operatorname{gr}^{j-e+el}(\hat\Omega_A^{q-1}/pd\hat\Omega_A^{q-2}), \\ &\operatorname{gr}^{j-e+el}(\hat\Omega_A^{q-1}/pd\hat\Omega_A^{q-2})/\langle \psi(M^{j-e+el})\rangle \cong \Omega_F^{q-1}/B_{s+l}^{q-1}. \end{aligned} \tag{69}$$

If s is odd and $s \geq 1$, let

$$\begin{aligned} M^{j-e} = {} & X^{j_0 p^{\frac{s-1}{2}}} \left(p f_{q-2}^{\frac{s-1}{2}} N^{q-2} \cup \bigcup_{m=0}^{((s-1)/2)-1} p \frac{d}{p^m} f_{q-3}^m N_0^{q-3} \right) \quad \dots (53) \\ & \cup \bigcup_{\frac{s}{2}<t\le s-1} \left(X^{j_0 p^t} f_{q-2}^{s-t} N_0^{q-2} \cup X^{j_0 p^t} f_{q-2}^{s-t-1} N_0^{q-2} \right) \quad \dots (56), (57) \\ & \cup \bigcup_{\frac{3s}{4}<t\le s-1} \left(X^{j_0 p^t} \frac{dX}{X} f_{q-3}^{s-t} N_0^{q-3} \right) \quad \dots (58) \\ & \cup \bigcup_{\frac{3s-2}{4}<t\le s-1} \left(X^{j_0 p^t} \frac{dX}{X} p f_{q-3}^{s-t-1} N_0^{q-3} \right) \quad \dots (59) \\ & \cup X^j N_0^{q-2} \quad \dots (60) \\ & \cup X^j \frac{dX}{X} N_0^{q-3} \quad \dots (62) \end{aligned} \tag{70}$$

and

$$\begin{aligned} M^{j-e+el} = & \bigcup_{\frac{s}{2}<t\le s-1} \left(X^{j_0p^t} f_{q-2}^{s-t} N_l^{q-2} \cup X^{j_0p^t} f_{q-2}^{s-t-1} N_l^{q-2} \right) \quad \dots (56), (57) \\ & \cup X^j N_l^{q-2} \quad \dots (60) \end{aligned} \tag{71}$$

for $l \ge 1$. By the similar calculation to the case s is even, we get the same result (68) and (69).

By the definition of M^{j+el},

$$\left(\bigcup_{i\in\Gamma_b} S_{i,1}^0 \cup S_{i,1}^1 \right) \cup \left(\bigcup_{i\in\Gamma_b} S'_{i,2} \right)$$

is equal to the union of M^{j-e+el} for all $e < j < e'$ and all $l \ge 0$.

7 Explicit Calculation, Case (c)

In this section, we compute $\psi(S_{i,1})$ and $\psi(S'_{i,2})$ for $i \in \Gamma_c$.

Γ_c has only two elements, $e/(p-1)$ and e'. At first let $i = e/(p-1)$. Then $S_{i,1}^0 = \emptyset$, $S'_{i,2} = \emptyset$ and

$$S_{i,1}^1 = X^i p N^{q-2}.$$

Note that this i has the property $i = ip - e$. Take $X^i p\omega \in X^i p N^{q-2}$, then

$$\begin{aligned} \psi(X^i p\omega) = & i\pi^i \frac{d\pi}{\pi} \wedge (\omega + f_{q-2}(\omega)) + \pi^i (d\omega + f_{q-1}(d\omega)) \\ & + \sum_{n\ge 2} \frac{1}{p^n} \pi^{ip^n} f_{q-1}^n (d\omega). \end{aligned}$$

If $\omega + f_{q-2}(\omega) \equiv 0 \bmod p$ then the leftmost term of the right hand side vanishes. But $\omega + f_{q-2}(\omega) \equiv 0$ means $\bar\omega + C^{-1}\bar\omega = 0$ in $\hat\Omega_F^{q-2}$, thus $d\omega = 0$ hence $\psi(X^i p\omega) = 0$ by the property of N_0^{q-2}, see (25). So we get

$$\psi(X^i p\omega) \begin{cases} \equiv i\pi^i \frac{d\pi}{\pi} \wedge (\omega + f_{q-2}(\omega)) + \pi^i (d\omega + f_{q-1}(d\omega)) \\ \qquad \bmod \operatorname{fil}^{i+1}(\hat\Omega_A^{q-1}/pd\hat\Omega_A^{q-2}) \\ \qquad (\text{ if } \omega + f_{q-2}(\omega) \not\equiv 0 \mod p) \\ = 0 \quad (\text{ if } \omega + f_{q-2}(\omega) \equiv 0 \mod p) \end{cases} \tag{72}$$

Next, let $i = e'$. Then $S_{i,1}^0 = X^i N_{q-2}$, $S'^0_{i,2} = X^{i-1} dX \wedge N_0^{q-3}$ and $S_{i,1}^1 = S'^1_{i,2} = \emptyset$. For $X^i\omega \in X^i N_{q-2}$,

$$\psi(X^i\omega) = \sum_{n\ge 0} \frac{1}{p^{n+1}} \pi^{ip^n} f_{q-1}^n (d\omega).$$

Thus if $\omega \in N_\infty^{q-2}$ then $\psi(X^i\omega) = 0$ and if $\omega = f_{q-2}^l\omega'$ for $\omega' \in N_0^{q-2}$ then

$$\psi(X^i f_{q-2}^l \omega') \equiv p^l \pi^{i-e} f_{q-1}^l(d\omega') \mod \mathrm{fil}^{i-e+el+1}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}). \tag{73}$$

For $X^{i-1}dX \wedge \omega \in X^{i-1}dX \wedge N_0^{q-3}$,

$$\psi(X^{i-1}dX \wedge \omega) = \pi^{i-e}\frac{d\pi}{\pi} \wedge d\omega. \tag{74}$$

Use (72), (73) and (74) to define M^{j+el} for $j = e/(p-1)$ and $l \geq 0$.

$$\begin{aligned} M^{\frac{e}{p-1}} =& X^{\frac{e}{p-1}} p N^{q-2} \setminus \left\{\omega \middle| \omega + f_{q-2}\omega \equiv 0 \mod p\right\} \ \dots (72) \\ &\cup X^{e'} N_0^{q-2} \ \dots (73) \\ &\cup X^{e'}\frac{dX}{X} \wedge N_0^{q-3} \ \dots (74) \end{aligned}$$

and let

$$M^{\frac{e}{p-1}+el} = X^{e'} N_l^{q-2} \ \dots (73).$$

By (3.1),

$$\mathrm{gr}^{e/(p-1)+el}\, \hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2} \cong \begin{cases} \Omega_F^{q-1} \oplus \Omega_F^{q-2} & (\text{ if } l = 0) \\ \Omega_F^{q-1}/B_l^{q-1} & (\text{ if } l \geq 1). \end{cases}$$

When $l = 0$, the image of (73) and (74) is

$$\Omega_F^{q-2}/Z_1^{q-2} \oplus \Omega_F^{q-3}/Z_1^{q-3} \xrightarrow{d \oplus d} \Omega_F^{q-1} \oplus \Omega_F^{q-2}$$

and the image of (72) modulo the subgroup generated by (73) and (74) is

$$\begin{aligned} &\Omega_F^{q-2}/Z_1^{q-2} \xrightarrow{\left((1+\mathrm{C}^{-1})d, \frac{e}{p-1}(1+\mathrm{C}^{-1})\right)} \\ &\Omega_F^{q-1}/B_1^{q-1} \oplus \Omega_F^{q-2}/B_1^{q-2} \cong \Omega_F^{q-1}/(1+\mathrm{C})B_1^{q-1} \oplus \Omega_F^{q-2}/(1+\mathrm{C})B_1^{q-2}. \end{aligned}$$

Here $\Omega_F^{q-1}/B_1^{q-1} \cong \Omega_F^{q-1}/(1+\mathrm{C})B_1^{q-1}$ follows from $\mathrm{C}(B_1^{q-1}) = 0$. Hence we get

$$\begin{aligned} &\psi(x) \neq 0 \text{ for } x \in M^{e/(p-1)} \text{ in } \mathrm{gr}^{e/(p-1)}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}), \\ &\mathrm{gr}^{e/(p-1)}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{e/(p-1)})\rangle \\ &\cong \mathrm{Coker}\begin{pmatrix} \Omega_F^{q-2} \longrightarrow \Omega_F^{q-1}/(1+\mathrm{C})B_1^{q-1} \oplus \Omega_F^{q-2}/(a+\mathrm{C})B_1^{q-2} \\ x \longmapsto \left((1+\mathrm{C})C^{-1}dx, \dfrac{e}{p-1}(1+\mathrm{C})\,\mathrm{C}^{-1}\,x\right) \end{pmatrix}. \end{aligned} \tag{75}$$

When $l \geq 1$, the image of (73) is

$$\Omega_F^{q-2}/Z_1^{q-2} \xrightarrow{\mathrm{C}^{-l}\,d} B_{l+1}^{q-1}/B_l^{q-1} \subset \Omega_F^{q-1}/B_l^{q-1}.$$

Hence we get

$$\begin{aligned}&\psi(x)\neq 0 \text{ for } x\in M^{e/(p-1)+el} \text{ in } \operatorname{gr}^{e/(p-1)+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2}),\\&\operatorname{gr}^{e/(p-1)+el}(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{e/(p-1)+el})\rangle\cong\Omega_F^{q-1}/B_{l+1}^{q-1}.\end{aligned}\tag{76}$$

By the definition of $M^{e/(p-1)+el}$,

$$\left(\bigcup_{i\in\Gamma_c}S_{i,1}^0\cup S_{i,1}^1\right)\cup\left(\bigcup_{i\in\Gamma_b}S'_{i,2}\right)$$

is equal to the union of $M^{e/(p-1)+el}$ for all $l\geq 0$.

8 THE STRUCTURE OF THE MILNOR K-GROUP

Proof of Theorem 1.1. At first, assume $\zeta_p\in K$, there exists a prime element π of K such that $\pi^e=p$ and the residue field F has a finite p-base. By the definition of M^n, the union of all M^n for $n\geq 1$ and $n/e\notin\mathbb{Z}$ generates $\psi(H^1(\mathbb{S}_q))$. M^{el} for $l\geq 0$ is not defined yet, so let $M^{el}=\emptyset$. Then M^n is defined for all $n\geq 1$. There is map

$$\left\langle\bigcup_{n\geq i}\psi(M^n)\right\rangle\Big/\left\langle\bigcup_{n\geq i+1}\psi(M^n)\right\rangle\longrightarrow\operatorname{gr}^i(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})\tag{77}$$

for each $i\geq 0$. By the exact sequence of (2.4), if (77) are injective for all $i\geq 0$ then

$$\langle\psi(M^i)\rangle\longrightarrow\operatorname{gr}^i(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})\xrightarrow{\exp_p}\operatorname{gr}^{i+e}K_q^M(K)$$

are also exact for all $i\geq 0$. We already know $\psi(x)\neq 0$ for $x\in M^i$ in $\operatorname{gr}^i(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})$ and what is the group $\langle\psi(M^i)\rangle$ in $\operatorname{gr}^i(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})$ for all $i\geq 0$ by (45), (46), (49), (50), (64), (65), (68), (69), (75) and (76). The results are as follows:

$$\operatorname{gr}^{j+el}(\hat{\Omega}_A^{q-1}pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{j+el})\rangle\cong\Omega_F^{q-1}/B_{s+l}^{q-1}\quad(\text{ if }\frac{e}{p-1}<j<e,l\geq 0),\tag{78}$$

where $s=v_p(j)+1$.

$$\begin{aligned}&\operatorname{gr}^{j-e+el}(\hat{\Omega}_A^{q-1}pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{j+el})\rangle\\&\cong\begin{cases}\operatorname{Coker}\begin{pmatrix}\Omega_F^{q-2}\to\Omega_F^{q-1}/B_s^{q-1}\oplus\Omega_F^{q-2}/B_s^{q-2}\\ x\mapsto(\mathrm{C}^{-s}\,dx,j_0\,\mathrm{C}^{-s}\,x)\end{pmatrix} & (\text{ if }e<j<e',l=0)\\ \Omega_F^{q-1}/B_{s+l}^{q-1} & (\text{ if }e<j<e',l\geq 1),\end{cases}\end{aligned}\tag{79}$$

where $s = v_p(j)$ and $j_0 = j/p^s$.

$$\begin{aligned}&\operatorname{gr}^{e/(p-1)+el}(\hat{\Omega}_A^{q-1} pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{e/(p-1)+el})\rangle\\ &\cong \begin{cases}\operatorname{Coker}\begin{pmatrix}\Omega_F^{q-2} \to \Omega_F^{q-1}/(1+\mathrm{C})B_1^{q-1} \oplus \Omega_F^{q-2}/(1+\mathrm{C})B_1^{q-2}\\ x \mapsto ((1+\mathrm{C})\,\mathrm{C}^{-1}\,dx, \dfrac{e}{p-1}(1+\mathrm{C})\,\mathrm{C}^{-1}\,x)\end{pmatrix} & (\text{ if } l = 0)\\ \Omega_F^{q-1}/B_{1+l}^{q-1} \quad (\text{ if } l \geq 1).\end{cases}\end{aligned} \tag{80}$$

$$\operatorname{gr}^{el}(\hat{\Omega}_A^{q-1} pd\hat{\Omega}_A^{q-2})/\langle\psi(M^{el})\rangle \cong \operatorname{gr}^{el}(\hat{\Omega}_A^{q-1} pd\hat{\Omega}_A^{q-2}) \cong \Omega_F^{q-1}/B_l^{q-1}(\text{ for } l \geq 0). \tag{81}$$

Let $n \geq 1$ and k be the integer which satisfies $e/(p-1) \leq n - ke < e'$. If $1 \leq n \leq e/(p-1)$, then the results of (79) with $l = 0$ and (80) with $l = 0$ is coincide with the result of [3] by $\operatorname{gr}^{n+e} K_q^M(K) \cong \operatorname{gr}^n(\hat{\Omega}_A^{q-1}/pd\hat{\Omega}_A^{q-2})/\langle\psi(M^n)\rangle$. Let $n > e/(p-1)$. Then (78), (79), (80) and (81) say

$$\operatorname{gr}^n(\hat{\Omega}_A^{q-1} pd\hat{\Omega}_A^{q-2})/\langle\psi(M^n)\rangle \cong \Omega_F^{q-1}/B_{s'+1+k}^{q-1},$$

where $s' = v_p(n - ke)$. Hence we have

$$\operatorname{gr}^{n+e} K_q^M(K) \cong \Omega_F^{q-1}/B_{s'+1+k}^{q-1}$$

and we get Theorem (1.1) by shifting degrees.

We prove Theorem (1.1) in the case K does not contain primitive p-th roots of unity ζ_p or K does not contain a prime element π such that $\pi^e = p$ as follows. Let $L = K(\zeta_p, \sqrt[e]{p})$ and let $m = [L : K]$. Then $p \nmid m$ and the extention L/K is unramified. By using standard norm argument, the composite map

$$\operatorname{gr}^i K_q^M(K) \longrightarrow \operatorname{gr}^{im} K_q^M(L) \xrightarrow{\text{Norm}} \operatorname{gr}^i K_q^M(K)$$

is the multiplication by m, hence injective. Furthermore, F_L/F_K is a finite separable extension, where F_L (resp. F_K) is the residue field of L (resp. K), we get $\Omega_{F_L}^{q-1}/B_l\Omega_{F_L}^{q-1} \cong \Omega_{F_K}^{q-1}/B_l\Omega_{F_K}^{q-1} \otimes_{F_K^{p^l}} F_L$. Thus Theorem (1.1) follows even if $\zeta_p \notin K$.

Lastly, do not assume that the residue field of K has a finite p-base. Then an inductive system of complete discrete valuation fields whose residue fields has a finite p-base and its limit is isomorphic to K exists by [9] Section 1.5. On the other hand, for a purely transcendental extension or a separable extension F'/F,

$$\Omega_F^q/B_l\Omega_F^q \longrightarrow \Omega_{F'}^q/B_l\Omega_{F'}^q$$

are injective for all q and l because, if F'/F is separable extension, then $\Omega_{F'}^q = F' \otimes_F \Omega_F^q$ and if F'/F is purely transcendental extension $F' = F(T)$ then $\Omega_{F'}^q = (F' \otimes_F \Omega_F^q) \oplus (F' \otimes_F \Omega_F^{q-1} \wedge dT)$. Hence we get Theorem (1.1) by taking inductive limit. □

To prove Corollary (1.2), we need the following

LEMMA 8.1. *Assume $\zeta_p \in K$. Let $V = \mathrm{Im}(\{\zeta_p, *\}\colon\ K_{q-1}^M(K)/p \to K_q^M(K)\hat{})$. Then the sequence*

$$\begin{aligned}&0 \longrightarrow V \cap U^i K_q^M(K)\hat{} \cap p^n K_q^M(K)\hat{} \longrightarrow U^i K_q^M(K)\hat{} \cap p^n K_q^M(K)\hat{} \\ &\longrightarrow U^{i+e} K_q^M(K)\hat{} \cap p^{n+1} K_q^M(K)\hat{} \longrightarrow 0\end{aligned}$$

is exact for $i > e/(p-1)$.

Proof. Restricting the bottom row of (3) to the filtration of $K_q^M(K)$, we have the exact sequence

$$0 \longrightarrow V \cap U^i K_q^M(K)\hat{} \longrightarrow U^i K_q^M(K)\hat{} \xrightarrow{\ p\ } U^{i+e} K_q^M(K)\hat{} \longrightarrow 0$$

and hence we get the exact sequence

$$\begin{aligned}&0 \longrightarrow V \cap U^i K_q^M(K)\hat{} \cap p^n K_q^M(K)\hat{} \longrightarrow U^i K_q^M(K)\hat{} \cap p^n K_q^M(K)\hat{} \\ &\longrightarrow U^{i+e} K_q^M(K)\hat{} \cap p^{n+1} K_q^M(K)\hat{}.\end{aligned} \tag{82}$$

We only have to show the surjectivity of the last arrow of (82). Take $p^{n+1}x \in U^{i+e} K_q^M(K)\hat{} \cap p^{n+1} K_q^M(K)\hat{}$. By the surjectivity of the multiplication by p map $U^i K_q^M(K)\hat{} \to U^{i+e} K_q^M(K)$, there exists $y \in U^i K_q^M(K)\hat{}$ such that $p(y - p^n x) = 0$. This $y - p^n x$ is a p-torsion element of $K_q^M(K)\hat{}$, thus $y - p^n x \in V \subset U^{e/(p-1)} K_q^M(K)\hat{}$. Hence $p^n x \in U^{e/(p-1)} K_q^M(K)\hat{}$ because $y \in U^i K_q^M(K)\hat{}$. Now $e/(p-1)$ is prime to p, thus $\mathrm{gr}^{e/(p-1)}\, K_q^M(K)\hat{} \cong \mathrm{gr}^{e/(p-1)}(K_q^M(K)/p^n)$ by [3], and $p^n x$ goes to zero on this map. Hence we get $p^n x \in U^{e/(p-1)+1} K_q^M(K)\hat{}$. Let $j = (e/(p-1)) + 1$. By the definition, all rows and columns in the following commutative diagram are exact:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{V\cap U_\infty^j \cap (p^n)}{V\cap U_\infty^i \cap (p^n)} & \longrightarrow & \frac{U_\infty^j \cap (p^n)}{U_\infty^i \cap (p^n)} & \xrightarrow{\ p\ } & \frac{U_\infty^{j+e} \cap (p^{n+1})}{U_\infty^{i+e} \cap (p^{n+1})} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{V\cap U_\infty^j}{V\cap U_\infty^i} & \longrightarrow & \frac{U_\infty^j}{U_\infty^i} & \xrightarrow{\ p\ } & \frac{U_\infty^{j+e}}{U_\infty^{i+e}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{V_{p^n}\cap U_{p^n}^j}{V_{p^n}\cap U_{p^n}^i} & \longrightarrow & \frac{U_{p^n}^j}{U_{p^n}^i} & \xrightarrow{\ p\ } & \frac{U_{p^{n+1}}^{j+e}}{U_{p^{n+1}}^{i+e}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where we denote $V_{p^n} = \mathrm{Im}(V \to K_q^M(K)/p^n)$, $U_\infty^m = U^m K_q^M(K)\hat{}$, $U_{p^n}^n = U^n(K_q^M(K)/p^n)$ and $(p^n) = p^n K_q^M(K)\hat{}$ only in this diagram. $p^n x$ is in the middle group of the top row and goes to zero by multiplication by p. Thus there exists $z \in V \cap U^j K_q^M(K)\hat{}\cap p^n K_q^M(K)\hat{}$ such that $p^n x - z \equiv 0$ modulo $U^i K_q^M(K)\hat{}$. Furthermore, $z \in p^n K_q^M(K)\hat{}$ implies $p^n x - z \in U^i K_q^M(K)\hat{} \cap p^n K_q^M(K)\hat{}$, thus

$$\begin{array}{rcl} U^i K_q^M(K)\hat{} \cap p^n K_q^M(K)\hat{} & \xrightarrow{p} & U^{i+e} K_q^M(K)\hat{} \cap p^{n+1} K_q^M(K)\hat{} \\ p^n x - z & \longmapsto & p^{n+1}x - pz = p^{n+1}x. \end{array}$$

Hence surjectivity of the last arrow of (82) follows. □

Corollary 8.2. *All rows and columns are exact in the following commutative diagram :*

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ \frac{U^i K_q^M(K)\hat{}\cap p^n K_q^M(K)\hat{}}{U^{i+1} K_q^M(K)\hat{}\cap p^n K_q^M(K)\hat{}} & \longrightarrow & \frac{U^{i+e} K_q^M(K)\hat{}\cap p^{n+1} K_q^M(K)\hat{}}{U^{i+e+1} K_q^M(K)\hat{}\cap p^{n+1} K_q^M(K)\hat{}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \operatorname{gr}^i K_q^M(K)\hat{} & \xrightarrow{p} & \operatorname{gr}^{i+e} K_q^M(K)\hat{} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \operatorname{gr}^i (K_q^M(K)/p^n) & \xrightarrow{p} & \operatorname{gr}^{i+e} (K_q^M(K)/p^{n+1}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0. & & \end{array} \tag{83}$$

Proof. Exactness of the top row comes from (8.1). □

Proof of Corollary 1.2. Denote $\operatorname{Ker}(\operatorname{gr}^i K_q^M(K)\hat{} \to \operatorname{gr}^i K_q^M(K)/p^{n+1})$ by $G_{i,n+1}$. At first, we prove Corollary (1.2) for $e' < i \leq e' + e$. Let $s = v_p(i-e)$ and $i_0 = (i-e)/p^s$. Then we know all $\operatorname{gr}^{i-e} K_q^M(K)\hat{}$ and $\operatorname{gr}^{i-e}(K_q^M(K)/p^n)$ by [3], thus (83) is, if $n \leq s$ and $i \neq e' + e$ then

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ Z_n^{q-1} \oplus Z_n^{q-2} & \longrightarrow & G_{i,n+1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \frac{\Omega_F^{q-1}/B_s^{q-1} \oplus \Omega_F^{q-2}/B_s^{q-2}}{(\mathrm{C}^{-s}\, d, i_0\, \mathrm{C}^{-s})\Omega_F^{q-2}} & \longrightarrow & \Omega_F^{q-1}/B_{s+1}^{q-1} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \frac{\Omega_F^{q-1}}{Z_n^{q-1}} \oplus \frac{\Omega_F^{q-2}}{Z_n^{q-2}} & \longrightarrow & \operatorname{gr}^i (K_q^M(K)/p^{n+1}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

here all maps are natural maps, and if $n \leq s$ and $i = e' + e$ then

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
(1+aC)Z_n^{q-1} \oplus (1+aC)Z_n^{q-2} & \longrightarrow & G_{i,n+1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\frac{\Omega_F^{q-1}/(1+aC)B_s^{q-1} \oplus \Omega_F^{q-2}/(1+aC)B_s^{q-2}}{((1+aC)\,\mathrm{C}^{-s}\,d, i_0(1+aC)\,\mathrm{C}^{-s})\Omega_F^{q-2}} & \longrightarrow & \Omega_F^{q-1}/B_{s+1}^{q-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\frac{\Omega_F^{q-1}}{(1+aC)Z_n^{q-1}} \oplus \frac{\Omega_F^{q-2}}{(1+aC)Z_n^{q-2}} & \longrightarrow & \mathrm{gr}^i(K_q^M(K)/p^{n+1}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

where a is the residue class of p/π^e. We get (1.2) in this case by these diagrams. If $n > s$ then $\mathrm{gr}^i K_q^M(K)\hat{} \to \mathrm{gr}^i(K_q^M(K)/p^n)$ is an isomorphism, thus $\mathrm{gr}^{i+e} K_q^M(K)\hat{} \to \mathrm{gr}^{i+e}(K_q^M(K)/p^{n+1})$ is also an isomorphism.

By induction on i and calculating the diagram (83) for each case, we get (1.2).

□

9 An application

THEOREM 9.1. *Let K be a Henselian discrete valuation field of mixed characteristics $(0, p > 2)$ with the residue field F. Assume $p \nmid e$ and $[F : F^p] = p^{q-1}$, where $e = v_K(p)$. Let L/K be a ferociously ramified cyclic extention of order p^n (i.e., the extention of the residue fields is inseparable of order p^n). Then $p^n \leq e'$, where $e' = ep/(p-1)$.*

Remark 9.2. In [15] and [6], they give the upper bounds of such extensions. If K has the property $p \nmid e$, our bound is stricter than them (or equal to [6] if e is small).

Proof. We use the notation $U^i_{p^n} = U^i(K_q^M(K)/p^n)$ for simplicity. The proof goes similarly to the argument of [15] Section 3. By the limit argument, we may assume F is a field of transcendental degree $q-1$ over $\mathbb{F}_p$. Then $H^{q+1}(K, \mathbb{Z}/p(q))$ is non zero by [10] and furthermore we know that $H^{q+1}(K, \mathbb{Z}/p^n(q))$ has an elements of order p^n by using Bockstein.

Let L/K be a cyclic extension of order p^n and let $\chi \in H^1(K, \mathbb{Z}/p^n)$ be the character which coincide with L/K. Let ϕ_χ be the homomorphism

$$\phi_\chi \colon K_q^M(K)/p^n \longrightarrow H^{q+1}(K, \mathbb{Z}/p^n(q))$$

which is induced by the pairing

$$H^1(K,\mathbb{Z}/p^n)\times K_q^{M}(K)/p^n \longrightarrow H^{q+1}(K,\mathbb{Z}/p^n(q))$$

by using $K_q^{M}(K)/p^n \cong H^q(K,\mathbb{Z}/p^n(q))$. If L/K is ferociously ramified, by [15] Section 3, we know

$$\phi_\chi : U^1_{p^n} \longrightarrow H^{q+1}(K,\mathbb{Z}/p^n(q))$$

is surjective and

$$\phi_\chi(\{1+\pi^i x, y_1,\ldots,y_{q-1}\}) \in \phi_\chi(U^{i+1}_{p^n}) \tag{84}$$

for any $x,y_1,\ldots,y_{q-1}\in\mathcal{O}_K^\times$ and $i\geq 1$. Theorem (1.1) says that $U^{e'+1}_{p^n}$ is generated by the elements of the form of the left hand side of (84), thus we get

$$\phi_\chi\colon U^p_{p^n}/U^{e'+1}_{p^n} \longrightarrow H^{q+1}(K,\mathbb{Z}/p^n(q))$$

is defined and surjective. Furthermore, for any element $\{1+\pi^i x,y_1,\ldots,y_{q-2},\pi\}\in U^1_{p^n}$ for $x,y_1,\ldots,y_{q-2}\in\mathcal{O}_K^\times$ and $i\geq p$, its order modulo $U^{e'+1}_{p^n}$ is less than or equal to p^l by [3] Theorem 1.4, where l be the maximal integer which satisfies $p^l\leq e'$. Thus he maximal order of the elements of $U^p_{p^n}$ modulo $U^{e'+1}_{p^n}$ is less than or equal to p^l. On the other hand, $H^{q+1}(K,\mathbb{Z}/p^n(q))$ has a element of order p^n, thus $n\leq l$. This is the inequality which we desired.

Note that there exists elements of $U^p_{p^n}/U^{e'+1}_{p^n}$ of order p^n, for example, $\{1+\pi^p T_1,T_2,\ldots,T_{q-1},\pi\}$, where $\{T_1,\ldots,T_{q-1}\}$ are the liftings of a p-base of F. Thus the maximal order of the elements of $U^p_{p^n}/U^{e'+1}_{p^n}$ is p^l. □

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