DISTRIBUTIVE COSET GRAPHS OF FINITE COXETER GROUPS

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Abstract. Let \( W \) be a finite Coxeter group, \( W_J \) a parabolic subgroup of \( W \) and \( X_J \) the set of distinguished coset representatives of \( W_J \) in \( W \) equipped with the induced weak Bruhat ordering of \( W \). All instances when \( X_J \) is a distributive lattice are known. In this note we present a short conceptual proof of this result.

1. Introduction.

Throughout, \( W \) denotes a finite Coxeter group, generated by a set of simple reflections \( S \subseteq W \). For \( J \subseteq S \), let \( W_J \) be the parabolic subgroup of \( W \) generated by \( J \). It is well-known that \( W \) is a lattice when equipped with the weak Bruhat order [2, Thm. 8], see Lemma 2.1. Let \( X_J \) denote the set of distinguished (right) coset representatives of \( W_J \) in \( W \) endowed with the induced (right) weak Bruhat ordering from \( W \). All instances when \( X_J \) is a distributive lattice are known:

1.1. Theorem. Let \( W \) be a finite irreducible Coxeter group and let \( J \subseteq S \). Then \( X_J \) is a distributive lattice if and only if one of the following holds:

(i) \( W \) is a Weyl group and \( W_J \) is minuscule;
(ii) \( W \) is dihedral and \( W_J \) is of type \( A_1 \);
(iii) \( W \) is of type \( H_3 \) and \( W_J \) is of type \( I_2(5) \).

In case \( W \) is a Weyl group we say that \( W_J \) is minuscule provided \( W_J \) is the stabiliser of a minuscule dominant weight of \( W \).

In case \( W \) is a Weyl group Theorem 1.1 was first proved by R.A. Proctor [6, Prop. 3.1, 3.2] in a case by case analysis. More specifically, in [6, Prop. 3.2] Proctor lists each \( X_J \) from Theorem 1.1 as the poset of all order ideals of some explicit poset. Thanks to [7, Thm. 3.4.1] the latter are known to be distributive lattices. In [8] J.R. Stembridge introduces the notion of fully commutative elements for arbitrary Coxeter groups. Using this concept he presents an a priori proof of Theorem 1.1, see [8, Thm. 7.1].

In this note we give a short conceptual proof of Theorem 1.1 utilising a Mackey type induction formula involving the structure constants of the descent algebra of \( W \).
approach is new and free of case by case considerations, showing that in all cases not listed above, $X_J$ is not distributive. We do make use of the classification of the irreducible Coxeter groups and the structure of the root systems of Weyl groups.

After a preliminary section we interpret the property that $X_J$ is not distributive in terms of the positivity of certain structure constants of the descent algebra of $W$. In Section 3 we derive inductive tools for the positivity of the relevant structure constants. Our proof of Theorem 1.1 is based on this study.

Let $J$ be the coset graph of the poset $X_J$, i.e., the corresponding Hasse diagram. Continuing to investigate the structure constants of the descent algebra of $W$ we obtain the surprising consequence that for each simple reflection $s \in S$, the coset graph $\Gamma_{[s]}$ contains the full Cayley graph of every maximal parabolic subgroup of $W$ as a subgraph, see Corollary 3.11.

Concerning the importance and ubiquity of minuscule lattices in Lie theory and combinatorial theory, see for instance [6, §11].

### 2. Notation and Preliminaries.

We maintain the notation from the introduction. For $W = (S)$ a Coxeter group, $|S|$ is its rank. For $J \subseteq S$, denote by $X_J = \{x \in W \mid l(wx) = l(w) + l(x) \text{ for all } w \in W_J\} = \{x \in W \mid l(sx) > l(x) \text{ for all } s \in J\}$ the set of distinguished right coset representatives of $W_J$ in $W$. Then, for $K \subseteq S$, the set $X_K^{-1}$ is a set of distinguished left coset representatives of $W_K$ in $W$, and $X_{JK} = X_J \cap X_K^{-1}$ is a set of double coset representatives of $W_J$ and $W_K$ in $W$. Denote $a_{JKL} = |\{x_{JKL}\}|$, where $X_{JKL} = \{d \in X_{JK} \mid J^d \cap K = L\} = \{d \in X_{IK} \mid W_J^d \cap W_K = W_L\}$ for $L \subseteq S$. Further, let $x_J = \sum_{x \in X_J} x^{-1} \in QW$, where $QW$ denotes the group algebra of $W$ over $Q$. Then, by a theorem of Solomon,

$$x_J x_K = \sum_{L \subseteq S} a_{JKL} x_L$$

(e.g. see [5, (2.1.10)]). The set $\{x_J \mid J \subseteq S\}$ thus forms a basis of a subalgebra of $QW$, the descent algebra of $W$.

By $\preceq$, we denote the weak (right) Bruhat order on $W$, i.e., for $x, y \in W$ we have $x \preceq y$ if $x$ is a prefix of $y$: $l(y) = l(x) + l(x^{-1}y)$. Thanks to [2, Thm. 8], the poset $(W, \preceq)$ is a lattice, i.e., any two elements $x, y \in W$ have a greatest lower and a least upper bound. Since $X_J$ is an interval in $(W, \preceq)$, the subposet $(X_J, \preceq)$, obtained by restricting $\preceq$ to $X_J$, is a lattice as well. It is well-known that a lattice is distributive if and only if it does not contain the so called “pentagon lattice” or the “diamond lattice” as a sublattice.

For the sake of completeness we include a proof of the lattice property.

**2.1. Lemma.** $(W, \preceq)$ is a lattice.
Proof. Denote by \((w) = \{u \in W \mid u \leq w\}\) the order ideal generated by \(w \in W\). Let \(w_1, w_2 \in W\). We claim that \((w_1) \cap (w_2) = (w)\) for some \(w \in W\). Let \(J = (w_1) \cap (w_2) \cap S\). Then every common prefix \(p \in (w_1) \cap (w_2)\) has a prefix in \(J\). If \(J = \emptyset\) then \(w = 1\) and we are done. Otherwise, choose \(s \in J\). Then, by induction on \(l(w_1)\), we have \((sw_1) \cap (sw_2) = (x)\) for some \(x \in W\) and \(sx\) is a common prefix of \(w_1\) and \(w_2\). It remains to show that \(p \leq sx\) for all \(p \in (w_1) \cap (w_2)\). If \(p \neq 1\) then we write \(p = t \cdot p'\) for some \(t \in S \cap (p)\). Again, by induction, \((tw_1) \cap (tw_2) = (y)\) for some \(y \in W\) and \(p \leq ty\). Let \(K = \{s, t\}\). Then the longest element \(w_K\) of \(W_K\) is a common prefix of \(w_1\) and \(w_2\) (see e.g., [5, (1.2.1)]). Once more by induction, we get \((w_Kw_1) \cap (w_Kw_2) = (z)\) for some \(z \in W\). The uniqueness of \(x\) and \(y\) requires \(x = sw_Kz\) and \(y = tw_Kz\). Therefore \(p \leq ty = w_Kz = sx\).

Let \(w_0\) be the longest element of \(W\). Let \(z\) be the greatest lower bound of \(w_0w_1^{-1}\) and \(w_0w_2^{-1}\). Then \(w_0z^{-1}\) is the least upper bound of \(w_1\) and \(w_2\). \(\square\)

Observe that the existence of greatest lower bounds in \(W\) in the proof of Lemma 2.1 does not require the finiteness assumption on \(W\).

2.2. Lemma. Let \(J \subseteq S\).

(i) The subgroup \(W_J\) is normal in \(W\) if and only if \(W = W_1 \times W_{S \setminus J}\).

(ii) If \(W\) is irreducible and \(J \subsetneq S\) then the action of \(W\) on the cosets of \(W_J\) is faithful.

Proof. (i) The Exchange Condition implies \(st = ts\) for all \(s \in J\) and all \(t \in S \setminus J\).

(ii) The kernel of the action of \(W\) on the cosets of \(W_J\) is the normal parabolic subgroup \(H = \bigcap_{w \in W} W_{w}^w\) of \(W\). By (i), we must have \(H = \{1\}\) since \(W\) is irreducible. \(\square\)

Now suppose that \(W\) is an irreducible Weyl group, i.e., a finite crystallographic reflection group. A non-zero dominant weight \(\lambda\) of \(W\) is called minuscule provided there is no other dominant weight \(\mu\) satisfying \(\mu < \lambda\), where \(\leq\) is the usual partial ordering on weights [4, Ch. VIII, §6.2]. The list of minuscule weights is well-known, see [4, Ch. VIII, §7.3]. For \(J \subseteq S\) we call \(W_J\) minuscule provided \(W_J\) is the stabiliser in \(W\) of a minuscule weight. Note this definition implies that Coxeter groups of type \(BC_r\) have two distinct minuscule parabolic subgroups; one arising from the minuscule weight of the \(B_r\) root system, the other from the minuscule weight of the \(C_r\) root system.

Let \(\Psi\) be a crystallographic root system. Let \(\Pi\) be a set of simple roots and \(\Psi^+\) the set of positive roots of \(\Psi\) with respect to \(\Pi\). For \(\beta \in \Psi^+\) we write \(\beta = \sum n_\alpha(\beta)\alpha\), where the sum is taken over \(\alpha \in \Pi\) and \(n_\alpha(\beta) \in \mathbb{Z}_{\geq 0}\). The highest root of \(\Psi\) is denoted by \(\rho\).

2.3. Remark. For every Weyl group \(W = \langle S \rangle\) and for every \(s \in S\) there exists a root system \(\Psi\) with Weyl group \(W\) such that \(s = s_\alpha\) where \(\alpha\) is a long simple root in \(\Psi\). Throughout, given a pair \(W\) and \(s \in S\), we make this choice of \(\Psi\). As a consequence,
\( W_f \) is minuscule if and only if \( J = S \setminus \{ s_\alpha \} \) with \( n_\alpha(p) = 1 \) in this choice of root system \( \Psi \), by \[4, \text{ Ch. VIII, §7.3}\].

Concerning further general facts and notation on Coxeter groups and root systems of Weyl groups we refer the reader to \[3], \[4], and \[5]\.

### 3. Coset Graphs.

The coset graph \( \Gamma_f \) on the set of \( W_f \)-cosets of \( W \) is the directed graph with vertex set \( X_f \) and labelled edges \( x \xrightarrow{s} y \) whenever \( y = xs \) and \( l(y) = l(x) + 1 \) for \( x, y \in X_f \) and \( s \in S \). Thus, \( \Gamma_f \) is the labelled Hasse diagram of the poset \((X_f, \leq)\). Note that \( \Gamma_\varnothing \) is the Cayley graph of \( W \) with respect to the generating set \( S \).

For \( J \subseteq M \subseteq S \) denote \( X_f^M = X_f \cap W_M \), the set of distinguished right coset representatives of \( W_f \) in \( W_M \), and let \( x_f^M = \sum_{x \in X_f^M} x^{-1} \). Then \( X_f = X_f^M X_M \) and \( x_f = x_M x_f^M \).

Also, denote \( a_{fKL}^M = |X_f^{KL} \cap W_M| \).

#### 3.1. Remark.
In terms of the coset graph \( \Gamma_f \), a Mackey decomposition of \( X_f \) with respect to \( K \subseteq S \) ((\[5, (2.1.9)\])) can be visualised by omitting the edges not labelled by elements of \( K \). This leads to the restricted coset graph

\[
\Gamma_f|_K = \bigcup_{d \in X_f^K} \Gamma_f^{dK} 
\]

where \( \Gamma_f^{dK} \) is the graph with vertex set \( X_f^K \) (see \[5, (2.2.12)\]). Therefore, \( \Gamma_f \) contains \( \Gamma_f^{dK} \) as a subgraph if \( a_{fKL} > 0 \).

#### 3.2. Remark.
If \( W \) is irreducible of rank 2, then the lattice afforded by \((W, \leq)\) contains the pentagon lattice as a sublattice, whence it is not distributive. Therefore, by Remark 3.1, \((X_f, \leq)\) is not distributive if \( a_{fKL} > 0 \) for some \( K \subseteq S \) with \( W_K \) irreducible of rank 2 (i.e., \( J^d \cap K = \varnothing \) for some \( d \in X_f^K \) for such a \( K \)).

The following induction formula appears in \[1\].

#### 3.3. Lemma.
Let \( J, K, L \subseteq S \) and let \( K \subseteq M \subseteq S \). Then

\[
a_{fKL} = \sum_N a_{fMN} a_{NKLM}^M.
\]

**Proof.** Using \( x_K = x_M x_K^M \), we get

\[
\sum_L a_{fKL} x_L = x_f x_K = x_f x_M x_K^M = \sum_N a_{fMN} x_N x_K^M = \sum_N a_{fMN} x_M x_N x_K^M 
\]

\[
= \sum_N a_{fMN} x_M \sum_L a_{NKLM}^M x_L = \sum_N \sum_L a_{fMN} a_{NKLM}^M x_M x_L = \sum_L \left( \sum_N a_{fMN} a_{NKLM}^M \right) x_L.
\]

Hence \( a_{fKL} = \sum_N a_{fMN} a_{NKLM}^M \), due to the linear independence of the \( x_L \), \( (L \subseteq S) \). \[ \Box \]
For the remainder of the paper suppose that $W$ is irreducible.

**3.4. Lemma.** Let $J, K \subseteq S$ such that $W_K$ is irreducible and $|J| + |K| \leq |S|$. Then $a_{JK\emptyset} > 0$.

*Proof.* Assume $K = S$. Then $J = \emptyset$ and by definition $a_{JK\emptyset} > 0$. If $K \subseteq S$, then there exists a maximal subset $M$ of $S$ containing $K$ such that $W_M$ is irreducible. If $d \in X_{JM}$ then $W_d \leq W^r_M$, the kernel of the action of $W$ on the cosets of $W_M$. But then by Lemma 2.2(ii), $W_d \leq \{1\}$, i.e., $J = \emptyset$ and we are done. So let $d \in X_{JM}$ such that $J^d \not\subseteq M$ and denote $N = J^d \cap M$ (whence $a_{JMN} > 0$). Then $|N| < |J|$ and thus $|N| + |K| \leq |M|$. By induction on the rank, $a_{NK\emptyset}^M > 0$. Therefore, by Lemma 3.3, $a_{JK\emptyset} \geq a_{JMN} a_{NK\emptyset}^M > 0$. \hfill \Box

**3.5. Corollary.** Let $J \subseteq S$ such that $|J| \leq |S| - 2$. Then $(X_J, \preceq)$ is not distributive.

*Proof.* Clearly there exists $K \subseteq S$ such that $W_K$ is irreducible and of rank 2. But then $a_{JK\emptyset} > 0$, by Lemma 3.4, and the result follows by Remark 3.2. \hfill \Box

Denote by $T = \{s^w \mid s \in S, w \in W\}$ the set of reflections in $W$.

**3.6. Lemma.** Let $J \subseteq S$. Suppose there exist elements $s, t \in S$ and $r \in T$ such that \{r, s, t\} generate a Coxeter group of type $A_3$ with $rt = tr$ and $r, s, r^s \notin W_J$. Then $(X_J, \preceq)$ is not distributive.

*Proof.* Denote $K = \{s, t\} \subseteq S$ and $\{d\} = W_JtsrW_K \cap X_{JK}$. We show that $L = J^d \cap K = \emptyset$. The claim then follows from Remark 3.2.

Note that $K^{rst} = \{r, s\}$. Thus $\langle r, s \rangle = W^{rst}_K$. We write $tsr = udv$ for $u \in W_J$ and $v \in W_K$. Then $W^{tsr}_J \cap W_K = W^{udv}_J \cap W_K = (W^{d}_J \cap W_K)^v = W^{rst}_L$ (cf. \[5, (2.1.12)\]). Hence $W_J \cap W^{rst}_K = (W^{tsr}_J \cap W_K)^{rst} = W^{rst}_L$ is a reflection subgroup of $\langle r, s \rangle$ contained in $W_J$. The reflections of $\langle r, s \rangle$ (which is a Coxeter group of type $A_2$) are $\{s, r, r^s\}$, none of which by assumption is contained in $W_J$. Therefore, $W^{rst}_L = \{1\}$ whence $W_L = \{1\}$ and thus $L = \emptyset$. \hfill \Box

**3.7. Lemma.** Let $W$ be an irreducible Weyl group of rank at least 3 and such that the highest root $\rho$ is a fundamental weight. Let $J = \{t \in S \mid ts_\rho = ts_\rho\}$. Then $(X_J, \preceq)$ is not distributive.

*Proof.* Observing [3, Planche I–IX], the choice of $W$ implies that there is a unique simple long root $\sigma$ which is not orthogonal to $\rho$; in particular, $J = S \setminus \{s_\sigma\}$ is maximal. Note, the condition on $\rho$ implies that $\Psi$ is not of type $A_r$ or $C_r$. In addition, since $|S| \geq 3$, there exists a long simple root $\tau$ which is different from and not orthogonal to $\sigma$. Set $r = s_\rho$, $s = s_\sigma$, and $t = s_\tau$. By construction, $\{r, s, t\}$ generates a reflection subgroup of $W$ of type $A_3$ and $r, s \notin W_J$. It follows that $r^s = s^r \notin W^r_J = W_J$. Hence the set $\{r, s, t\}$ satisfies the conditions of Lemma 3.6 and the result follows. \hfill \Box
3.8. Proposition. Let $W$ be an irreducible Weyl group of rank at least 3. Let $J = S \setminus \{s_\alpha\}$ be maximal such that $n_\alpha(\rho) \geq 2$. Then $(X_J, \leq)$ is not distributive.

Proof. Recall that $\alpha$ is long by our choice of $\Psi$, cf. Remark 2.3. Let $\beta$ be the unique minimal root in $\Psi^+$ with $n_\alpha(\rho) = 2$. Let $\Pi' = \{\delta \in \Pi \mid n_\delta(\beta) \neq 0\}$ and $\Psi' = \Pi' \cap \Psi$. Then $\beta$ is the highest root in $\Psi'$ and $\alpha$ is the unique simple root in $\Pi'$ not orthogonal to $\beta$. Let $M = \{s_\delta \mid \delta \in \Pi' \setminus S\}$. Setting $N = J \cap M$, we have $a_{JM} > 0$ (for $1 \in X_{JMN}$). By choice of $W$ and our construction, $W_M$ satisfies the conditions of Lemma 3.7. In particular, there is a simple long root $\tau \in \Pi'$ which is not orthogonal to $\alpha$. Set $K = \{s_\alpha, s_\tau\}$. Then Lemma 3.7 implies $a_{KN}^M > 0$ and by Lemma 3.3 we have $a_{JK} \geq a_{JMN}^M > 0$. The result follows by Remark 3.2.

Example. We illustrate the inductive method from the proof of Proposition 3.8 by the following example. Here $W$ is of type $B_5$, $\alpha = \alpha_3$, $\tau = \alpha_2$, and $\beta = 2\alpha_3 + 2\alpha_4 + 2\alpha_5$. Thus $J = S \setminus \{s_\alpha\}$ and $\Pi' = \{\alpha_2, \ldots, \alpha_5\}$; thus $W_M$ is of type $B_4$, indicated by the dashed box. The reflection subgroup of $W$ of type $A_3$ generated by $s_{\alpha_2}$, $s_{\alpha_3}$, and $s_\beta$ is indicated by the marked nodes.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}\caption{An example in type $B_5$}
\end{figure}

Proof of Theorem 1.1. As stated in the introduction, $(X_J, \leq)$ is distributive for each of the cases listed in the theorem.

Now suppose the pair $W_J \leq W$ is not on this list. Thanks to Corollary 3.5, $J$ is maximal. Since $W$ is not dihedral, $|S| \geq 3$.

Suppose $W$ is a Weyl group. Then $W_J$ is not minuscule. By Remark 2.3, we have $J = S \setminus \{s_\alpha\}$, where $\alpha$ is a long simple root and $n_\alpha(\rho) \geq 2$. The result then follows from Proposition 3.8.

A construction similar to the one used in Lemma 3.7 can be employed to treat the non-crystallographic cases. In all cases we consider in Lemma 3.7, the reflection $r = s_\rho$ can be written as $r = s^x$ where $s = s_\sigma$ and $x = w_N w_J$ for $N = \{u \in J \mid us = su\}$. Therefore, and since $t \in J$, we have $W_J tsrW_K = W_Jsxsw_K$.

We return to the non-crystallographic instances. For $W$ of type $H_3$, the parabolic $W_J$ is of type $A_2$ or $A_1 \times A_1$. If $W$ is of type $H_4$, then $W_J$ is of type $H_3$ or $I_2(5) \times A_1$. 

All other $H_4$ cases can be reduced to $H_3$. We proceed as follows. Let $J = S \setminus \{s\}$ and let $t \in S$ such that $ts \neq st$. Let $N = \{u \in J \mid us = su\}$ and $r = sx^s$ for $x = w_N w_J$. Choose $K \subseteq S$ such that $W_K$ is of type $A_2$. Then, for $\{d\} = W_J x s W_K \cap X_{JK}$, we get $J^d \cap K = \emptyset$.

Although this is not needed for the proof of Theorem 1.1, the condition that $K$ be irreducible in Lemma 3.4 can be dropped, as we show in Proposition 3.10 below. Corollary 3.11 suggests that this result is of independent interest.

3.9. Lemma. Let $K \subseteq S$ such that $|K| \leq |S|/2$. Then there exist $L \subseteq M \subseteq S$ such that $W_L$ is conjugate to $W_K$ in $W$ and $W_M$ is irreducible.

Proof. The case $|S| \leq 1$ is trivial. For $|S| \geq 2$ we write $S$ as a disjoint union

$$S = \bigsqcup_{i \in I} S_i$$

where, for $i \in I$, $S_i = \{s_{i,1}, \ldots, s_{i,r_i}\}$ generates a Coxeter group of type $A_{r_i}$ such that $(s_{i,j-1} s_{i,j})^3 = 1$ ($j = 2, \ldots, r_i$), where $s_{i,1}$ commutes with $s_{i,j}$ whenever $i \neq i'$ unless $j = j' = 1$ ($1 \leq j \leq r_i$, $1 \leq j' \leq r_i'$, $i, i' \in I$), where $s_{i,r_i}$ commutes with all but one $s \in S$ ($i \in I$), and where $\langle s_{i,1} \mid i \in I \rangle$ is of type $A_3$ or irreducible of rank 2. (This is always possible since the Coxeter graph of $W$ contains at most one vertex of degree more than 2 or at most one edge not of type $A_2$.) In this situation the maximal parabolic subgroup $W_{M_i}$ of $W$, where $M_i = S \setminus \{s_{i,r_i}\}$, is irreducible ($i \in I$).

Now let $K \subseteq S$ be such that no conjugate of $W_L$ ($L \subseteq S$) of $W_K$ is contained in an irreducible maximal parabolic subgroup of $W$. We have to show that then $|K| > |S|/2$. Let $i \in I$ and suppose that $K \cap S_i$ does not contain both of $s_{i,j-1}$ and $s_{i,j}$ for some $1 < j \leq r_i$. Let $x = s_{i,j} s_{i,j-1} \cdots s_{i,r_i}$. Then $\langle s_{i,j+1}, \ldots, s_{i,r_i} \rangle^x = \langle s_{i,j}, \ldots, s_{i,r_i-1} \rangle$ and $x$ leaves $K \setminus \{s_{i,j}, \ldots, s_{i,r_i-1}\}$ fixed. Hence $K^x \subseteq S$ is a conjugate of $K$ that does not contain $s_{i,r_i}$ and therefore lies in $M_i$ contradicting our choice of $K$. It follows for each $i \in I$ that $K$ contains at least half of the elements of $S_i$, whence $|K| \geq \sum_{i \in I} |S_i| / 2 = |S| / 2$.

Suppose $K \cap \{s_{i,1} \mid i \in I\} = \emptyset$. By the preceding argument we then must have $r_i \geq 2$ for all $i \in I$. Let $x = s_{1,1} s_{1,2} \cdots s_{1,r_1}$. Then $\langle s_{1,2}, \ldots, s_{1,r_1} \rangle^x = \langle s_{1,1}, \ldots, s_{1,r_1-1} \rangle$ and $x$ leaves $K \setminus S_1$ fixed. Hence $K^x \subseteq S$ is a conjugate of $K$ that does not contain $s_{1,r_1}$ and therefore lies in $M_1$ contradicting our choice of $K$. It follows that $|K \cap S_i| > |S_i| / 2$ for at least one $i \in I$ whence the result.

Example. We illustrate a suitable decomposition $S = \bigsqcup_i S_i$ from the proof of Lemma 3.9 by the following example. Here $W$ is of type $E_6$. The parabolic subgroup of $W$ of type $A_3$ is indicated by the marked nodes.

3.10. Proposition. Let $J, K \subseteq S$ such that $|J| + |K| \leq |S|$. Then $a_{JK\emptyset} > 0$. 

Proof. From $X_{K\varnothing} = X_{\varnothing}^{-1}$ it follows that $a_{K\varnothing} = a_{\varnothing\varnothing}$. Thus we may assume that $|K| \leq |S|/2$. Note also that $a_{KL\varnothing} = a_{K\varnothing}$ for all conjugates $L \subseteq S$ of $K$ (for, if $W_L = W_K^x$ for some $x \in W$, then right multiplication by $x$ induces a bijection $W_j d W_K = W_j d W_K = W_j d W_L$ of double cosets and $W_j d W_K = \varnothing$ implies $W_j d W_L = \varnothing$). Therefore, using Lemma 3.9, we may assume that there exists $M \subseteq S$ such that $W_M$ is an irreducible maximal parabolic subgroup of $W$ containing $W_K$. Now we proceed as in the proof of Lemma 3.4.

As a special case, we immediately obtain from Proposition 3.10 and Remark 3.1:

3.11. Corollary. For each $s \in S$ the coset graph $\Gamma_{[s]}$ contains the full Cayley graph of every maximal parabolic subgroup of $W$ as a subgraph.

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References


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