Length and reflections: odds and ends

Angela Carnevale

Ollscoil na Gaillimhe
University of Galway

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What this talk is about

odd analogues + k-analogues

$\boxed{\text{Permutations}}$ + other Coxeter groups

$\boxed{\text{Length}}$ + other lengthlike statistics

$\boxed{\text{Bruhat order}}$ + other posets old and new

Permutations and length

Coxeter gp generated by $\{(i\ i+1) : i=1,..n-1\}$

S_n = permutations of $\{1,\dots n\}$

$\overset{\psi}{\sigma}$ one-line $\qquad \sigma \in S_5 \qquad \sigma = 51432$

$$\mathrm{inv}(\sigma) = \left|\{(i\ j) : 1 \le i < j \le n : \sigma(i) > \sigma(j)\}\right|$$

(W, S) $\quad W$ generated by S (involutions) + other rel.

$$\ell_S(w) = \min\{r : w = s_{i_1} \dots s_{i_r},\ s_k \in S\}$$

if $W = S_n$

$\ell_S(\sigma) = \mathrm{inv}(\sigma)$

Bruhat order

$T = \{ w s w^{-1} \mid w \in W, s \in S \}$

$W = S_n$

$T =$ all transpositions

Bruhat graph

$B(W) = (W, E)$ with $u \to v$ if $\ell(u) < \ell(v)$ & $uv^{-1} \in T$

Bruhat order

$(W, \leq)$ transitive closure of $B(W)$

Right weak order $(W, \leq_R)$

transitive closure of $u \lessdot_R v$ if $\ell(u) = \ell(v) - 1$ & $uv^{-1} \in S$

“Odds”…

Joint with Francesco Brenti, and with Francesco Brenti and Bridget Tenner.

Odd Length

$\sigma \in S_n$

$$L(\sigma) = |\{ \ldots : \sigma(i) > \sigma(j) \ \& \ j-i \equiv 1 \pmod 2 \}|$$

$$L(51432) = 4$$

Some facts

- Odd length introduced in the context of **zeta functions** in algebra.

  (Klopsch - Voll '09, Stasinski - Voll '13)

- Applications to the **enumeration of matrices** over finite fields.

  (Stasinski - Voll '13, Brenti - C. '17)

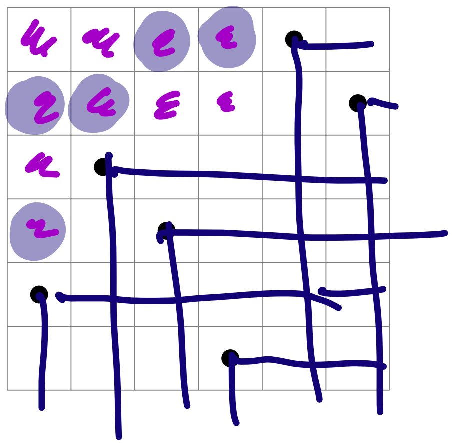
- Generalised to **classical Weyl groups** and **finite Coxeter groups**.

  (Brenti - C. '19)

- **Odd and even major indices** were recently studied.

  (Brenti - Sentinelli '21)

~~(Odd)~~ Rothe diagrams



$\sigma = 562314$

● $=$ graph of σ

size of ξ $= \ell(\sigma)$

Odd diagrams

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			$\bullet$		

$\sigma = 562314$

$\bullet$ = graph of σ

$*$ = odd diagram of σ

Odd diagrams (Brenti - C. 21)

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$\sigma = 562314$

$\bullet$ = graph of σ

$*$ = odd diagram of σ

- $L(\sigma) = |\text{odd diagram of } \sigma|$
- "The usual kind of magic" works to define an **odd Schubert variety associated with a permutation** σ. of dim $L(\sigma)$
- The diagram of a permutation knows everything about the permutation. . .

Odd diagrams

…how much does an **odd** diagram know about a permutation?

Not so much!

213

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312

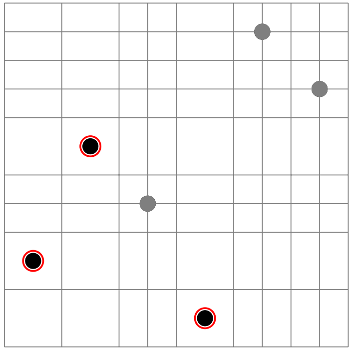
Questions:

- How many odd diagrams are there?
- How do odd diagram classes look like?

How many odd diagrams are there?

The first values of the sequence $|\{\text{odd diagram of } \sigma : \sigma \in S_n\}|$ are:

$$1, 2, 5, 17, 70, 351, 2041, 13732, 103873, 882213.$$



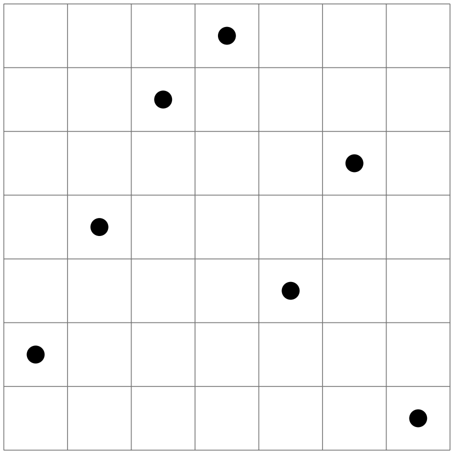
$\sigma = 562314$

σ contains the pattern 213

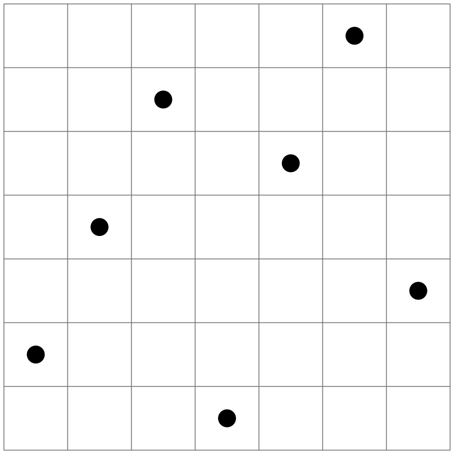
Finding patterns

A **key fact** is the following:

If $\sigma \sim \tau$ with $\sigma \neq \tau$ then σ contains the pattern 213 and τ contains the pattern 312.



$\sigma = 4362517$



$\tau = 6352714$

Odd diagrams and permutation patterns

Theorem (Brenti - C. - Tenner '22)

Every odd diagram class contains

- at most one permutation avoiding the pattern 213, and
- at most one permutation avoiding 312.

If these permutations exist, they are the **longest** and **shortest** elements of the class, respectively.

Corollary. There are at least n-th-Catalan-many (and in fact, at least n-th-Bell-many) odd diagrams arising from permutations in S_n.

Odd diagrams and Bruhat order

Every odd diagram class contains a **unique minimal** and a **unique maximal element**.

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$\sigma = 6731425$

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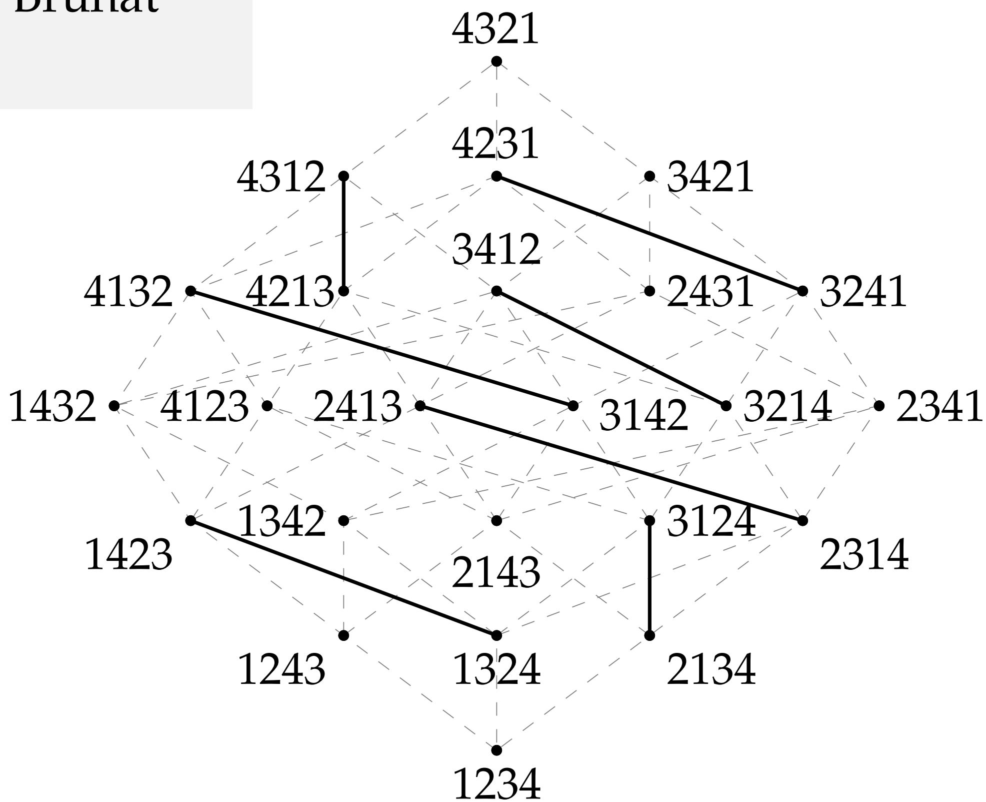
$\tau = 6751423$

Odd diagrams and Bruhat order

Theorem (Brenti - C. - Tenner ’22)

Odd diagram classes are Bruhat intervals.



Odd diagrams and Bruhat order

Odd diagram classes are:

- Antichains in right weak order.
- Not self-dual in general.
- Rank-symmetric (Fan - Guo '22).
  They show that the Poincaré polynomials of odd diagram classes are products of finite geometric progressions.

Length and reflections: odds and ends

?

Length and reflections

Fact. The Coxeter length of an element can be written in terms of reflections:

$$\boxed{\ell(\sigma) = |\{t \in T : \ell(\sigma t) < \ell(\sigma)\}|}$$

Similarly, one can interpret the odd length written in terms of **odd reflections**:

$$\boxed{L(\sigma) = |\{t \in T : \ell(t) \equiv 1 \pmod 4,\ \ell(\sigma t) < \ell(\sigma)\}|}$$

. . . and “ends”

Joint with Matthew Dyer and Paolo Sentinelli.

k-analogues

Intermediate orders

Let (W, S) be a Coxeter system and T its set of reflections.

For $k \geqslant 0$ we let
$$T_k := \left\{ t \in T : \frac{\ell(t) - 1}{2} \leqslant k \right\}.$$

$k = 0 \quad T_k = S$

$k >> 0$ & W finite $\quad T_k = T$

Intermediate orders

Set $u \lhd_k v$ if and only if $\ell(u) = \ell(v) - 1$ and $uv^{-1} \in T_k$.

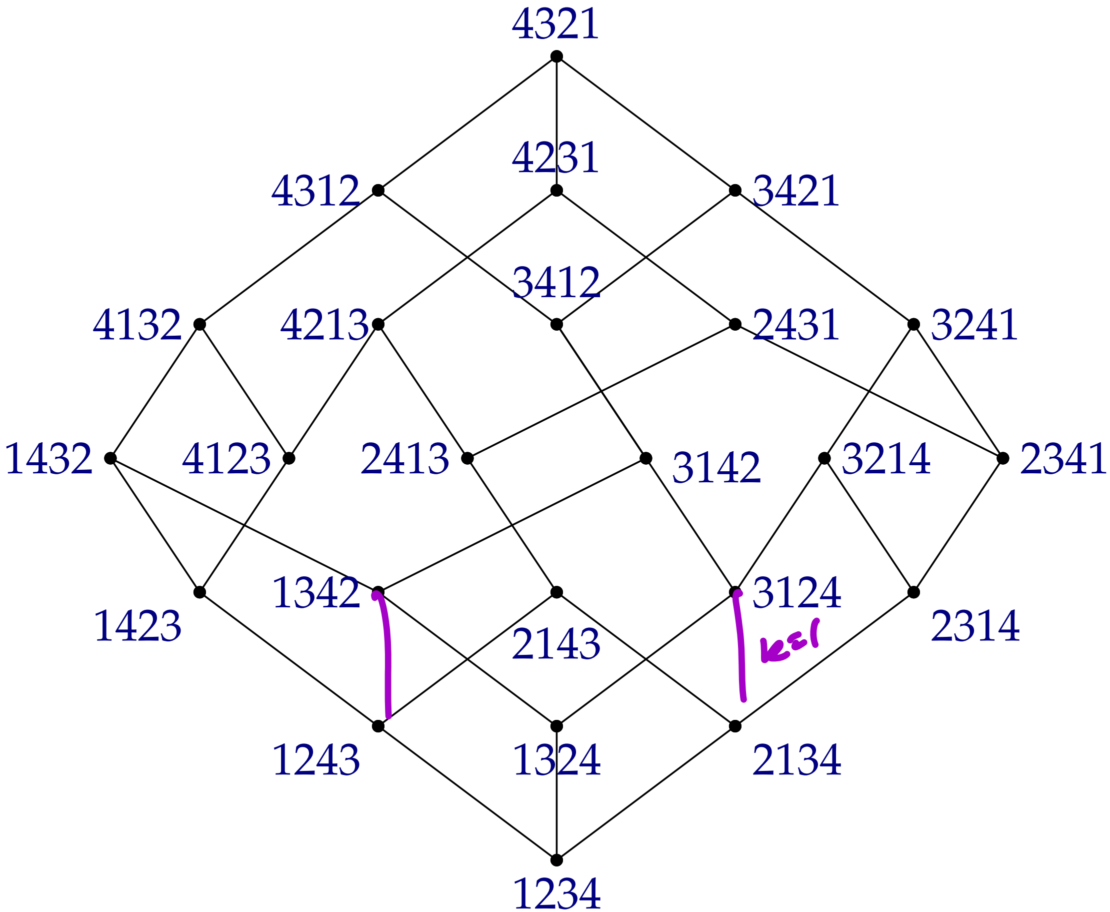
Define $\leqslant_k$ to be the transitive closure of $\lhd_k$.

We call $\leqslant_k$ a k-**intermediate order** on W.

$\leq_0$ = weak order

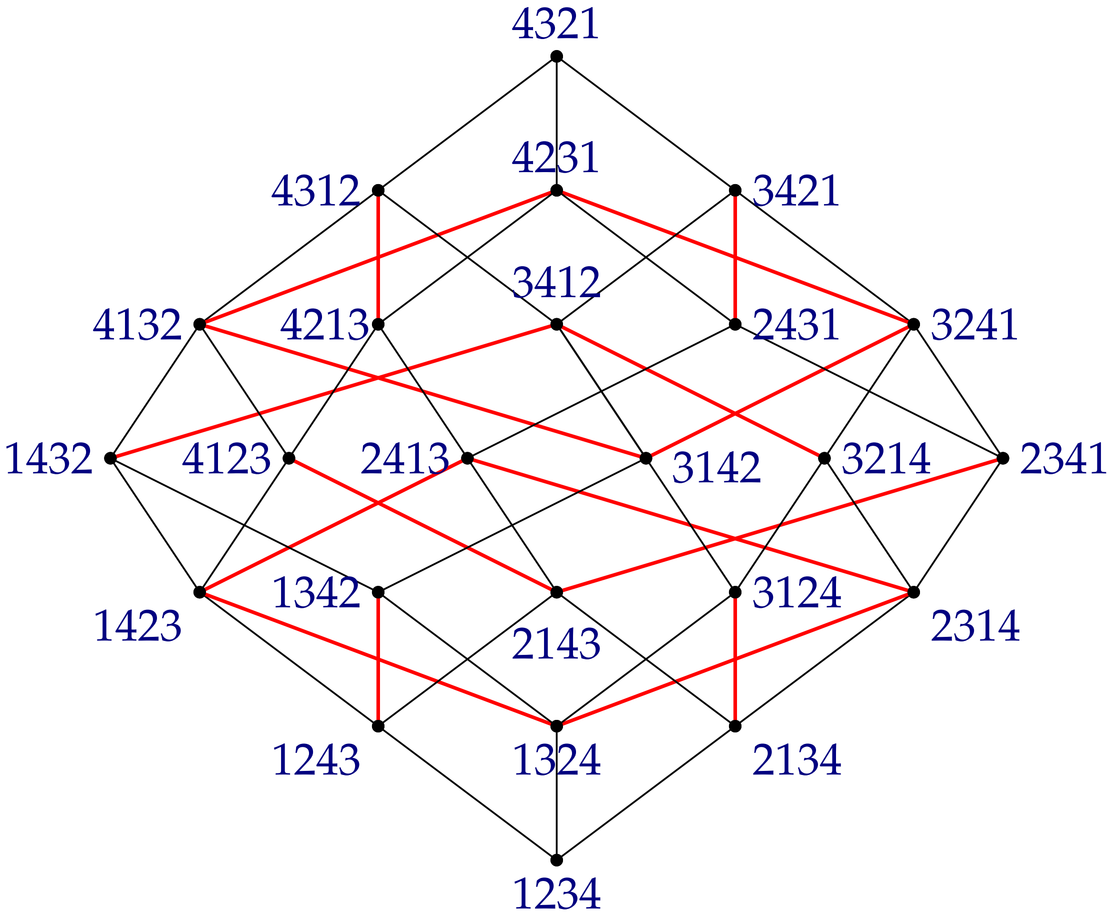
Intermediate orders

k = 0

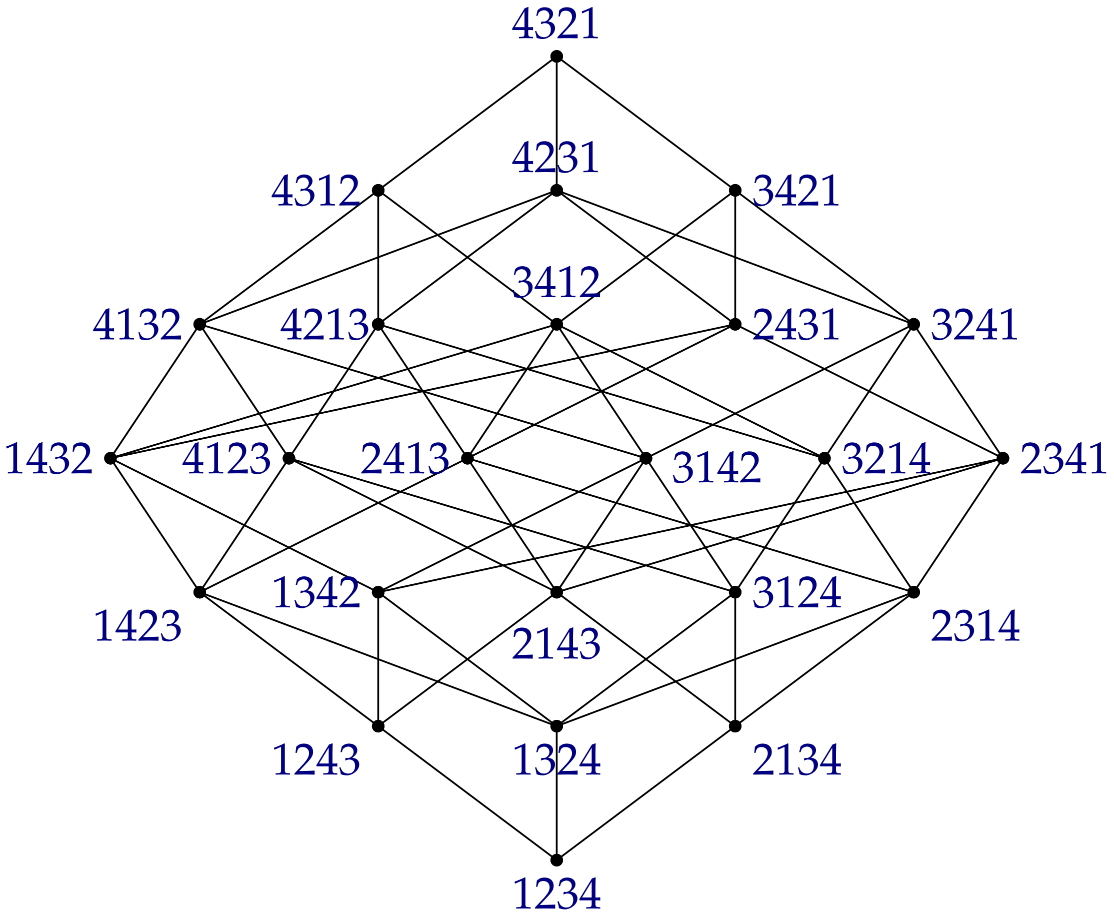


Intermediate orders

$k = 1$



Intermediate orders



Intermediate orders

Theorem (C. - Dyer - Sentinelli '22)

The k-intermediate orders are graded by the Coxeter length ℓ.

- Special case of a more general result, pertaining to orders defined starting with more general sets of reflections.
- Projections to (left) parabolic quotients are order preserving.
- k-intermediate orders on S_n are Sperner.

k-Bruhat graphs, k-absolute length

Using the same idea, we define other k-analogues.

$\boxed{k\text{-Bruhat graph}}$ $B_k(w)$

$\boxed{k\text{-absolute length}}$ $\ell_k(w) = \vec{d}_k(e, w)$

Unimodality?

Example

$$\sum_{\sigma \in S_4} x^{\ell_1(\sigma)} = 1 + 5x + 10x^2 + 7x^3 + x^4$$

Conjecture (Brenti - C. '21)

$\sum_{w \in S_n} x^{L(w)}$ is unimodal for all $n \geqslant 5$.

Conjecture (C. - Dyer - Sentinelli '22)

$\sum_{w \in W} x^{\ell_k(w)}$ is unimodal for any finite Coxeter group W and all $k \geqslant 0$.

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