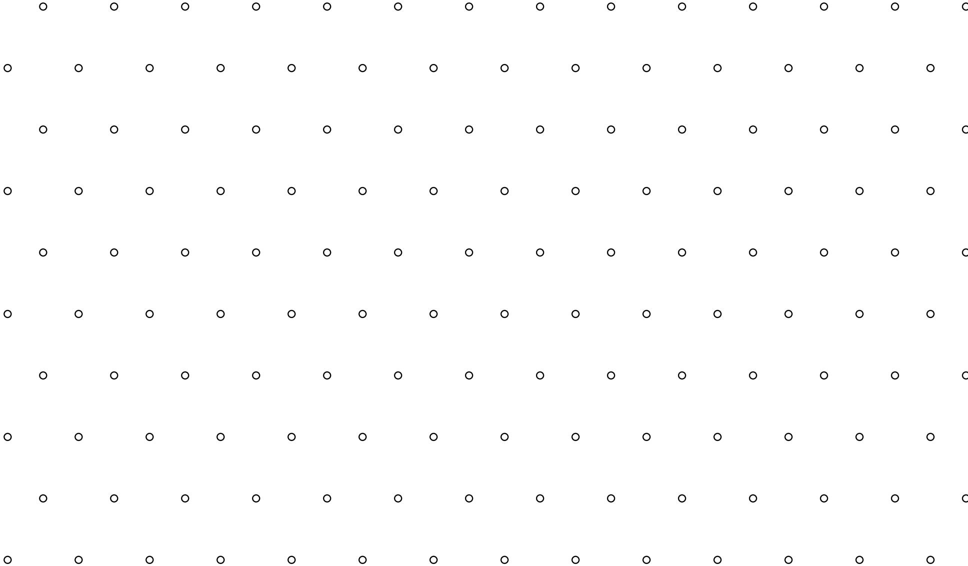
Flatness constants and lattice-reduced bodies

Giulia Codenotti

September 7th 2022

Geometry meets Combinatorics in Bielefeld

Lattices



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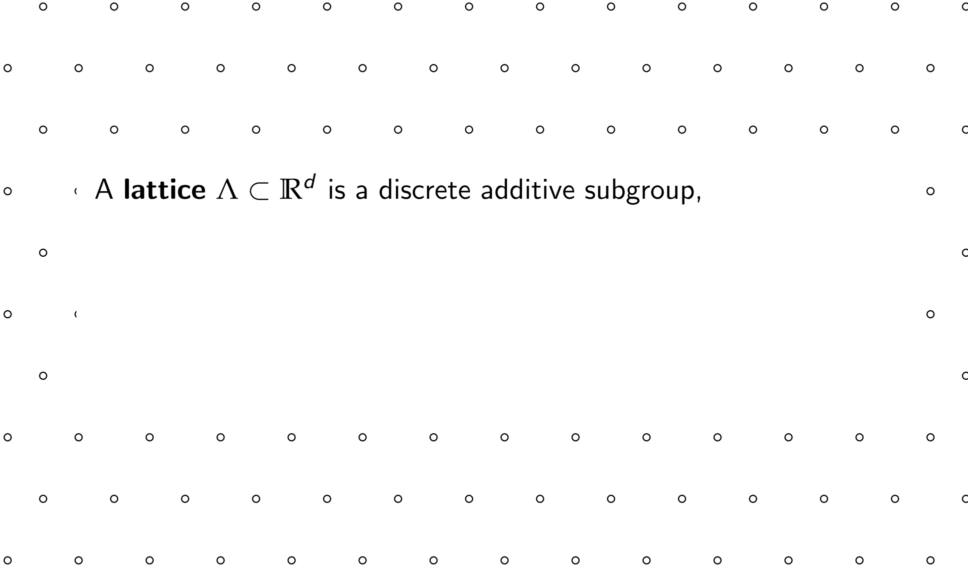

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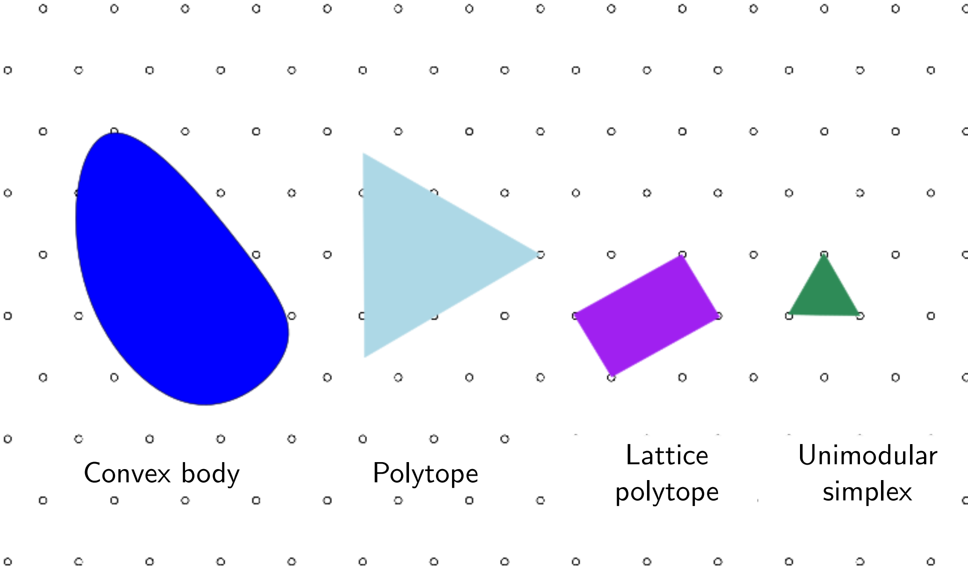
- The **dual lattice** $\Lambda^* \subseteq (\mathbb{R}^d)^*$ of Λ is the lattice of functionals taking integer values on points of Λ.
- A **unimodular tranformation** ϕ is a linear transformation of $\mathbb{R}^d$ with $\phi(\Lambda) = \Lambda$.

Convex bodies and polytopes



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Given K a convex body in $\mathbb{R}^d$; and $\mathbb{Z}^d \cong \Lambda \subset \mathbb{R}^d$ a lattice

Definition

- The **width** of K w.r.t. a functional $c \in (\mathbb{R}^d)^*$ is

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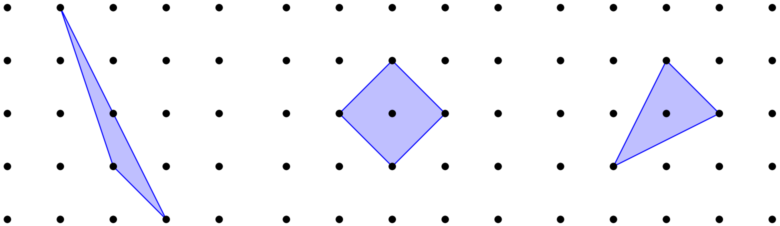

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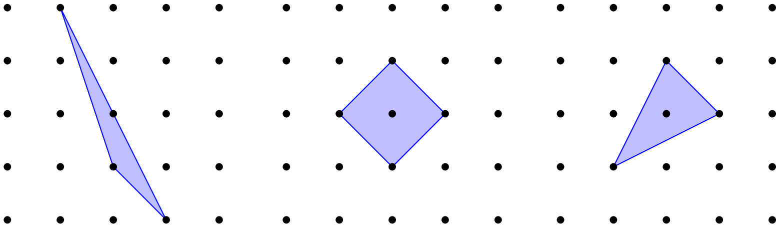

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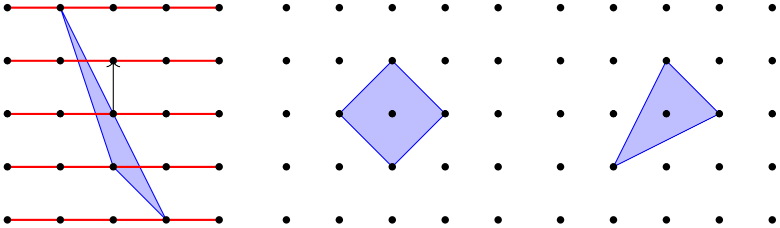

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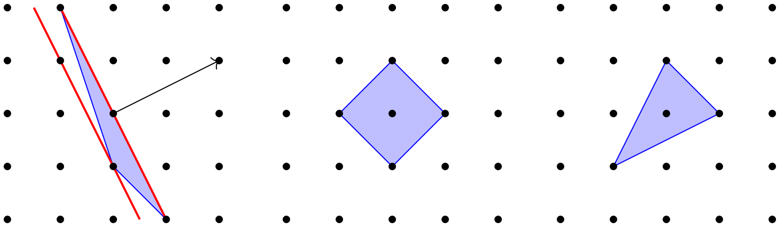

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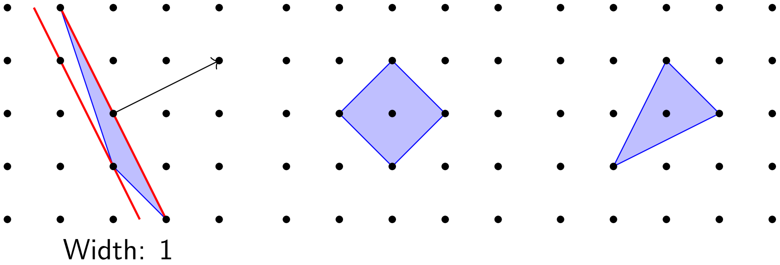

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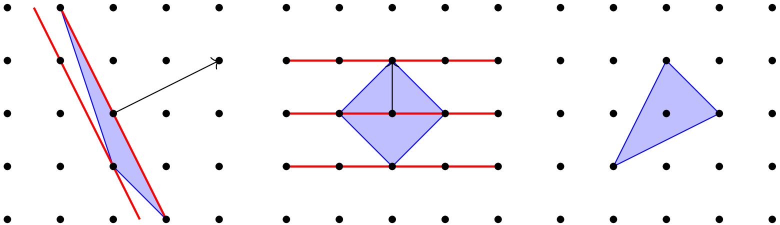

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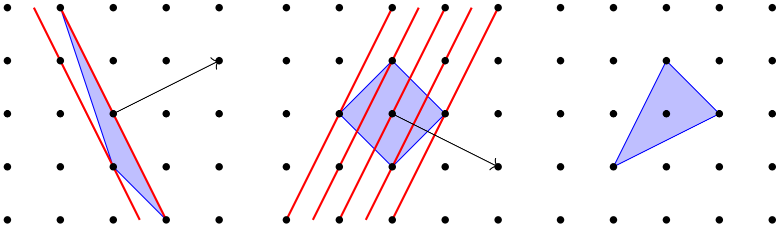

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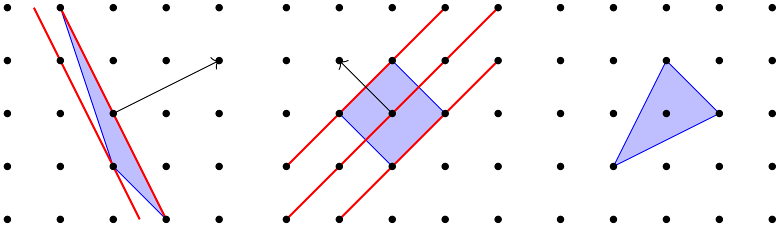

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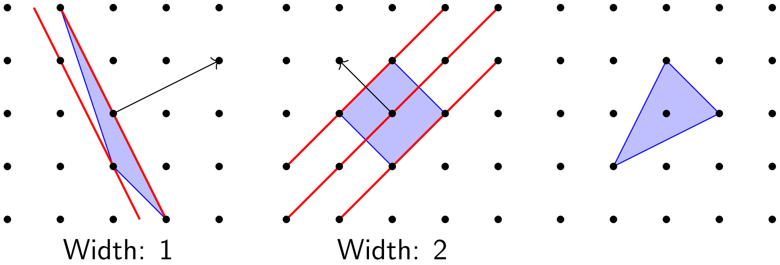

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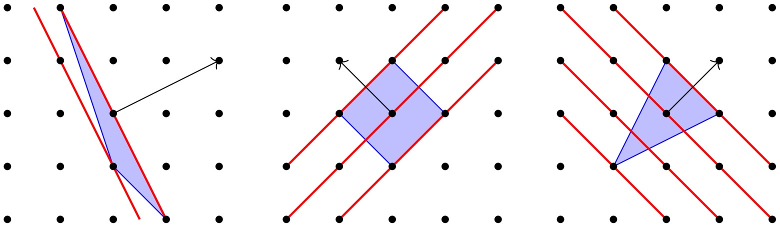

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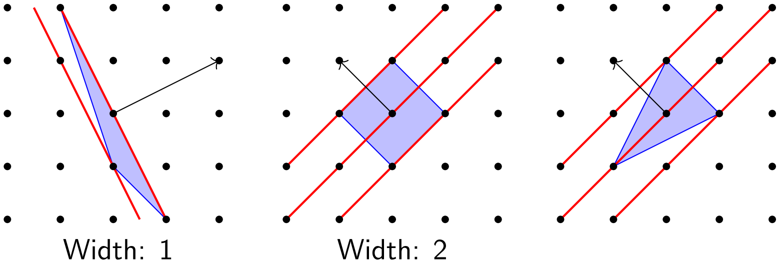

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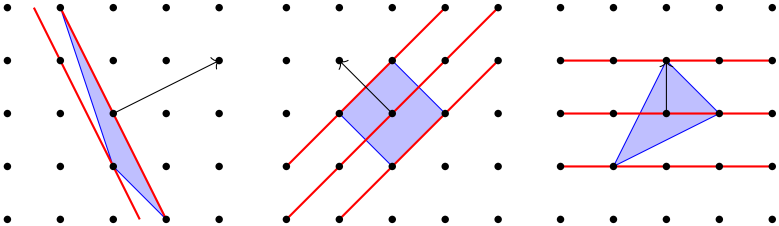

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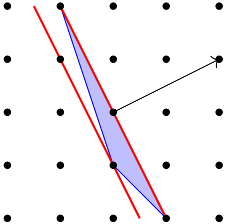

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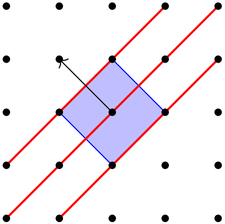

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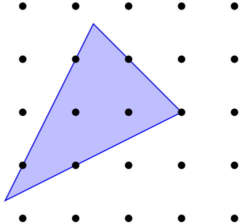

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Flatness theorem

A convex body is hollow if there are no lattice points in its interior.

Theorem (Kinchine 1948)

If $K \subset \mathbb{R}^d$ is a hollow convex body, then its width is bounded by a constant $\mathrm{Flt}(d)$.

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Exact values are mostly unknown!

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- Introduce **lattice-reduced** convex bodies and explore their connection to flatness.

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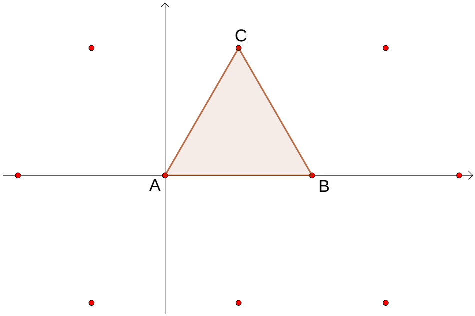
  🔑: study the width of *inclusion-maximal* hollow convex sets.

Theorem (Lovász '89)

Bounded inclusion-maximal hollow convex sets in $\mathbb{R}^d$ are polytopes with $\leq 2^d$ facets and a lattice point in the interior of each facet.

$\mathrm{Flt}(2) = 1 + \frac{2}{\sqrt{3}}$: Hurkens' construction



A triangular lattice and a unimodular triangle ABC.

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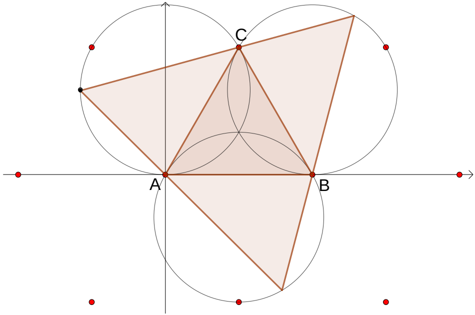

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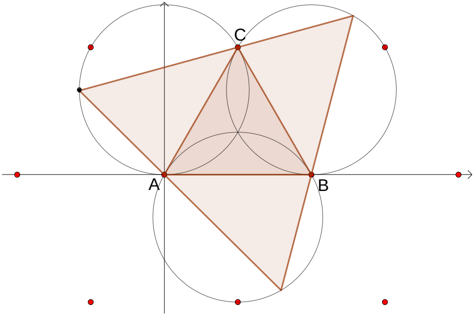


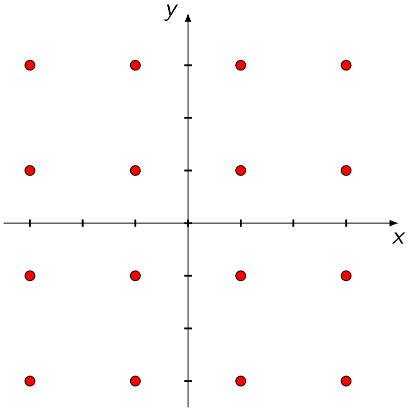
Among triangles with vertices on the circles and containing A, B, and C on the boundary, this triangle has largest lattice width, equal to $1 + \frac{2}{\sqrt{3}}$.

Theorem (Hurkens 1990)

This triangle has the largest lattice width of any hollow convex body in $\mathbb{R}^2$*; that is,* $\mathrm{Flt}(2) = 1 + \frac{2}{\sqrt{3}}$.

Flt(3): A wide tetrahedron

In the (affine) lattice
$\{(a, b, c) : a, b, c \in 1 + 2\mathbb{Z}, a + b + c \in 1 + 4\mathbb{Z}\}$,



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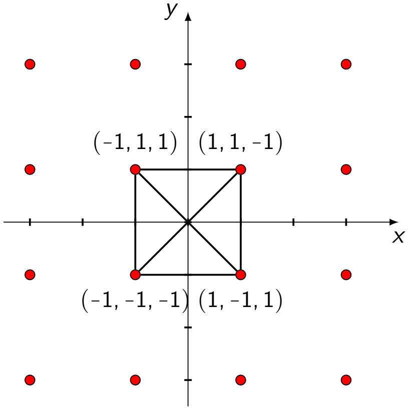

$$\Gamma = \text{conv}\{(-1, 1, 1), (-1, -1, -1), (1, -1, 1), (1, 1, -1)\}$$

is a unimodular simplex.

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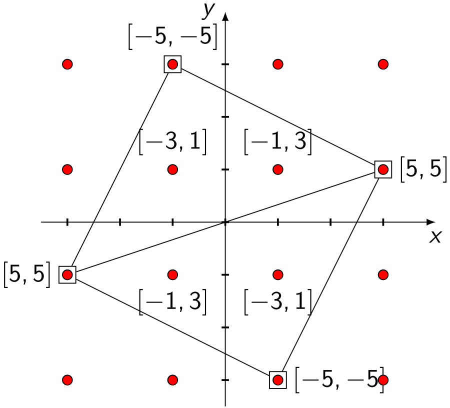
Consider the family of tetrahedra $\Delta(x, y, z)$ circumscribed to Γ and with vertices of the form

$$A = (x, y, z),$$
$$B = (-y, x, -z),$$
$$C = (-x, -y, z),$$
$$D = (y, -x, -z).$$



$\Delta(3, 1, 5)$ is a hollow lattice tetrahedron

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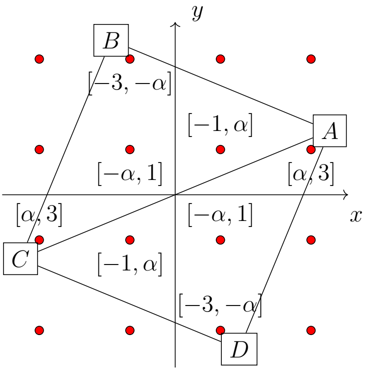

Figure:
$\bar{\Delta} := \Delta(2+\sqrt{2}, \sqrt{2}, 2+\sqrt{2})$ has width $2+\sqrt{2}$

Theorem (C.-Santos, 2020)

The width of any $\Delta(x, y, z)$ in this family is at most $2+\sqrt{2}$, with equality if and only if $(x, y, z) = \left(2+\sqrt{2}, \sqrt{2}, 2+\sqrt{2}\right)$, or $(x, y, z) = \left(\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}\right)$.

Thus,

Corollary (C.-Santos, 2020)

$\mathrm{Flt}(3) \geq 2+\sqrt{2}$.

$\mathrm{Flt}(3) = 2 + \sqrt{2}$?

Conjecture (C.-Santos)

$\bar{\Delta}$ is the hollow 3-body of maximum width. That is, $\mathrm{Flt}(3) = 2 + \sqrt{2}$.

We can prove a local version of the conjecture, namely:

Theorem (Averkov-C.-Macchia-Santos)

$\bar{\Delta}$ is a strict local maximizer for width among hollow tetrahedra. That is, every small perturbation of $\bar{\Delta}$ is either non-hollow or has width strictly smaller than $2 + \sqrt{2}$.

Corollary

$\bar{\Delta}$ is a strict local maximizer for width among hollow convex bodies.

$\mathrm{Flt}(d)$?

Recently, wide hollow simplices in arbitrary dimension were constructed:

Theorem (Mayrhofer-Schade-Weltge)

$\mathrm{Flt}(d) \geq 2d - O(d)$, *attained by a family of hollow simplices.*

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The examples seen so far bring us to the following conjecture:

Conjecture (1)

Hollow width-maximising convex bodies in any dimension d are always simplices.

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Conjecture (1')

There is a hollow width-maximising simplex in any dimension d.

A generalization of flatness

Definition

Let $K \subset \mathbb{R}^d$ be a convex body, and let $X \subset \mathbb{R}^d$.

K is hollow $\overset{\text{def}}{\Leftrightarrow}$ its interior doesn't contain any affine unimodular transformation of the origin.

A generalization of flatness

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Let $K \subset \mathbb{R}^d$ be a convex body, and let $X \subset \mathbb{R}^d$.

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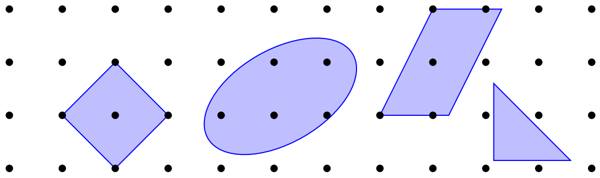

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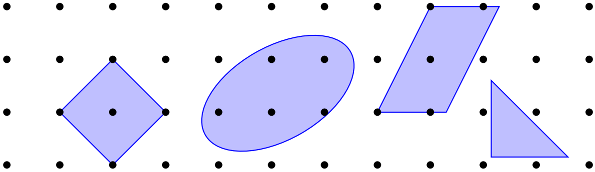

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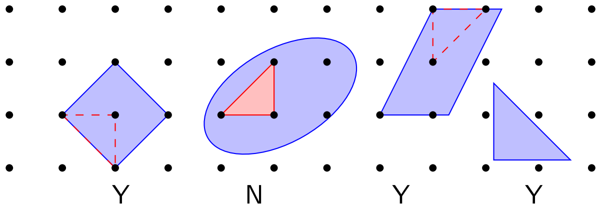
K is $\mathbb{R}$-X-free $\overset{\text{def}}{\Leftrightarrow}$ its interior doesn't contain any unimodular transformation + real translation of X.

are these...
$\mathbb{Z}$-Δ_2-free?
$\mathbb{R}$-Δ_2-free?



A generalization of flatness

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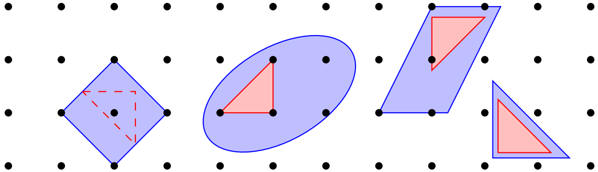

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Generalization of flatness

Theorem (Averkov-Hofscheier-Nill '19)

For fixed $d \in \mathbb{N}$, $X \subset \mathbb{R}^d$, $A \in \{\mathbb{Z}, \mathbb{R}\}$, there exists a constant $\mathrm{Flt}_X^A(d)$ larger than the width of any A-X-free convex body.

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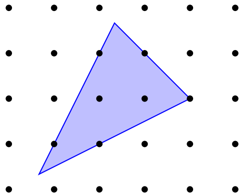
Inclusion-maximal A-X-free convex bodies are special:

Theorem (C.-Hall-Hofscheier)

If $X \subset \mathbb{R}^d$ *is a full-dimensional polytope, then every inclusion-maximal A-X-free convex body* $K \subset \mathbb{R}^d$ *is a polytope.*

A dimension 2 case for $X = \Delta_2$

Consider the unimodular simplex $X = \Delta_2$ inside of $\mathbb{R}^2$.



Theorem

We have $\mathrm{Flt}^{\mathbb{Z}}_{\Delta_2}(2) = \frac{10}{3}$*, achieved uniquely by the triangle to the right.*

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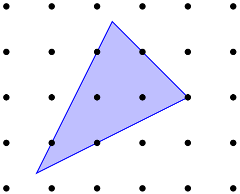

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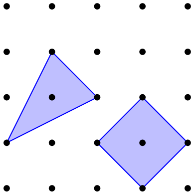

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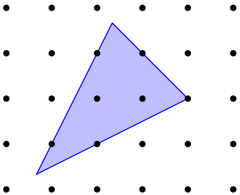

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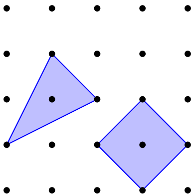


Theorem

We have $\mathrm{Flt}^{\mathbb{R}}_{\Delta_2}(2) = 2$ and it is achieved by the polygons on the left.

Question

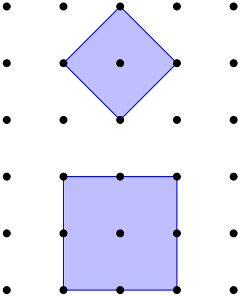
Is there always at least one simplex among width maximizers of $\mathrm{Flt}^{A}_{\Delta_d}(d)$? If $A = \mathbb{Z}$, are all maximizers simplices?

Lattice-reduced convex bodies

Definition

A convex body K is **lattice-reduced** if it is inclusion-minimal among convex bodies of the same width, i.e.,

$$K' \subsetneq K \implies \text{width}(K) > \text{width}(K').$$



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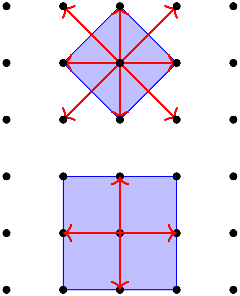
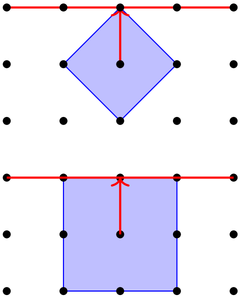


Lattice-reduced convex bodies



Definition

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Proposition

A lattice-reduced convex body K is always a polytope and for every vertex v of K there is a direction $c \in \Lambda^*$ such that

(i) $\text{width}_c(K) = \text{width}_\Lambda(K)$,

(ii) $c^T \cdot v > c^T \cdot x$ *for any* $x \in K \setminus \{v\}$.

Lattice-reduced and flatness

Recall our bold conjecture:

Conjecture (1)

All hollow width-maximising convex bodies are simplices.

Lattice-reduced and flatness

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Lattice-reduced and flatness

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All hollow width-maximising convex bodies are simplices.

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All hollow width-maximising convex bodies are lattice-reduced.

(1) implies (2), thanks to the following result:

Theorem (C., 22+)

Hollow simplices which are width-maximisers are lattice-reduced.

Lattice-reduced convex bodies

So what can we say about lattice-reduced convex bodies? How many vertices can they have?

Lattice-reduced convex bodies

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The following is a Lovász-style result:

Theorem (C.-Freyer, 22+)

A lattice-reduced convex body K is always a polytope with at most $2^{d+1}-2$ vertices. Furthermore, this bound is tight: the dual of the permutohedron has exactly this many vertices and is lattice-reduced.

Lattice-reduced convex bodies

Why should we care about lattice-reduced polytopes?

Lattice-reduced convex bodies

Why should we care about lattice-reduced polytopes?

- They are cool and as far as I can tell haven't been studied!
- Analogue of the well-studied analogue for Euclidean width.
- Could all hollow width-maximisers be lattice-reduced?
- If not, studying those that are might still yield new constructions for lower bounds on flatness constant(s)...

Thank you for your attention!

Giulia Codenotti, Francisco Santos. *Hollow polytopes of large width.* In Proceedings of the AMS.
`http://arxiv.org/abs/1812.00916`

Gennadiy Averkov, Giulia Codenotti, Antonio Macchia, Francisco Santos. *A local maximizer for lattice width of 3-dimensional hollow bodies.* In Discrete Applied Mathematics.
`http://arxiv.org/abs/1907.06199`

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Preprint: https://arxiv.org/abs/2110.02770.